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Two aspects of the theta divisor associated with the autonomous Garnier system of type 9/2

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Abstract. In a previous paper [**16**], we considered the 40 types of autonomous 4-dimensional Painlevé-type equations and studied the degenerations of their spectral curves. In this paper, we treat the autonomous Garnier system of type $\frac{9}{2}$, one of the most degenerated systems, as an example to illustrate another important curve (or union of curves) associated with the system: the Painlevé divisor [**2**]. These curves can be considered as the theta divisor of the Liouville tori, which in turn is the Jacobian of these curves. Some of the possible applications are construction of 2×2 Lax pair using separation of variables [**24**], and making identification or distiction with other systems.

1. Introduction

The Painlevé equations are 8 types of second-order nonlinear equations with the Painlevé property: the positions of multi-valued singularities of any of the solutions are independent of the initial conditions. They were discovered by Painlevé and Gambier around the year 1900. The Painlevé equations are now recognized to define useful special functions called the Painlevé transcendents. One way to think about these functions is they are non-autonomous generalization of the elliptic functions. These 8 equations constitute a family linked by degeneration process and the sixth Painlevé equation is the source equation of all the others.

Various generalization of the Painlevé equations have been proposed. The most classical generalization is the ones called the Garnier systems. They are derived by Garnier as a generalization of the sixth Painlevé equation to higher orders from isomonodromic point of view, in a modern terminology. For the fourth order Garnier system, the degenerations from the Garnier's original system were studied by Kimira [**12**] and Kawamuko [**11**] and there are 16 systems in their list.

There are other generalizations of the Painlevé equations to higher dimensions as well. For instance, Noumi and Yamada [**18**] proposed generalizations form representations of affine Weyl group of type $A_n^{(1)}$. Many generalizations have been derived by reductions of different soliton hierarchies. However, the overall situation or the relation between systems of different origins were not clear until recently.

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The situation has improved for the 4-dimensional case thanks to the development of the classification theory of linear equations. Let us call the isomonodromic equations as the Painlevé-type equations. Oshima [**20**] classified the Fuchsian linear equation with 4 accessory parameters up to Katz operations. Sakai [**21**] derived the Hamiltonians of the isomonodromic systems for Oshima's list and he gave all the 4-dimensional generalizations of the sixth Painlevé equation. Degenerations to unramified non-Fuchsian cases are computed by Kawakami, Nakamura and Sakai [**10**], and ramified cases are studied by Kawakami [**9**]. In total, there are 40 types of 4-dimensional Painlevé-type equations. Some of the 4-dimensional generalizations of the Painlevé equations such as the Noumi-Yamada system of type $A_4^{(1)}$ and $A_5^{(1)}$, can now be understood in the framework of degeneration scheme.

Despite the recent developments, we still do not know the relations between some of the alleged 4-dimensional generalizations of the Painlevé equations and those in the degeneration scheme. When such systems have corresponding Lax pairs, it is helpful to study their spectral types of the linear equations in order to identify with the systems in the degeneration scheme. However, the systems in Cosgrove's list [**5**], for example, are not a priori given their Lax pairs. This situation is one of the background of our work.

In a previous paper [**16**], we derived 40 types of integrable systems as the autonomous (isospectral) limit of the 4-dimensional Painlevé-type equations and studied the degenerations of their spectral curves, which are curves of genus two. In this paper, we take the autonomous Garnier system of type $\frac{9}{2}$ as an example to illustrate another important curve associated with the integrable system. These two types of curves, the spectral curve and (irreducible components of) the Painlevé divisor, are two incarnation of the theta divisor of the Liouville torus, which is an abelian surface. We take coordinates to derive explicit equations of these isomorphic curves. The reason why we dare to compute such abstractly trivial result is the following. If integrable systems can be characterized by the degeneration of their spectral curves, as we hope, and when the integrable system in consideration is of dimension 4, we can utilize the isomorphic curves, (irreducible components of) the Painlevé divisor, to identify the system with known ones by studying their degenerations.

2. Preliminaries

In this section, we summarize the background of the computations in the following sections. The main reference for this section is a thorough exposition on algebraically integrable systems by Adler, van Moerbeke and Vanhaecke [**2**].

2.1. Algebraically completely integrable systems

We often consider integrable systems with complex variables and those satisfying integrability in a stronger sense than Liouville integrability as in the following definition.

Definition 2.1 ([**1**]). *Let* $(M, \{\cdot,\cdot\}, \mathbf{H})$ *be a complex integrable system, where* $(M, \{\cdot,\cdot\})$ *is an affine nonsingular Poisson variety of rank* $2r$ *and* $\mathbf{H} = (H_1, \dots, H_n)$ *are* $n(= \dim M - r)$ *functions in involution. We say* $(M, \{\cdot,\cdot\}, \mathbf{H})$ *is an algebraically completely integrable, or a.c.i. system for short, if for generic* $h \in \mathbb{C}^n$ *the fiber* $\mathbf{H}_h = \mathbf{H}^{-1}(h)$ *is an affine part of an Abelian variety and if the Hamiltonian vector fields* X_{H_i} *are translation invariant, when restricted to the fibers. In the particular case when* M *is an affine space* $\mathbb{C}^{n+r}$*, we call* $(\mathbb{C}^{n+r}, \{\cdot,\cdot\}, \mathbf{H})$ *a polynomial a.c.i. system.*

The phase space of an a.c.i. system admits, locally on the base space, a partial compactification, and that the integrable vector fields extend holomorphically to this partial compactification [**2**].

Definition 2.2 ([**2**]). *Let* $\mathcal{D}$ *be the analytic hypersurface adjoined in the partial compactification of the phase space of a polynomial a.c.i system* $\mathbf{H}\colon \mathbb{C}^{n+r} \to \mathbb{C}^n$*. The restriction* $\mathcal{D}_h$ *of* $\mathcal{D}$ *to generic fiber* $\mathbf{H}_h$ *for* $h \in \mathbb{C}^n$ *is called the Painlevé divisor at* h*.*

2.2. Weight-homogeneous system

We will restrict our attention to weight-homogeneous systems, which is defined as follows.

Definition 2.3 ([**2**]). *Let* $\nu = (\nu_1, \dots, \nu_n)$ *be a weight vector comprised of positive integers without a common divisor and denote* $\nu_i = \varpi(x_i)$*. A polynomial* $f \in \mathbb{C}[x_1, \dots, x_n]$ *is weight-homogeneous of weight* k *with respect to* $\nu = (\nu_1, \dots, \nu_n)$ *if*

$$f(t^{\nu_1} x_1, \dots, t^{\nu_n} x_n) = t^k f(x_1, \dots, x_n).$$

A polynomial vector $\mathcal{V}$ *field on* $\mathbb{C}^n$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n), \end{aligned}$$

is called a weight-homogeneous vector field of weight k *with respect to* ν *if each of the polynomial* $f_1, \dots, f_n$ *is weight-homogeneous with respect to* $\nu = (\nu_1, \dots, \nu_n)$ *and if* $\varpi(f_i) = \nu_i + k$*, where* $\varpi(f)$ *stands for the weight of* f*.*

Proposition 2.4 ([**2**]). *Suppose that* $\mathcal{V}$ *is a weight-homogeneous vector field* *on $\mathbb{C}^n$, given by*

$$\dot{x}_i = f_i(x_1, \ldots, x_n),\ (i = 1, \ldots, n),$$

and suppose that

$$x_i(t) = \sum_{k=0}^{\infty} x_i^{(k)} t^{-\nu_i + k},\ (i = 1, \ldots, n)$$

is a weight-homogeneous Laurent solution for this vector field. Then the leading coefficients $x_i^{(0)}$ satisfy the non-linear algebraic equations

$$\begin{aligned} \nu_1 x_1^{(0)} + f_1(x_1^{(0)}, \ldots, x_n^{(0)}) &= 0, \\ &\vdots \\ \nu_n x_n^{(0)} + f_n(x_1^{(0)}, \ldots, x_n^{(0)}) &= 0. \end{aligned} \tag{1}$$

The subsequent terms $x_i^{(k)}$ of the Laurent solution satisfy

$$(k \operatorname{Id}_n - \mathcal{K}(x^{(0)}))x^{(k)} = R^{(k)},$$

where

$$x^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix},\ R^{(k)} = \begin{pmatrix} R_1^{(k)} \\ \vdots \\ R_n^{(k)} \end{pmatrix},$$

and each $R_i^{(k)}$ is a polynomial of $x_1^{(l)}, \ldots, x_n^{(l)}$ with $1 \leq l \leq k-1$. The (i,j)-th entry of the $(n \times n)$-matrix $\mathcal{K}$ is the regular function on $\mathbb{C}^n$ defined by

$$\mathcal{K}_{i,j} = \frac{\partial f_i}{\partial x_j} + \nu_i \delta_{i,j}. \tag{2}$$

Definition 2.5 ([**2**]). *The set of equations (1) for $x_i^{(0)}$'s is called the indicial equation of $\mathcal{V}$. Its solution set, which is an algebraic set in $\mathbb{C}^n$, is called the indicial locus, and is denoted by $\mathcal{I}$. Then matrix $\mathcal{K}$ defined in (2) is called the Kowalevski matrix. For any $m \in \mathcal{I}$, the eigenvalues of $\mathcal{K}$ evaluated for $x = m$ are called the Kovalevski exponents for m.*

When a nonnegative integer k is one of the Kowalevski exponents, $x^{(k)}$ is not determined by the coefficients of the previous terms and $x^{(k)}$ has at least one new free parameter. The Laurent solutions are organized in families, each family depending on a certain number of independent parameters. Each of these irreducible families of formal Laurent solutions is called a balance.

Proposition 2.6 ([**2**]). *Let $\mathcal{V}$ be a weight-homogeneous vector field on $\mathbb{C}^n$. The set of all weight-homogeneous Laurent solutions to $\mathcal{V}$ is parametrized by a finite number of affine varieties $\Gamma^{(i)}$. For any one of these affine varieties $\Gamma^{(i)}$ the coefficients that appear in the corresponding weight-homogeneous balance are regular functions on $\Gamma^{(i)}$.*

Definition 2.7 ([**2**]). *Let $\mathcal{V}$ be a weight-homogeneous vector field on $\mathbb{C}^n$. The weight-homogeneous balance that corresponds to the affine variety $\Gamma^{(i)}$, as in proposition 2.6, is denoted by $x(t;\Gamma^{(i)})$. For $m \in \mathcal{I}$ the balance specializes to a Laurent series that will be denoted by $x(t;m)$. Each of the affine varieties $\Gamma^{(i)}$ that corresponds to a principal balance, i.e., depends on $n-1$ parameters, is called an abstract Painleve wall of $\mathcal{V}$.*

2.3. Embedding and separation of varialbes

The Liouville tori of a 4-dimensional algebraically completely integrable system is an affine part of an abelian surface, that is, a complex 2-dimensional torus that can be embedded in a projective space. We use explicit embedding of the Kummer surface of the Jacobian to $\mathbb{P}^3$ in order to find a suitable coordinate for separation of variables. In this section, we introduce notations necessary to compute separation of variables following Vanhaecke [**24**].

Let $\nu = (\nu_1, \ldots, \nu_n)$ be a weight vector. We define for $k \in \mathbb{N}$,

$$\mathcal{F}^{(k)} := \{F \in \mathbb{C}\,[x_1, \ldots, x_n] \mid \varpi(F) = k\},$$

where $\varpi(F)$ stands for the weighted-degree of F. Note that the dimension of $\mathcal{F}^{(k)}$ has the following generating function when $\varpi(x_i) = \nu_i$;

$$\prod_{i=1}^{n} \frac{1}{1-t^{\nu_i}} = \sum_{k=0}^{\infty} \left(\dim \mathcal{F}^{(k)}\right) t^k.$$

Let us define

$$\mathcal{H} = \{F \in \mathbb{C}\,[x_1, \ldots, x_n] \,|\dot{F} = 0\},$$

where we have written $\dot{F}$ for $\mathcal{V}(F)$ for $F \in \mathbb{C}[x_1, \ldots, x_n]$. The algebra of constants of motion $\mathcal{H}$ is graded by weighted degree, $\mathcal{H} = \oplus_{j\in\mathbb{N}}\mathcal{H}^{(j)}$, where

$$\mathcal{H}^{(j)} := \{F \in \mathbb{C}\,[x_1, \ldots, x_n] \mid \dot{F} = 0 \text{ and } \varpi(F) = j\}.$$

The following generating function computes the dimension $\dim \mathcal{H}^{(j)}$

$$\prod_{j=1}^{n} \frac{1}{1 - t^{\varpi(H_j)}} = \sum_{k=0}^{\infty} \left(\dim \mathcal{H}^{(k)}\right) t^k,$$

where $H_1, \ldots, H_n$ are weight-homogeneous constant of motion of $\mathcal{V}$ which generate $\mathcal{H}$ as an algebra.

Let $x(t;\Gamma^{(1)}), \ldots, x(t;\Gamma^{(d)}))$ denote the weight-homogeneous principal balances of weight-homogeneous vector field $\mathcal{V}$ on $\mathbb{C}^n$. Let $\rho = (\rho_1, \ldots, \rho_d)$ be a pole vector with $\rho_i \geq 0$ for $i = 1, \ldots, d$. We introduce the vector space of polynomials in $x_1, \ldots, x_n$ with no harder pole than ρ_i when evaluated on $x(t;\Gamma^{(i)})$, where $1 \leq i \leq d$. Let us define

$$\mathcal{Z}_\rho \colon = \{F \in \mathbb{C}[x_1, \ldots, x_n] | \mathrm{ord}_{t=0} F(x(t;\Gamma^{(i)})) \geq -\rho_i \text{ for } 1 \leq i \leq d\}.$$

$\mathcal{Z}_\rho$ is an $\mathcal{H}$-module. We denote $\mathcal{Z}_\rho^{(k)} \colon = \mathcal{Z}_\rho \cap \mathcal{F}^{(k)}$. The number ζ_l of independent elements of $\bigoplus_{k=0}^{l} \mathcal{Z}_\rho^{(k)}$ that are added at level l is given by:

$$\begin{aligned}\zeta_l &\coloneqq \dim \left(\mathcal{Z}_\rho^{(l)} / \bigoplus_{k=0}^{l-1} \mathcal{H}^{(l-k)} \mathcal{Z}_\rho^{(k)} \right) \\ &= \dim \mathcal{Z}_\rho^{(l)} - \sum_{k=0}^{l-1} \zeta_k \dim \mathcal{H}^{(l-k)}.\end{aligned}$$

Let $\mathbb{T}^r$ be an abelian variety and $(x_1, \ldots, x_r)$ be its linear coordinate induced from $\mathbb{C}^r$. The quotient $K = \mathbb{T}^r/(-1)$ of $\mathbb{T}^n$ by the involution $(-1) \colon (x_1, \ldots, x_r) \to (-x_1, \ldots, -x_r)$, is called the Kummer variety. When Γ is a curve of genus two in $\mathbb{T}^2$, sections of linear system $|2\Gamma|$ embeds the Kummer surface of $\mathbb{T}^2$ in $\mathbb{P}^3$. The Kummer surface can be expressed by a quartic equation. As Vanhaecke [**25**] showed, we can carry out separation of variables using a special basis of $|2\Gamma|$.

THEOREM 2.8 ([**25**]). *Suppose we are given an affine part of a generic Abelian surface equipped with a holomorphic vector field $\dot{x} = \{H, x\}$ and principally polarised by one of the irreducible components of the divisor at infinity Γ. A base $\{1, z_1, z_2, z_3\}$ of $|2\Gamma|$ can be chosen so as the equation of the Kummer surface $\mathrm{Jac}\,\Gamma/(-1)$ in $\mathbb{P}^3$ takes the form*

$$(z_1^2 - 4z_0z_2)z_3^2 + r_3(z_0, z_1, z_2)z_3 + r_4(z_0, z_1, z_2) = 0,$$

where r_3 and r_4 are polynomials of degree 3 and 4 respectively. Let μ_1, μ_2 be the roots of the quadratic equation $z_0x^2 + z_1x + z_2 = 0$. Using these variables, the *vector field $\{H, \cdot\}$ is expressed in the Jacobi form*

$$\frac{\dot{\mu}_1}{\sqrt{g(\mu_1)}} + \frac{\dot{\mu}_2}{\sqrt{g(\mu_2)}} = \alpha_1, \quad \frac{\mu_1\dot{\mu}_1}{\sqrt{g(\mu_1)}} + \frac{\mu_2\dot{\mu}_2}{\sqrt{g(\mu_2)}} = \alpha_2,$$

for some constants α_1 and α_2 and some equation $y^2 = g(x)$ for the curve Γ. Therefor, the roots of the polynomial $z_0x^2 + z_1x + z_2$ are variables under which the vector field linearizes.

3. The Painlevé Divisor

In the following sections, we consider the autonomous Garnier system of type $\frac{9}{2}$ as an example. The Garnier system of type $\frac{9}{2}$ is also known as the second member of the first Painlevé hierarchy [**13, 23**].

3.1. The degeneration of the Painlevé divisor of the autonomous Garnier system of type 9/2

We first consider the Painlevé divisor of the Liouville torus. The autonomous Garnier system of type 9/2 is a Hamiltonian system with the Hamiltonians

$$\begin{aligned} H^{9/2}_{\mathrm{Gar},s_1}(q_1,p_1,q_2,p_2) =& p_1^4 + 3p_1^2p_2 + p_1q_2^2 - 2q_1q_2 - p_1s_1 + p_2s_2 + p_2^2, \\ H^{9/2}_{\mathrm{Gar},s_2}(q_1,p_1,q_2,p_2) =& p_1^2q_2^2 - 2p_1q_1q_2 + p_2q_2^2 + p_1^3s_2 + p_1s_2^2 + p_2p_1s_2 + p_2s_1 - p_2p_1^3 \\ & - 2p_2^2p_1 - q_2^2s_2 + q_1^2, \end{aligned}$$

where s_1, s_2 are constants in this autonomous setting [**16**]. The Hamiltonians are weight-homogeneous of weights

$$(\varpi(H_{s_1}), \varpi(H_{s_2})) = (8, 10)$$

with respect to the weights

$$\begin{aligned} &(\varpi(q_1), \varpi(p_1), \varpi(q_2), \varpi(p_2)) = (5, 2, 3, 4) =: (\nu_1, \nu_2, \nu_3, \nu_4), \\ &(\varpi(s_1), \varpi(s_2)) = (6, 4). \end{aligned}$$

The Hamiltonian system is given as follows:

$$\begin{aligned} \frac{dq_1}{dt_1} &= \frac{\partial H_{s_1}}{\partial p_1} = 4p_1^3 + 6p_2p_1 + q_2^2 - s_1, \\ \frac{dp_1}{dt_1} &= -\frac{\partial H_{s_1}}{\partial q_1} = 2q_2, \\ \frac{dq_2}{dt_1} &= \frac{\partial H_{s_1}}{\partial p_2} = 3p_1^2 + 2p_2 + s_2, \\ \frac{dp_2}{dt_1} &= -\frac{\partial H_{s_1}}{\partial q_2} = 2\left(q_1 - p_1q_2\right). \end{aligned}$$

Let us write the canonical variables by x_i

$$x_1 = q_1,\ x_2 = p_1,\ x_3 = q_2,\ x_4 = p_2,$$

and denote the vector field by f_i's

$$\frac{dx_i}{dt_1} = f_i(x_1, x_2, x_3, x_4),$$

where

$$\begin{aligned} f_1 =& 4x_2^3 + 6x_4x_2 + x_3^2 - s_1, \quad f_2 = 2x_3, \\ f_3 =& 3x_2^2 + 2x_4 + s_2, \quad f_4 = 2(x_1 - x_2x_3). \end{aligned}$$

Let us denote $t = t_1$ for the brevity. We compute the Laurent series solutions of the following form

$$x_i(t) = \frac{1}{t^{\nu_i}} \sum_{j=0}^{\infty} x_{i,j} t^j.$$

The initial terms $(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0})$ are one of the followings:

$$m_1 = (0,0,0,0),\ m_2 = (-1,1,-1,0),\ m_3 = (9,3,-3,-9).$$

The Kowalevski matrix is given by

$$\mathcal{K} = \begin{pmatrix} 5 & 12x_2^2 + 6x_4 & 2x_3 & 6x_2 \\ 0 & 2 & 2 & 0 \\ 0 & 6x_2 & 3 & 2 \\ 2 & -2x_3 & -2x_2 & 4 \end{pmatrix}.$$

The Kowalevski exponents for each indicial locus is as follows.

indicial locus	Kowalevski exponents
$m_1 = (0,0,0,0)$	$(2,3,4,5)$
$m_2 = (-1,1,-1,0)$	$(-1,2,5,8)$
$m_3 = (9,3,-3,-9)$	$(-1,-3,8,10)$

We first consider the balance corresponding to the initial term $m_1 = (0,0,0,0)$:

$$\begin{aligned} x_1(t;m_1) =& \alpha + \beta t + \gamma t^2 + \delta t^3 + \frac{1}{12} t^4 \left(-20\alpha^3 + 20\alpha\gamma + 5\beta^2 - 12\alpha s_2 + 4s_1\right) + O\left(t^5\right), \\ x_2(t;m_1) =& \left(\frac{3\delta}{4} - 2\alpha\beta\right) + t\left(-5\alpha^3 + \alpha\gamma - \frac{3\beta^2}{4} - 3\alpha s_2 + s_1\right) \\ &+ \frac{1}{2} t^2 \left(-15\alpha^2\beta + 3\alpha\delta - 2\beta\gamma - 3\beta s_2\right) + O\left(t^3\right), \end{aligned}$$

$$x_3(t;m_1) =\frac{1}{2}\left(-\alpha^2+\gamma-s_2\right)+t\left(\frac{3\delta}{2}-\alpha\beta\right)+t^2\left(-5\alpha^3+4\alpha\gamma+\frac{3\beta^2}{4}-3\alpha s_2+s_1\right)$$
$$+t^3\left(-5\alpha^2\beta+4\alpha\delta+\frac{7\beta\gamma}{3}-\beta s_2\right)+O\left(t^4\right),$$
$$x_4(t;m_1) =\frac{\beta}{2}+\gamma t+\frac{3\delta t^2}{2}+\frac{1}{6}t^3\left(-20\alpha^3+20\alpha\gamma+5\beta^2-12\alpha s_2+4s_1\right)+O\left(t^4\right).$$

These are Taylor series with 4 parameters α, β, γ, and δ. Now let us consider the affine part of the fiber of the momentum map corresponding to the balance m_1:

$$H_{s_1}(x_1(t;m_1),x_2(t;m_1),x_3(t;m_1),x_4(t;m_1)) =h_1,$$
$$H_{s_2}(x_1(t;m_1),x_2(t;m_1),x_3(t;m_1),x_4(t;m_1)) =h_2,$$

which is equivalent to

$$\frac{1}{4}\left(5\alpha^4-5\alpha\beta^2+3\beta\delta-\gamma^2+6\alpha^2 s_2-4\alpha s_1+s_2^2\right)=h_1,$$
$$\frac{1}{16}\left(-48\alpha^5+40\alpha^3\gamma+30\alpha^2\beta^2-36\alpha\beta\delta-8\alpha\gamma^2+2\beta^2\gamma\right.$$
$$\left.+9\delta^2-6s_2\left(8\alpha^3-4\alpha\gamma+\beta^2\right)+8s_1\left(3\alpha^2-\gamma+s_2\right)\right)=h_2.$$

The balance starting from the initial term $m_1=(0,0,0,0)$ corresponds to the affine part of the Liouville tori, which is an abelian surface.

Next we consider the principal balance starting from $m_2=(-1,1,-1,0)$.

$$x_1(t;m_2)=-\frac{1}{t^5}+\frac{\alpha}{t^3}+\beta+t\left(-\frac{\alpha^3}{2}-\frac{9\alpha s_2}{35}+\frac{s_1}{7}\right)-\frac{15}{2}t^2(\alpha\beta)+\gamma t^3$$
$$+t^4\left(\frac{18\beta s_2}{7}-\frac{15\alpha^2\beta}{2}\right)+O\left(t^5\right),$$
$$x_2(t;m_2) =\frac{1}{t^2}+\frac{\alpha}{2}+t^2\left(-\frac{3\alpha^2}{4}-\frac{3s_2}{5}\right)-4\beta t^3+\frac{1}{28}t^4\left(-35\alpha^3-24\alpha s_2+4s_1\right)+O\left(t^5\right),$$
$$x_3(t;m_2)=-\frac{1}{t^3}+t\left(-\frac{3\alpha^2}{4}-\frac{3s_2}{5}\right)-6\beta t^2+\frac{1}{14}t^3\left(-35\alpha^3-24\alpha s_2+4s_1\right)-\frac{15}{2}t^4(\alpha\beta)$$
$$+O\left(t^5\right),$$
$$x_4(t;m_2)=-\frac{3\alpha}{2t^2}+\left(\frac{3\alpha^2}{2}+s_2\right)+6\beta t+t^2\left(\frac{9\alpha^3}{8}+\frac{9\alpha s_2}{10}\right)$$
$$+\frac{3t^4\left(1925\alpha^4+1680\gamma-120\alpha^2 s_2-400\alpha s_1-1008s_2^2\right)}{12320}+O\left(t^5\right),$$

where α,β,γ are parameters. The level set of the moment map is

$$H_{s_1}(x_1(t;m_2),x_2(t;m_2),x_3(t;m_2),x_4(t;m_2)) =h_1,$$
$$H_{s_2}(x_1(t;m_2),x_2(t;m_2),x_3(t;m_2),x_4(t;m_2)) =h_2.$$

These are equivalent to the followings

$$\frac{405\alpha^4}{32}+\frac{81\gamma}{22}+\frac{648\alpha^2 s_2}{77}-\frac{150\alpha s_1}{77}-\frac{23s_2^2}{110}=h_1,$$
$$s_1\left(s_2-\frac{207\alpha^2}{308}\right)+\frac{81\left(35\left(99\alpha^5+48\alpha\gamma+704\beta^2\right)+760\alpha^3 s_2-1008\alpha s_2^2\right)}{24640}=h_2.$$

From the first equation, γ can be expressed in terms of α:

$$\gamma=-\frac{55\alpha^4}{16}+\frac{22h_1}{81}-\frac{16\alpha^2 s_2}{7}+\frac{100\alpha s_1}{189}+\frac{23s_2^2}{405}.$$

Substituting this expression of γ in the second equation, we obtain

$$-\frac{243\alpha^5}{32}+81\beta^2+\frac{3\alpha h_1}{2}-\frac{81\alpha^3 s_2}{8}+s_1\left(\frac{9\alpha^2}{4}+s_2\right)-3\alpha s_2^2=h_2.$$

By replacing $\alpha=\frac{2}{3}x$, $\beta=\frac{1}{9}y$, the equation reads

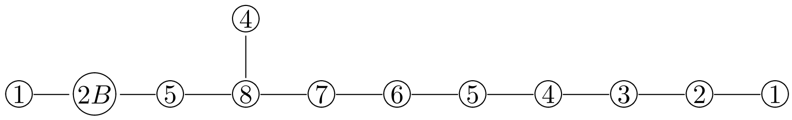
$$y^2=x^5+3s_2x^3-s_1x^2+(2s_2^2-h_1)x+h_2-s_1s_2.$$

Let us consider the affine equation around $h_1=\infty$ by introducing $\tilde{h}_1=1/h_1^2$, $\tilde{y}=y/h_1^5$, $\tilde{x}=x/h_1$:

$$\tilde{y}^2=\tilde{x}^5+3\tilde{h}_1^4s_2\tilde{x}^3+\tilde{h}_1^6s_1x^2+(2\tilde{h}_1^8s_2^2+\tilde{h}_1^7)\tilde{x}+\tilde{h}_1^{10}(s_1s_2+h_2).$$

The degenerations of genus two curves can be studied using Liu's algorighm [**15**]. At $\tilde{h}_1=0$, the curve has the degeneration of type VII* in Namikawa-Ueno's notation with the following dual graph.

VII*: $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$



$$\begin{pmatrix}0&-1&-1&0\\-1&1&0&-1\\1&-1&-1&0\\0&1&0&0\end{pmatrix}$$

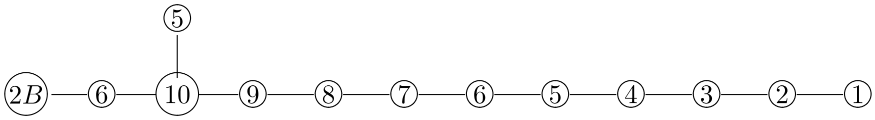
The numbers in circles indicate the multiplicities of components in the reducible fibers. All curves are (-2)-curves except the one expressed as "B", which has -3 as its self-intersection number. The matrix expresses the monodromy. For more details, see Namikawa and Ueno [**17**].

Similarly, we can consider the degeneration of the Painlevé divisor with respect to the other Hamiltonian "h_2". After replacing $\tilde{x} = x/h_2^2$, $\tilde{y} = y/h_2^5$, $\tilde{h}_2 = 1/h_2$ in the above example, we obtain an affine equation around $h_2 = \infty$;

$$\tilde{y}^2 = \tilde{x}^5 + 3\tilde{h}_2^4 s_2 \tilde{x}^3 + \tilde{h}_2^6 s_1 \tilde{x}^2 + 2\tilde{h}_2^8 s_2^2 \tilde{x} + \tilde{h}_2^{10} s_1 s_2 + h_1 \tilde{h}_2^8 \tilde{x} + \tilde{h}_2^9.$$

From Liu's algorithm, the fiber at $h_2 = \infty$ is of type $\mathrm{VIII} - 4$ in Namikawa-Ueno's notation with the following dual graph and monodromy matrix.

$\mathrm{VIII} - 4$: $H^{\frac{9}{2}}_{\mathrm{Gar},s_2}$



$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

Now let us consider the lowest balance, which starts with the initial term $m_3 = (9, 3, -3, -9)$:

$$\begin{aligned} x_1(t; m_3) =& \frac{9}{t^5} + \frac{s_1 t}{21} + \alpha t^3 + \beta t^5 + \left(-\frac{138\alpha s_2}{455} - \frac{54}{15925} s_2^3 - \frac{10}{17199} s_1^2\right) t^7 \\ &+ \left(-\frac{55\alpha s_1}{3213} - \frac{11\beta s_2}{238} - \frac{4 s_1 s_2^2}{12495}\right) t^9 + O\left(t^{10}\right), \\ x_2(t; m_3) =& \frac{3}{t^2} - \frac{3 s_2 t^2}{35} - \frac{s_1 t^4}{63} + t^6 \left(\frac{\alpha}{3} + \frac{6 s_2^2}{1225}\right) + t^8 \left(\frac{\beta}{24} + \frac{s_1 s_2}{2940}\right) + O\left(t^{10}\right), \\ x_3(t; m_3) =& -\frac{3}{t^3} - \frac{3 s_2 t}{35} - \frac{2 s_1 t^3}{63} + t^5 \left(\alpha + \frac{18 s_2^2}{1225}\right) + t^7 \left(\frac{\beta}{6} + \frac{s_1 s_2}{735}\right) \\ &+ t^9 \left(-\frac{11\alpha s_2}{273} - \frac{57 s_2^3}{111475} + \frac{5 s_1^2}{154791}\right) + O\left(t^{10}\right), \\ x_4(t; m_3) =& -\frac{9}{t^4} + \frac{8 s_2}{35} + \frac{2 s_1 t^2}{21} + t^4 \left(-\frac{\alpha}{2} - \frac{9 s_2^2}{490}\right) + t^6 \left(\frac{5\beta}{24} - \frac{s_1 s_2}{420}\right) \\ &+ t^8 \left(-\frac{3\alpha s_2}{130} - \frac{27 s_2^3}{222950} - \frac{5 s_1^2}{17199}\right) + O\left(t^{10}\right). \end{aligned}$$

If we substitute these Laurent series solutions with parameters α and β into the equality

$$H_{s_1}(x_1(t;m_3),x_2(t;m_3),x_3(t;m_3),x_4(t;m_3)) = h_1,$$
$$H_{s_2}(x_1(t;m_3),x_2(t;m_3),x_3(t;m_3),x_4(t;m_3)) = h_2,$$

we obtain the following equations:

$$\frac{297\alpha}{2} + \frac{29s_2^2}{350} = h_1, \ 1287\beta + \frac{214s_1s_2}{245} = h_2.$$

Therefore, the balance starting from m_3 corresponds to a point

$$(\alpha,\beta) = \left(\frac{350h_1 - 29s_2^2}{51975}, \frac{245h_2 - 214s_1s_2}{315315}\right)$$

in the fiber of the momentum map.

Remark 3.1. *If we consider other integrable systems such as other types of autonomous Painlevé-type equations, they are usually equipped with symmetries of the Bäcklund transformations and the symmetries are reflected in the components of the Painlevé divisors. The genus of the Painlevé divisor is not necessary two for the 4-dimensional a.c.i., as we can observe in the case of matrix Painlevé equations.*

3.2. A remark for the 2-dimensional case

We explain the 2-dimensional case to aid an understanding of the 4-dimensional case. Geometrical treatment of the autonomous 2-dimensional Painlevé-type equations from different perspective can be found in Sakai [**22**] or [**16**]. Let us consider the autonomous H_{I} given by the Hamiltonian

$$H_{\mathrm{I}}(q,p) = p^2 - q^3 - sq.$$

The Hamiltonian system is thus

$$\dot{q} = 2p,$$
$$\dot{p} = 3q^2 + s.$$

This is a weight-homogeneous system with the weight

$$(\varpi(q),\varpi(p)) = (2,3), \ (\varpi(H),\varpi(s)) = (6,4).$$

The balance associated to $m_1 = (0,0)$ is a Taylor series

$$q(t;m_1) = \alpha + \beta t + t^2\left(3\alpha^2 + s\right) + 2\alpha\beta t^3 + t^4\left(3\alpha^3 + \frac{\beta^2}{2} + \alpha s\right) + O\left(t^5\right),$$
$$p(t;m_1) = \frac{\beta}{2} + t\left(3\alpha^2 + s\right) + 3\alpha\beta t^2 + t^3\left(6\alpha^3 + \beta^2 + 2\alpha s\right) + O\left(t^4\right).$$

Since the Kowalevski exponents are $2, 3$, the balance contains two free parameters α and β. The level set of the momentum map is

$$H(q(t;m_1), p(t;m_2)) = -s\alpha - \alpha^3 + \frac{\beta^2}{4} = h.$$

If we write $\alpha = x$, $\beta = 2y$, the equation is

$$y^2 = x^3 + sx + h.$$

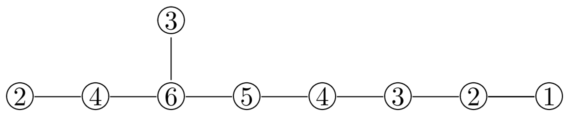
The degeneration of the affine Liouville tori (elliptic curve, in this case) at $h = \infty$ can be studied in the following manner. The affine equation around $h = \infty$ is derived by transforming to $h = 1/\tilde{h}$, $y = \tilde{y}/\tilde{h}^3$, $x = \tilde{x}/\tilde{h}^2$:

$$\tilde{y}^2 = \tilde{x}^3 + s\tilde{h}^4\tilde{x} + \tilde{h}^5.$$

The discriminant and the j-invariant of the cubic are

$$\begin{aligned}\Delta =& 4(s\tilde{h}^4)^3 + 27(\tilde{h}^5)^2 = \tilde{h}^{10}(4s^3\tilde{h}^2 + 27),\\ j =& \frac{4(s\tilde{h}^4)^3}{\Delta} = \frac{4s^4\tilde{h}^2}{4s^3\tilde{h}^2 + 27}.\end{aligned}$$

At $\tilde{h} = 0$, using Tate's algorithm, we can see that the elliptic curve has the degeneration of Kodaira-type II* or E_8 in Dynkin's notation.



Compare this result with two other derivations of the elliptic surface treated in[**16**]. While we used the spectral curve or the explicit form of the Hamiltonian in [**16**], we used the Taylor series solution in this paper.

Now let us consider the principal balance associated with $m_2 = (1, -1)$. The Kowalevski exponents are $-1, 6$, and the principal balance contains a free parameter α:

$$\begin{aligned}q(t;m_2) &= \frac{1}{t^2} - \frac{s}{5}t^2 + \alpha t^4 + \frac{s^2}{75}t^6 - \frac{3s\alpha}{55}t^8 + O\left(t^{10}\right),\\ p(t;m_2) &= -\frac{1}{t^3} - \frac{s}{5}t + 2\alpha t^3 + \frac{s^2}{25}t^5 - \frac{12s\alpha}{55}t^7 + O\left(t^9\right).\end{aligned}$$

The level set of the momentum map is

$$H(q(t;m_2), p(t;m_2)) = -7\alpha = h.$$

Therefore, the principal balance starting from m_2 corresponds to a point given by

$$\alpha = -\frac{h}{7}.$$

4. The Spectral Curve

Algebraically completely integrable systems are often endowed with Lax pairs

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0, \quad A(x), B(x) \in \mathfrak{gl}(N)[\![x]\!]$$

with spectral parameter x. The flow is linearized on the Jacobian of the spectral curve, defined by the characteristic polynomial of $A(x)$.

4.1. The spectral curve of the autonomous Garnier system of type $\frac{9}{2}$

Let us recall the result on the degeneration of the spectral curve of the autonomous Garnier system of type $\frac{9}{2}$ [**16**]. The linear equations of the Garnier system of type $\frac{9}{2}$ is written in Kimura [**12**], and Kawakami [**9**]. Using Kawakami's result, the Lax equation for the autonomous equation is expressed as

$$\frac{dA(x)}{dt_i} + [A(x), B_i(x)] = 0, \quad i = 1, 2$$

for

$$\begin{aligned} A(x) &= A_0x^3 + A_1x^2 + A_2x + A_3, \\ B_1(x) &= A_0x^2 + A_1x + B_{10} = \frac{A(x)}{x} + C_1 - \frac{A_3}{x}, \\ B_2(x) &= -A_0x + B_{20}, \end{aligned}$$

where

$$\begin{aligned} A_0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2s_1 \\ -p_1 & -q_2 \end{pmatrix}, \\ A_3 &= \begin{pmatrix} q_1 - p_1q_2 & p_1^3 + 2p_1p_2 - q_2^2 + s_1p_1 - s_2 \\ -p_2 + s_1 & -q_1 + p_1q_2 \end{pmatrix}, \\ B_{10} &= \begin{pmatrix} q_2 & p_1^2 + 2p_2 + s_1 \\ -p_1 & -q_2 \end{pmatrix}, B_{20} = \begin{pmatrix} 0 & -2p_2 \\ -1 & 0 \end{pmatrix}, \\ C_1 &= \begin{pmatrix} 0 & p_2 - s_1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The characteristic polynomial of a Lax equation of the Garnier system of type $\frac{9}{2}$ is expressed as

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

The equation for the spectral curve is exactly the same equation as that of the Painlevé divisor. Therefore, the degeneration type at $h_1 = \infty$ and $h_2 = \infty$ are the same.

4.2. Genus of the spectral curve

In the classification of the 4-dimensional Painlevé-type equations [**21**], Sakai started from the linear equations with 4 accessory parameters classified by Oshima [**20**]. Therefore, the autonomous systems considered in [**16**] always have spectral curves of geneus two. See also Hiroe [**7**] for the genus of the spectral curves.

However, integrable systems often come equipped with Lax pairs with higher genus than the half the dimension of the phase space. One famous example is the Lax pair for the Kowalevski top given in [**3**].

We note an example for people interested in the Painlevé-type equations. Noumi and Yamada [**19**] gave a Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}}(8)$. While the system is 2-dimensional and the autonomous case can be solved by an elliptic function, the spectral curve has genus two. The Lax pair with a parameter δ is

$$\frac{\partial A(x)}{\partial t} - \delta \frac{\partial B(x)}{\partial x} = [B(x), A(x)],$$

where

$$\begin{aligned}
A(x) =& \frac{A_0}{x} + A_1, \quad B(x) = \frac{B_0}{x} + B_1, \\
A_0 =& \begin{pmatrix} \epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & -p & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_3 & q-1 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_4 & 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_4 & 1-q & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\epsilon_3 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1, \end{pmatrix} \\
A_1 =& E_{8,3} - E_{6,1} + (q-t)(E_{8,2} - E_{7,1}),
\end{aligned}$$

$$
\begin{aligned}
B_0 =& \begin{pmatrix}
u_1 & x_1 & y_1 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 & x_2 & -y_3 & -y_4 & 0 & 0 & 0 \\
0 & 0 & u_3 & x_3 & x_4 & 0 & 0 & 0 \\
0 & 0 & 0 & u_4 & 0 & -x_4 & y_4 & 0 \\
0 & 0 & 0 & 0 & -u_4 & -x_3 & y_3 & 0 \\
0 & 0 & 0 & 0 & 0 & -u_3 & -x_2 & -y_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -u_2 & -x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_1
\end{pmatrix}, \\
B_1 =& E_{8,2} - E_{7,1}, \\
E_{i,j} =& (\delta_{i,a}\delta_{j,b})_{a,b=1}^{8}.
\end{aligned}
$$

We consider the isospectral case $\delta = 0$ and t in the Lax pair being a constant s. The spectral curve C is defined by the characteristic polynomial for $A(x)$

$$C\colon \det(yI_8 - A(x)) = 0.$$

This is equivalent to

$$
\begin{aligned}
&y^8 - y^6\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2\right) + y^4\left(4sx - 2x + \epsilon_1^2\epsilon_2^2 + \epsilon_3^2\epsilon_2^2 + \epsilon_4^2\epsilon_2^2 + \epsilon_1^2\epsilon_3^2 + \epsilon_1^2\epsilon_4^2 + \epsilon_3^2\epsilon_4^2\right) \\
&+ xy^2\left(4h(s-1)s + 4s(\epsilon_1\epsilon_2 - \epsilon_1\epsilon_3 - \epsilon_2\epsilon_3) + 2(-\epsilon_1\epsilon_2 + \epsilon_3\epsilon_2 + \epsilon_4\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_1\epsilon_4 - \epsilon_3\epsilon_4)\right) \\
&- y^2\left(\epsilon_1^2\epsilon_2^2\epsilon_3^2 + \epsilon_1^2\epsilon_4^2\epsilon_3^2 + \epsilon_2^2\epsilon_4^2\epsilon_3^2 + \epsilon_1^2\epsilon_2^2\epsilon_4^2\right) + x^2 - 2x\epsilon_1\epsilon_2\epsilon_3\epsilon_4 + \epsilon_1^2\epsilon_2^2\epsilon_3^2\epsilon_4^2 = 0,
\end{aligned}
$$

where h is the Hamiltonian of the autonomous sixth Painlevé equation

$$
\begin{aligned}
h = \frac{1}{s(s-1)}\big(&p^2 q(q-1)(q-s) - p(\alpha_0 q(q-1) + \alpha_3 q(q-s) + \alpha_4(q-1)(q-s)) \\
&+ \alpha_2(\alpha_1 + \alpha_2)(q-s)\big),
\end{aligned}
$$

with s being a constant and

$$\alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = -\epsilon_1 - \epsilon_2,\ \alpha_3 = \epsilon_3 - \epsilon_4,\ \alpha_4 = -\epsilon_3 + \epsilon_4.$$

Let us consider the following change of variables

$$
\begin{aligned}
x_1 =& y, \\
y_1 =& x + ((2s-1)y^4 + (2s(s-1)h + (2s-1)(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_2 - \epsilon_1\epsilon_3) \\
&+ \epsilon_1\epsilon_4 + \epsilon_2\epsilon_4 - \epsilon_3\epsilon_4)y^2 - \epsilon_1\epsilon_2\epsilon_3\epsilon_4).
\end{aligned}
$$

Then the curve C is expressed as

$$y_1^2 = a_0(h)x_1^8 + a_1(h)x_1^6 + a_2(h)x_1^4 + a_3(h)x_1^2,$$

where

$$\begin{aligned}
a_0(h) =&4s(s-1),\\
a_1(h) =&8hs^3+(-12h+8(\epsilon_1\epsilon_2-\epsilon_1\epsilon_3-\epsilon_2\epsilon_3))s^2\\
&+4(h+2(-\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\epsilon_2\epsilon_3)+\epsilon_4(\epsilon_1+\epsilon_2-\epsilon_3))s+(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)^2,\\
a_2(h) =&4h^2s^4+8(-h^2+h(\epsilon_1\epsilon_2-\epsilon_1\epsilon_3-\epsilon_2\epsilon_3))s^3+4(h^2+h(3(-\epsilon_1\epsilon_2+\epsilon_3\epsilon_2+\epsilon_1\epsilon_3)\\
&+(\epsilon_1+\epsilon_2-\epsilon_3)\,\epsilon_4)+\epsilon_1^2\epsilon_2^2+\epsilon_1^2\epsilon_3^2+\epsilon_2^2\epsilon_3^2-2\epsilon_1\epsilon_2\,(\epsilon_1+\epsilon_2-\epsilon_3)\,\epsilon_3)t^2\\
&+4(h(\epsilon_1\epsilon_2-\epsilon_3\epsilon_2-\epsilon_4\epsilon_2-\epsilon_1\epsilon_3-\epsilon_1\epsilon_4+\epsilon_3\epsilon_4)-\epsilon_1^2\epsilon_2^2-\epsilon_3^2\epsilon_2^2\\
&+2\epsilon_1(\epsilon_1+\epsilon_2-\epsilon_3)\epsilon_3\epsilon_2-4\epsilon_1\epsilon_3\epsilon_4\epsilon_2-\epsilon_1^2\epsilon_3^2\\
&+(\epsilon_2\epsilon_1^2-\epsilon_3\epsilon_1^2+\epsilon_2^2\epsilon_1+\epsilon_3^2\epsilon_1+\epsilon_2\epsilon_3^2-\epsilon_2^2\epsilon_3)\epsilon_4)s\\
&+2(2\epsilon_1\epsilon_2\epsilon_3\epsilon_4+\epsilon_3(\epsilon_1^2-\epsilon_3\epsilon_1-\epsilon_4\epsilon_1+\epsilon_2^2-\epsilon_2\epsilon_3-\epsilon_2\epsilon_4)\epsilon_4\\
&-\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)(\epsilon_3+\epsilon_4),\\
a_3(h) =&4\epsilon_1\epsilon_2\epsilon_3\epsilon_4 s(h(1-s)-\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_1\epsilon_3).
\end{aligned}$$

If we blowup the singular point $(0,0)$ by introducing

$$Y=x_1y_1,\quad X=x_1,$$

the curve turns into a smooth hyperelliptic curve of genus 2 in the Weierstrass form

$$C\colon Y^2=a_0(h)X^6+a_1(h)X^4+a_2(h)X^2+a_3(h).$$

The curve C has an involution $\sigma\colon (x,y)\mapsto(-x,y)$, other than the hyperelliptic involution $\iota\colon (x,y)\mapsto(x,-y)$, Let us denote $\pi\colon C\to E=C/\sigma$, then

$$E\colon y^2=a_0(h)x^3+a_1(h)x^2+a_2(h)x+a_3(h).$$

We write the cubic $f(x)=a_0(h)x^3+a_1(h)x^2+a_2(h)x+a_3(h)$. In order to write an affine equation around ∞, we introduce the following variables

$$\tilde{y}=\frac{y}{h^3},\quad \tilde{x}=\frac{x}{h^2},\tilde{h}=\frac{1}{h}.$$

We obtain

$$\tilde{y}^2=\tilde{a_0}(\tilde{h})\tilde{x}^3+\tilde{a_1}(\tilde{h})\tilde{x}^2+\tilde{a_2}(\tilde{h})\tilde{x}^2+\tilde{a_3}(\tilde{h})\tilde{x}+\tilde{a_4}(\tilde{h}),\quad \tilde{a_i}(\tilde{h})=\tilde{h}^{2i}a_i(1/\tilde{h}).$$

The discriminant and the j-invarinat are

$$\begin{aligned}
\Delta =&s(s-1)\tilde{h}^6+O(\tilde{h^7})\\
j =&\frac{256(s^2-s+1)^3}{s^2(s-1)^2}+O(\tilde{h}).
\end{aligned}$$

At $\tilde{h}=0$, the order of Δ and j are

$$\mathrm{ord}(\Delta)=6,\ \mathrm{ord}(j)=0.$$

Therefore, using Tate's algorithm, we can tell that E has degeneration type I_0^* (or $D_4^{(1)}$ in Dynkin's notation) over $\tilde{h}=0$. The other 6 zeros of the discriminant are generically simple zeros. This can be checked by computing the discriminant of the polynomial Δ in h. Therefore, type of the rational elliptic surface associated with E is $\mathrm{I}_0^*+6\mathrm{I}_1$ in Kodaira's notation, as expected.

5. Separation of varibles and construction of a Lax pair

5.1. Separation of variables

In this section, we consider separation of the variables following Vanhaecke [**24**].

Let us return to the case of autonomous Garnier system of type $\frac{9}{2}$. Recall that the canonical variables q_1, p_1, q_2, p_2 have weights $5, 2, 3, 4$, respectively. The number of independent polynomials of weight k, $\dim(\mathcal{F}^{(k)})$, are computed from

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\dim\mathcal{F}^{(k)}\right)t^k &= \frac{1}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)}\\ &=1+t^2+t^3+2t^4+2t^5+3t^6+3t^7+5t^8+5t^9+7t^{10}+O(t^{11}).\end{aligned}$$

Since the two functionally independent constants of motion H_1, H_2 have weights 8, 10 respectively, the number of constants of motion of weight k, $\dim(\mathcal{H}^{(k)})$, are the coefficients of the following series

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\dim\mathcal{H}^{(k)}\right)t^k &= \frac{1}{(1-t^8)(1-t^{10})}\\ &=1+t^8+t^{10}+t^{16}+t^{18}+t^{20}+O(t^{24}).\end{aligned}$$

Let ρ be 2. The number of linearly independent polynomials which have a double pole at most when the principal balance $x(t;m_2)$ is substituted in them, $\dim(\mathcal{Z}_\rho^{(k)})$, can be computed from direct computations. Let us introduce other notations in the following table. $\sharp$dependent denotes the number of elements in $\mathcal{Z}_\rho^{(k)}$ that are dependent of the previous ones over $\mathcal{H}$. This can be computed from the previous data by the formula $\sum_{j=0}^{i-1}\zeta_j\dim\mathcal{H}^{(i-j)}$. The number of independent elements that are added at degree k is denoted by ζ_k. Our situation can be summarized as in the following table.

The basis of $|2\Gamma|$ is

$$z_0=1,\ z_1=x_2,\ z_2=-x_4+s_2,\ z_3=x_2^3-x_3^2+x_2x_4.$$

k	$\dim\mathcal{F}^{(k)}$	$\dim\mathcal{H}^{(k)}$	$\dim\mathcal{Z}_\rho^{(k)}$	$\sharp$dependent	ζ_k	independent functions
0	1	1	1	0	1	$z_0 = 1$
1	0	0	0	0	0	–
2	1	0	1	0	1	$z_1 = x_2$
3	1	0	0	0	0	–
4	1	0	1	0	1	$z_2 = -x_4 + s_2$
5	2	0	0	0	0	0
6	2	0	1	0	1	$z_3 = x_2^3 - x_3^2 + x_2x_4$
7	3	0	0	0	0	–
8	5	1	1	1	0	–
9	5	0	0	0	0	–
10	7	1	1	1	0	–

Table 1. The polynomials of weighted-degree less than 10 which have a double pole at most when $x(t;m_2)$ is substituted in them.

Let us consider the Kodaira map $\phi_{|2\Gamma|}\colon \operatorname{Jac}(\Gamma)\setminus\Gamma$ defined by

$$\phi_{|2\Gamma|}(P) = (z_0(P) : z_1(P) : z_2(P) : z_3(P)).$$

for any $P = (x_1, x_2, x_3, x_4) \in \operatorname{Jac}(\Gamma)\setminus\Gamma$. The Kummer surface of the Jacobian $\operatorname{Jac}(\Gamma)$ can be expressed in the form

$$(z_1^2 - 4z_2)z_3^2 + r_3(z_1, z_2)z_3 + r_4(z_1, z_2) = 0,$$

where r_3 and r_4 are polynomials of degree 3 and 4 respectively.

$$\begin{aligned} r_3(z_1, z_2) =& -2h_1z_1 + 4h_2 + 4s_2z_1^3 - 2s_1z_1^2 + 4s_2^2z_1 - 10s_2z_2z_1 + 4s_1z_2 \\ & -4s_1s_2 + 2z_2^2z_1, \\ r_4(z_1, z_2) =& z_2^4 + 4s_2^2z_1^4 + 4s_2z_1^2z_2^2 - 4h_2z_1^3 - 6s_2z_2^3 + 4h_1z_1^2z_2 - 12s_2^2z_1^2z_2 + 2s_1z_1z_2^2 \\ & -4h_1s_2z_1^2 + 8s_2^3z_1^2 + s_1^2z_1^2 - 2h_1z_2^2 + 13s_2^2z_2^2 + 4h_2z_1z_2 - 2s_1s_2z_1z_2 \\ & -4h_2s_2z_1 + 6h_1s_2z_2 - 12s_2^3z_2 + 2h_1s_1z_1 - 4h_1s_2^2 + h_1^2 + 4s_2^4. \end{aligned}$$

A system of linearizing variables μ_1, μ_2 is given by the roots of the quadratic equation $z_0x^2 + z_1x + z_2 = 0$, that is,

$$\mu_1 + \mu_2 = -z_1, \quad \mu_1\mu_2 = z_2,$$

since $z_0 = 1$. We also have $(\mu_1 - 1)(\mu_2 - 1) = z_1 + z_2 + 1$. Differentiating these equation with respect to the vector field X_{h_1}, we have

$$\frac{\dot\mu_1}{\mu_1} + \frac{\dot\mu_2}{\mu_2} = \frac{\dot z_2}{z_2},$$
$$\frac{\dot\mu_1}{\mu_1 - 1} + \frac{\dot\mu_2}{\mu_2 - 1} = \frac{\dot z_1 + \dot z_2}{z_1 + z_2 + 1}.$$

These equations further give

$$\dot\mu_i^2 = -\frac{4g(\mu_i)}{(\mu_1 - \mu_2)^2},\ (i = 1, 2)$$

where

$$g(x) = x^5 + 3s_2x^3 + s_1x^2 + (2s_2^2 - h_1)x - h_2 + s_1s_2.$$

It follows that in terms of the coordinates μ_1, μ_2, the differential equation reduces to the Jacobi form

$$\frac{\dot\mu_1}{\sqrt{g(\mu_1)}} + \frac{\dot\mu_2}{\sqrt{g(\mu_2)}} = 0, \quad \frac{\mu_1\dot\mu_1}{\sqrt{g(\mu_1)}} + \frac{\mu_2\dot\mu_2}{\sqrt{g(\mu_2)}} = 1.$$

The Jacobi inversion problem can be solved using genus two $\wp$ function associated with the Jacobian of the hyperelliptic curve $\Gamma\colon y^2 = g(x)$ as in [**4**].

5.2. Construction of a Lax pair

According to Mumford's description of hyperelliptic Jacobian [**14**], the affine part $\mathrm{Jac}(\Gamma) \setminus \Gamma$ of Jacobian of genus two curve $\Gamma\colon y^2 = g(x)$ is isomorphic to the space of polynomials $(u(x), v(x))$ such that $u(x)$ is monic of degree 2, $v(x)$ has degree less than 2 and $g(x) - v(x)^2$ is divisible by $u(x)$.

$$\begin{aligned} u(x) &= = (x-\mu_1)(x-\mu_2) = x^2 + u_1x + u_2,\\ v(x) &= \frac{\sqrt{g(\mu_2)}}{\mu_2 - \mu_1}(x - \mu_1) + \frac{\sqrt{g(\mu_1)}}{\mu_2 - \mu_1}(x - \mu_2),\\ w(x) &= \frac{g(x) - v(x)^2}{u(x)} = x^3 + (\mu_1 + \mu_2)x^2 + (\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + 3s_2)x + w_3, \end{aligned}$$

where

$$\begin{aligned} w_3 = &-\frac{1}{(\mu_1 - \mu_2)^2}\big((\mu_1\mu_2(\mu_1^3 + \mu_2^3 + 3s_2(\mu_1 + \mu_2) + 2s_1)\\ &- (h_1 - 2s_2^2)(\mu_1 + \mu_2) - 2h_2 + 2s_1s_2 + \sqrt{g(\mu_1)g(\mu_2)}\big). \end{aligned}$$

Let us set

$$A(x) = \begin{pmatrix} v(x) & u(x) \\ w(x) & -v(x) \end{pmatrix}.$$

The Poisson bracket can be defined and the Hamiltonian H is determined by the coefficient of $u(x)w(x)+v(x)^2$. The flow $\dot{A}(x) = \{H, A(x)\}$ is expressed in the Lax form

$$\dot{A}(x) = [A(x), B(x)],\ B(x) = \left(\frac{A(x)}{x^2}\right)_+ - \begin{pmatrix} 0 & 0 \\ u_1 & 0 \end{pmatrix}.$$

In this way, a 2×2 Lax expression can be constructed without knowing one in advance. Construction of 2×2 Lax representations for the Weierstrass algebraically completely integrable systems is also treated in [**6**]. For the first Painlevé hierarchy including the Garnier system of type 9/2, such Lax form is treated by Takasaki [**23**].

6. Comments

In this paper, we have seen two genus two curves, the Painlevé divisor and the spectral curve, associated with the autonomous Garnier system of type $\frac{9}{2}$ with the following equation

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

These are just translates of the theta divisor of the Liouville torus. One practical benefit of having explicit expressions of these curves is one might be able to distinguish the integrable systems by studying the degeneration of (irreducible components of) the Painlevé divisor. When an integrable system is not equipped with its Lax pair in advance, such comparisons might be meaningful.

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References

[1] M. Adler, P. van Moerbeke, The complex geometry of the Kowalewski-Painlevé analysis, *Invent. Math.*, 97, 1989, 3-51.

[2] M. Adler, P. van Moerbeke, P. Vanhaecke, Algebraic integrability, Painlevé geometry and Lie algebras, *A Series of Modern Surveys in Mathematics*, 47, 2004.

[3] A.I. Bobenko, A.G. Reyman, M.A. Semenov-Tian-Shansky, The Kowalewski top 99 years later: a Lax pair, generalizations and explicit solutions, *Comm. Math. Phys.*, 122, 1989, 2, 321–354.

[4] V.M. Buchstaber, V.Z. Enolskiĭ, D.V. Leĭkin, Hyperelliptic Kleinian functions and applications, Solitons, geometry, and topology: on the crossroad, *Amer. Math. Soc. Transl. Ser. 2*, 179, 1997, 1–33.
[5] C. Cosgrove, Christopher, Higher-order Painlevé equations in the polynomial class. II., *Stud. Appl. Math.*, 116, 2006, 4, pp.321–413.
[6] D. Fairbanks, Lax equation representation of certain completely integrable systems, *Compositio Math.*, 68, 1988, 1, pp.31–40.
[7] K. Hiroe, Ramified irregular singularities of meromorphic connections and plane curve singularities, in this volume.
[8] K. Hiroe, T. Oshima, A classification of roots of symmetric Kac-Moody root systems and its application, Symmetries, integrable systems and representations, *Springer Proc. Math. Stat.*, 40, pp.195–241, 2013.
[9] H. Kawakami, Four-dimensional Painlevé-type equations associated with ramified linear equations III (in preparation).
[10] H. Kawakami, A. Nakamura, H. Sakai, Degeneration scheme of 4-dimensional Painlevé-type equations, *arXiv*:1209.3836, 2012.
[11] H. Kawamuko, On the Garnier system of half-integer type in two variables *Funkcial. Ekvac.* 52, 2009, 2, pp.181–201.
[12] H. Kimura, The degeneration of the two-dimensional Garnier system and the polynomial Hamiltonian structure, *Ann. Mat. Pura Appl. (4)* 155, 1989, 25-74.
[13] N. A. Kudryashov, Nicolai A., The first and second Painlevé equations of higher order and some relations between them, *Phys. Lett. A*, 224, 1997, 6, 353-360.
[14] D. Mumford, Tata lectures on theta. II, *Modern Birkhäuser Classics*, 1984.
[15] Q. Liu, Modèles minimaux des courbes de genre deux, *J. Reine Angew. Math.*, 453, 1994, 137-164.
[16] A. Nakamura, Autonomous limit of 4-dimensional Painlevé-type equations and degeneration of curves of genus two, *arXiv*1505.00885, 2015.
[17] Y. Namikawa, K. Ueno, The complete classification of fibres in pencils of curves of genus two, *Manuscripta Math.*, 9, 1973, 143-186.
[18] M. Noumi, Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Comm. Math. Phys.*, 199, 1998, 2, pp.281–295.
[19] M. Noumi, Y. Yamada, A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}}(8)$, *Microlocal analysis and complex Fourier analysis*, 2002, 238–252.
[20] T. Oshima, Fractional calculus of Weyl algebra and Fuchsian differential equations, *MSJ memoirs*, 28, 2012.
[21] H. Sakai, Isomonodromic deformation and 4-dimensional Painlevé type equations, preprint, University of Tokyo, Graduate School of Mathematical Sciences, 2010.
[22] H. Sakai, Ordinary differential equations on rational elliptic surfaces, *Springer Proc. Math. Stat.*, 40, pp.515–541.
[23] K. Takasaki, Hamiltonian structure of PI hierarchy, *SIGMA*, 3, 2007, 042, 32.
[24] P. Vanhaecke, Integrable systems in the realm of algebraic geometry, *LNM* 1638, 2001.
[25] P. Vanhaecke, Linearising two-dimensional integrable systems and the construction of action-angle variables, *Math. Z.*, 211, 1992, 2, pp.265–313.

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