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On convergence of basic hypergeometric series

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Abstract. We examine the convergence of q-hypergeometric series when $|q| = 1$. We give a condition so that the radius of the convergence is positive and get the radius. We also show that the numbers q with the positive radius of the convergence are densely distributed in the unit circle of the complex plane of q and so are those with the radius 0. First we give an elementary proof of this result and then an extended result.

1. Introduction

Basic hypergeometric series (cf. [**5**]) with a base q is defined by

$$
\begin{aligned}
&{}_r\phi_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right] \\
&\quad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \big((-1)^n q^{\frac{n(n-1)}{2}}\big)^{s+1-r} z^n,
\end{aligned}
\tag{1}
$$

where

$$
(a;q)_n = \prod_{j=1}^{n} (1 - aq^{j-1})
$$

is the q-Pochhammer symbol. Here $a_1, \dots, a_r$, $b_1, \dots, b_s$ and q are complex parameters. In this paper we always assume

$$
a_i q^n \neq 1 \ \text{ and } \ b_j q^n \neq 1 \qquad (i = 1, \dots, r,\ j = 1, \dots, s,\ n = 0, 1, 2, \dots) \tag{2}
$$

so that the factors $(a_i; q)_n$ and $(b_j; q)_n$ in the terms of the series are never zero.

Let v_n be the terms of the series ${}_r\phi_s$ which contain z^n. Then we have

$$
\begin{aligned}
\frac{v_{n+1}}{v_n} &= \frac{(1 - a_1 q^n)(1 - a_2 q^n) \cdots (1 - a_r q^n)}{(1 - q^{n+1})(1 - b_1 q^n) \cdots (1 - b_s q^n)} (-q^n)^{s+1-r} z \\
&= \frac{(a_1 - q^{-n})(a_2 - q^{-n}) \cdots (a_r - q^{-n})}{(1 - q^{-n-1})(b_1 - q^{-n}) \cdots (b_s - q^{-n})} \frac{z}{q}.
\end{aligned}
$$

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If $0 < |q| < 1$, the radius of convergence of the series ${}_r\phi_s$ equals ∞ if $r \le s$ and equals 1 if $r = s + 1$. If $|q| > 1$ and

$$a_1 \cdots a_r b_1 \cdots b_s \neq 0, \tag{3}$$

the radius of convergence of the series equals

$$\frac{|b_1 b_2 \cdots b_s q|}{|a_1 a_2 \cdots a_r|}. \tag{4}$$

In this paper we discuss the convergence of the series when $|q| = 1$. The convergence of ${}_2\phi_1$ is assumed in [**6**] but it is a subtle problem depending on the base

$$q = e^{2\pi i\theta}. \tag{5}$$

We assume that θ is not a rational number, namely, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ so that $(q; q)_n$ never vanish and we have the following theorem.

Theorem 1.1. *Retain the notation above and assume the conditions* (2) *and* (3)*.*

i) *Assume that there exists a positive number C such that*

$$\left|\theta - \frac{k}{m}\right| > \frac{C}{m^2} \qquad (\forall k \in \mathbb{Z},\ m = 1, 2, 3, \ldots). \tag{6}$$

Then we have

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(e^{2\pi i\theta}; e^{2\pi i\theta}\right)_n\right|} = 1. \tag{7}$$

Suppose moreover that every parameter a_i or b_j has an absolute value different from 1 *or equals $e^{2\pi i\alpha} q^{\beta}$ with suitable rational numbers α and β which may depend on a_i and b_j. Then the radius of convergence of the series ${}_r\phi_s$ equals*

$$\frac{\max\{|b_1|, 1\} \cdots \max\{|b_s|, 1\}}{\max\{|a_1|, 1\} \cdots \max\{|a_r|, 1\}}. \tag{8}$$

ii) *In general, we have*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|(e^{2\pi i\theta}; e^{2\pi i\theta})_n\right|} \le 1 \qquad (\forall \theta \in \mathbb{R} \setminus \mathbb{Q}). \tag{9}$$

The set of irrational real numbers θ satisfying

$$\underline{\lim_{n\to\infty}} \sqrt[n]{\left|(e^{2\pi i\theta}; e^{2\pi i\theta})_n\right|} = 0 \tag{10}$$

is dense in $\mathbb{R}$ *and uncountable. If* θ *satisfies* (10) *and the absolute value of any parameter* a_i *or* b_j *is not* 1*, the radius of convergence of the series* ${}_r\phi_s$ *equals* 0*.*

Note that it is known that an irrational number θ satisfies the assumption in Theorem 1.1 i) if and only if the positive integers appearing in its expansion of continued fraction are bounded and hence the set of real numbers θ satisfying it is uncountable and dense in $\mathbb{R}$.

Suppose $\theta \in \mathbb{R} \setminus \mathbb{Q}$ satisfies the assumption in Theorem 1.1 i). Then

$$\left|\frac{k_1}{m_1}\theta - \frac{k_2}{m_2} - \frac{k}{m}\right| = \left|\frac{k_1}{m_1}\right| \cdot \left|\theta - \frac{m_1(k_2 m + k m_2)}{k_1 m_2 m}\right| > \frac{C}{|k_1 m_1| m_2^2 \cdot m^2} \tag{11}$$

for integers k, k_1, k_2, m, m_1, m_2 with $k_1 m m_1 m_2 \neq 0$ and therefore the number $r_1\theta + r_2$ with $r_1 \in \mathbb{Q} \setminus \{0\}$ and $r_2 \in \mathbb{Q}$ also satisfies the assumption.

Example 1.2. *The estimate*

$$\left|\sqrt{2} - \frac{k}{m}\right| > \frac{1}{3m^2} \qquad (\forall k \in \mathbb{Z},\ m = 1, 2, 3, \ldots) \tag{12}$$

shows that the real number $\theta = \sqrt{2} + r$ *with* $r \in \mathbb{Q}$ *satisfies the assumption in Theorem 1.1 i).*

We will prove (12). We assume the existence of integers k and m satisfying $m \geq 1$ and $|\sqrt{2} - \frac{k}{m}| \leq \frac{1}{3m^2}$. Then we may moreover assume $m \geq 2$ and therefore

$$\begin{aligned} 1 &\leq \left|2m^2 - k^2\right| = \left|\left(\sqrt{2}m - k\right)\left(\sqrt{2}m + k\right)\right| \\ &= \left|m^2\left(\sqrt{2} - \frac{k}{m}\right)\right| \cdot \left|2\sqrt{2} - \left(\sqrt{2} - \frac{k}{m}\right)\right| \\ &\leq \frac{1}{3}\left(2\sqrt{2} + \frac{1}{3m^2}\right) \leq \frac{2\sqrt{2}}{3} + \frac{1}{9 \cdot 4} = 0.9705\cdots < 1, \end{aligned}$$

which leads a contradiction.

We will show Theorem 1.1 i) in §4 and Theorem 1.1 ii) in §5.

2. Preliminary results

First we review the following theorem which claims that $k\theta \mod \mathbb{Z}$ for $k = 1, 2, \ldots$ are uniformly distributed on $\mathbb{R}/\mathbb{Z}$.

Theorem 2.1 (Bohl, Sierpiński and Weyl). *Let* $f(x)$ *be a periodic function on* $\mathbb{R}$ *with period* 1*. If* $f(x)$ *is Riemann integrable in* $[0, 1]$ *, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\theta) = \int_0^1 f(x)\, dx \tag{13}$$

for any $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

This theorem is proved by approximating $f(x)$ by a finite Fourier series since the theorem is directly proved if $f(x)$ is a finite Fourier series with the fact

$$\sum_{k=1}^{n} \frac{e^{2\pi i m k\theta}}{n} = \frac{e^{2\pi i m\theta}}{n} \left(\frac{1 - e^{2\pi i m n\theta}}{1 - e^{2\pi i m\theta}} \right) \xrightarrow{n\to\infty} 0 \qquad (m \in \mathbb{Z} \setminus \{0\}).$$

We also prepare the following integral formula.

$$\int_0^1 \log\left|1 - re^{2\pi i x}\right| dx = \begin{cases} 0 & (0 \le r \le 1), \\ \log r & (r \ge 1). \end{cases} \tag{14}$$

The series $-\frac{\log(1-z)}{z} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots$ converges when $|z| < 1$ and therefore

$$0 = \int_{|z|=r} \frac{\log(1-z)}{z}\, dz = 2\pi i \int_0^1 \log(1 - re^{2\pi i x})\, dx \qquad (0 \le r < 1,\ z = re^{2\pi i x})$$

by Cauchy's integral formula. Since

$$\mathrm{Re} \int_0^1 \log(1 - re^{2\pi i x})\, dx = \int_0^1 \mathrm{Re} \log(1 - re^{2\pi i x})\, dx = \int_0^1 \log|1 - re^{2\pi i x}|\, dx,$$

we have (14) when $0 \le r < 1$. Moreover the relation

$$\log|1 - re^{2\pi i x}| = \log r + \log|r^{-1} - e^{2\pi i x}| = \log r + \log|1 - r^{-1}e^{2\pi i x}|$$

assures (14) when $r > 1$.

Note that the expansion $e^\sigma - 1 = \sigma(1 + \frac{\sigma}{2!} + \frac{\sigma^2}{3!} + \cdots)$ assures $|1 - e^\sigma| \ge \frac{|\sigma|}{2}$ when $|\sigma| < 1$. Hence if $r = 1$, the improper integral (14) converges because

$$\big|\log|1 - e^{2\pi i z}|\big| < \big|\log|\pi z|\big| \quad \text{for} \quad 0 < |2\pi z| < 1 \tag{15}$$

and we obtain (14) by taking the limit $r \to 1 - 0$ (cf. [**1**, 5.3.5]).

3. A lemma

We prepare a lemma to prove Theorem 1.1 i).

LEMMA 3.1. *Let* $\theta \in \mathbb{R}\setminus\mathbb{Q}$ *and* $f(x)$ *be a periodic function on* $\mathbb{R}$ *with period* 1. *Suppose that* $f(x)$ *is continuous in* $[0,1]$ *except for finite points* $c_1, \ldots, c_p \in [0,1)$. *Suppose there exist* $r_j \in \mathbb{Q}$ *for* $j = 1, \ldots, p$ *such that*

$$c_j - r_j\theta \in \mathbb{Q} \quad \textit{and} \quad k\theta - c_j \notin \mathbb{Z} \quad \textit{for} \quad k = 1, 2, \ldots. \tag{16}$$

Suppose moreover that there exist a positive number ϵ *and a continuous function* $h(t)$ *on* $(0,1]$ *such that*

$$\begin{gathered} |f(x)| < h(|x-c_j|) \quad \textit{for} \;\; 0 < |x-c_j| < \epsilon, \\ \int_0^1 h(t)\,dt < \infty \quad \textit{and} \;\; h(t_1) \geq h(t_2) \geq 0 \quad \textit{if} \;\; 0 < t_1 < t_2 \leq 1. \end{gathered} \tag{17}$$

If $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *satisfies the assumption in Theorem 1.1 i), then* (13) *is valid. Here we note that the condition* (17) *assures that the improper integral in* (13) *converges.*

Proof. Put

$$J(j,n,\epsilon) = \bigl\{k \mid 1 \leq k \leq n,\ \min_{m\in\mathbb{Z}} |k\theta - c_j - m| < \epsilon\bigr\}$$

and

$$I_\epsilon = \bigl\{x \in [0,1] \mid \min_{m\in\mathbb{Z}} |x - c_j - m| \geq \epsilon \;\; \text{for} \;\; j = 1,\ldots,p\bigr\}.$$

Then Theorem 2.1 shows

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{k\notin J(1,n,\epsilon)\cup\cdots\cup J(p,n,\epsilon)\\ 1\leq k\leq n}} f(k\theta) = \int_{I_\epsilon} f(x)\,dx$$

and therefore we have only to show

$$\lim_{\epsilon\to+0} \lim_{n\to\infty} \frac{1}{n} \sum_{k\in J(j,n,\epsilon)} |f(k\theta)| = 0 \tag{18}$$

to get this lemma.

Fix j. Since Theorem 2.1 shows

$$\lim_{n\to\infty} \frac{1}{n} \# J(j,n,\epsilon) = 2\epsilon,$$

there exists a positive integer N_ϵ such that

$$\# J(j,n,\epsilon) \leq 3\epsilon n \quad (\forall n \geq N_\epsilon).$$

Put

$$J(j,n,\epsilon) = \{k_1, k_2, \ldots, k_L\}$$

with $L = \#J(j,\epsilon,n)$ so that

$$\min_{m\in\mathbb{Z}} |k_\nu\theta - c_j - m| \le \min_{m\in\mathbb{Z}} |k_{\nu'}\theta - c_j - m| \quad \text{if} \quad 1 \le \nu < \nu' \le L.$$

Note that $c_j = \frac{k_1}{m_1}\theta + \frac{k_2}{m_2}$ with integers k_1, k_2, m_1, m_2. In view of (11), we have

$$\left| k\theta - \frac{k_1}{m_1}\theta - \frac{k_2}{m_2} - m \right| = \left| \frac{m_1k - k_1}{m_1}\theta - \frac{k_2 + mm_2}{m_2} \right| > \frac{C}{m_1 m_2^2 |m_1k - k_1|}$$

for $k = 1, 2, 3, \ldots$ satisfying $m_1k \ne k_1$. If $\frac{k_1}{m_1}$ is a positive integer, the assumption implies $\frac{k_2}{m_2} \notin \mathbb{Z}$. Hence replacing C by a small positive number if necessary, we may assume

$$\min_{m\in\mathbb{Z}} |k\theta - c_j - m| > \frac{C}{k} \quad (k = 1, 2, \ldots,\ j = 0, 1, \ldots, p),$$

where we put $c_0 = 0$. In particular, we have

$$\min_{m\in\mathbb{Z}} |k\theta - k'\theta - m| > \frac{C}{k' - k} \ge \frac{C}{n} \quad (0 \le k < k' \le n).$$

Thus we have

$$\min_{m\in\mathbb{Z}} |k_\nu\theta - c_j - m| > \frac{C\nu}{2n} \quad (1 \le \nu \le L)$$

and

$$|f(k_\nu\theta)| < h\bigl(\tfrac{C\nu}{2n}\bigr) \quad (1 \le \nu \le L < 3\epsilon n).$$

Hence if $n \ge N_\epsilon$, we have

$$\begin{aligned}\frac{1}{n}\sum_{k\in J(j,n,\epsilon)} |f(k\theta)| &= \frac{1}{n}\sum_{\nu=1}^{L} |f(k_\nu\theta)| \le \frac{1}{n}\sum_{\nu=1}^{[3\epsilon n]} h(\tfrac{C\nu}{2n}) \\ &\le \int_0^{\frac{[3\epsilon n]}{n}} h\bigl(\tfrac{Cx}{2}\bigr)\,dx \le \frac{2}{C}\int_0^{\frac{3\epsilon C}{2}} h(t)\,dt,\end{aligned}$$

which implies (18). Here $[3\epsilon n]$ denotes the largest integer satisfying $[3\epsilon n] \le 3\epsilon n$. □

4. Estimate I

Let $a = re^{2\pi i\tau}$ and $q = e^{2\pi i\theta}$ with τ, $\theta \in \mathbb{R}$ and $r > 0$. Then

$$\sqrt[n]{\bigl|(a;q)_n\bigr|} = \exp\left(\frac{1}{n}\sum_{k=0}^{n-1} \log\bigl|1 - re^{2\pi i(k\theta+\tau)}\bigr|\right). \tag{19}$$

If $r \neq 1$, Theorem 2.1 and (14) imply

$$\lim_{n\to\infty} \sqrt[n]{\left|(a;q)_n\right|} = \max\{|a|,1\} \qquad (|a|\neq 1,\ q=e^{2\pi i\theta} \text{ with } \theta\in\mathbb{R}\setminus\mathbb{Q}). \tag{20}$$

Assume $r = 1$. Since

$$\sum_{\substack{\min_{m\in\mathbb{Z}}|k\theta-m|<\epsilon\\ 1\le k\le n}} \log\left|1-e^{2\pi i k\theta}\right| \le 0$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{\min_{m\in\mathbb{Z}}|k\theta-m|\ge\epsilon\\ 1\le k\le n}} \log\left|1-e^{2\pi i k\theta}\right| = \int_{\epsilon}^{1-\epsilon}\log\left|1-e^{2\pi i x}\right| dx$$

for any small positive number ϵ, we have (9) in view of (14) and (19).

Now assume moreover that θ satisfies the assumption in Theorem 1.1 i). Then Lemma 3.1 with $f(x)=\log|1-e^{2\pi i x}|$ and $h(t)=|\log\pi t\,|$ (cf. (15)) proves (7).

Suppose $\tau = \frac{k_1}{m_1}\theta+\frac{k_2}{m_2}$ with integers k_1,k_2,m_1,m_2 with $m_1>0$, $m_2>0$ and

$$\left(\frac{k_1}{m_1}+k-1\right)\theta+\frac{k_2}{m_2}\notin\mathbb{Z} \qquad (k=1,2,3,\ldots)$$

corresponding to (2), (3) and (16). Lemma 3.1 with $f(x)=\log\left|1-e^{2\pi i(x-c_1)}\right|$, $h(t)=|\log\pi t\,|$, $p=1$ and $c_1=-(\frac{k_1}{m_1}-1)\theta-\frac{k_2}{m_2}$ implies

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(e^{2\pi i\tau};e^{2\pi i\theta}\right)_n\right|}=1. \tag{21}$$

Thus we have Theorem 1.1 i) by the estimates (20), (7) and (21).

5. Estimate II

Define a series of rapidly increasing positive integers $\{a_n\}$ by

$$a_1=2,\quad a_{n+1}=k_{n+1}\cdot a_n\cdot a_n! \qquad (k_{n+1}=2 \text{ or } 3,\ n=1,2,3,\ldots) \tag{22}$$

and put $\theta=\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ and $q=e^{2\pi i\theta}$. Then $\theta\notin\mathbb{Q}$ and we have

$$\min_{m\in\mathbb{Z}}\left|a_n\cdot\theta-m\right| \le \sum_{k=n+1}^{\infty}\frac{a_n}{a_k}<\frac{1}{a_n!}, \tag{23}$$

$$\left|1-e^{2\pi i a_n\theta}\right| \le \frac{2\pi}{a_n!}, \quad \prod_{j=1}^{a_n}\left|1-q^j\right| \le \frac{2^{a_n}\pi}{a_n!}, \quad \lim_{n\to\infty}\sqrt[a_n]{\prod_{j=1}^{a_n}\left|1-q^j\right|}=0$$

and (10) in Theorem 1.1 ii). We may choose $k_n \in \{2,3\}$ for $n=1,2,\ldots$, we get uncountably many θ's. Moreover if we put $\theta=\sum_{n=1}^{\infty}\frac{1}{a_n}+r$ so that there exists a positive integer N satisfying $rN\in\mathbb{Z}$, then θ also satisfies (23) for $n\ge N$ and hence θ satisfies (10).

The remaining claim in Theorem 1.1 ii) is clear from (20).

6. An extended result

We have proved the following statement for the series ${}_r\phi_s$ given by (1):

> There exist a dense subset $J\subset\mathbb{R}$ and a dense subset $K_\theta\subset\mathbb{R}$ for $\theta\in J$ such that for any $\theta\in J$ and $\alpha_i,\ \beta_j\in(\mathbb{C}\setminus\mathbb{R})\cup K_\theta$, the radius of convergence of the series ${}_r\phi_s$ with $q=e^{2\pi\sqrt{-1}\theta}$, $a_i=e^{2\pi\sqrt{-1}\alpha_i}$ and $b_j=e^{2\pi\sqrt{-1}\beta_j}$ equals the number given by (8).

Here we have the following theorem.

THEOREM 6.1. *We can choose $\mu(\mathbb{R}\setminus J)=0$ and $\mu(\mathbb{R}\setminus K_\theta)=0$ in the above. Here $\mu(L)$ denotes the Lebesgue measure of a subset L of $\mathbb{R}$.*

Let $f(x)$ be a periodic function with period 1 and suppose that $f(x)$ defines L^1 function on $[0,1]$. Then Khinchin [**3**] proves that if θ is irrational, the equality

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(k\theta+\beta)=\int_0^1 f(x)dx \tag{24}$$

is valid for almost all β. Hence it is an interesting problem to find conditions for the identity (24) to hold if $\int_0^1 f(x)dx$ exists only as an improper Riemann integral. First results to this problem are as follows.

Giving a beautiful algebraic identity between $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{\sin\pi\theta\cdot\sin 2\pi\theta\cdots\sin n\pi\theta}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{\sin n\theta}$, Hardy-Littlewood [**4**] proves that if

$$\int_0^1 f(x)\left|\log^2 x+\log^2(1-x)\right|dx<\infty,$$

the identity (13) is valid for almost all θ, in particular, for θ with bounded continued fraction expansions. Note that the arguments in §2–4 and the above results assure Theorem 6.1.

There has been several results related to the identity (24). In particular Baxa [**2**] proves that the identity (24) is valid if and only if

$$\lim_{k\to\infty} \frac{f(k\theta+\beta)}{n} = 0$$

under the condition:

> The number θ has either a bounded fraction expansion or β is rational or
>
> $$\varliminf_{k\to\infty} \min_{m\in\mathbb{Z}} |\beta q_k - m| > 0$$
>
> where q_k denotes the denominator of the k-th convergent of the continued fraction expansion of θ.

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