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On the behavior of solutions for Lanchester square-law models with time-dependent coefficients

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Abstract. This paper concerns an ordinary differential system which is a so-called Lanchester square-law model with time-dependent coefficients. We study qualitative properties of solutions and, among other things, discuss the relation between behavior of solutions and their initial data.

1. Introduction

In this paper, we are concerned with the differential system of the form:

$$\text{(S)} \qquad \begin{cases} x'(t) = -a(t)y(t), \\ y'(t) = -b(t)x(t), \end{cases}$$

where $a(t)$ and $b(t)$ are positive continuous functions on $[0, \infty)$. Throughout this paper we will consider nonnegative solutions $(x(t), y(t))$ for (S) with positive initial data satisfying $x(0) > 0$ and $y(0) > 0$. We will imposed additional conditions on $a(t)$ and $b(t)$ later.

System (S) is known as one of Lanchester-type model, which describes many phenomena appearing in economics, logistics, biology, and so on. It was Lanchester [8] who first proposed system (S) to describe combat situations.

Let us consider the special case where $a(t) \equiv \alpha$ and $b(t) \equiv \beta$ for some constants α, $\beta > 0$, namely the case where the system (S) is in the form

$$(\mathrm{S}_0) \qquad \begin{cases} x'(t) = -\alpha y(t), \\ y'(t) = -\beta x(t). \end{cases}$$

Then, a solution $(x(t), y(t))$ of (S_0) with initial data $x(0) = x_0$ and $y(0) = y_0$

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satisfies

$$\frac{d}{dt}\left(\beta x^2 - \alpha y^2\right) = 2\beta x \frac{dx}{dt} - 2\alpha y \frac{dy}{dt} = 0.$$

Therefore, if we put $E = \alpha/\beta$, then

$$x(t)^2 - x_0^2 = E\left(y(t)^2 - y_0^2\right), \tag{1}$$

namely, the so-called Lanchester square law holds. More precisely, by a simple calculation a solution $(x(t), y(t))$ can be written as

$$\begin{aligned} x(t) &= \frac{1}{2}\left\{(x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}t} + (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}t}\right\}, \\ y(t) &= \frac{1}{2\sqrt{E}}\left\{(x_0 + \sqrt{E}y_0)e^{-\sqrt{\alpha\beta}t} - (x_0 - \sqrt{E}y_0)e^{\sqrt{\alpha\beta}t}\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} &x_0^2 > Ey_0^2, \quad \text{implies} \quad x(T) > 0,\ y(T) = 0 \ \text{ for } T = \frac{1}{2\sqrt{\alpha\beta}} \log \frac{x_0 + \sqrt{E}y_0}{x_0 - \sqrt{E}y_0}, \\ &x_0^2 = Ey_0^2, \quad \text{implies} \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0, \\ &x_0^2 < Ey_0^2, \quad \text{implies} \quad x(T) = 0,\ y(T) > 0 \ \text{ for } T = \frac{1}{2\sqrt{\alpha\beta}} \log \frac{\sqrt{E}y_0 + x_0}{\sqrt{E}y_0 - x_0}. \end{aligned}$$

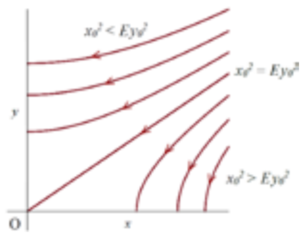

Figure 1. Behavior of solutions for (S_0).

2. Main results

For system (S), we impose the following conditions:

(A1) $0 < \inf_{t \geq 0} \dfrac{a(t)}{b(t)} < \sup_{t \geq 0} \dfrac{a(t)}{b(t)} < \infty$,

(A2) $\displaystyle\int_0^\infty a(t)\,dt = \infty$.

Remark 2.1. *Under assumption* (A1), *assumption* (A2) *is equivalent to* $\int_0^\infty b(t)\,dt = \infty$.

For $x_0 > 0$ and $y_0 > 0$, let $(x(t), y(t)) = (x(t; x_0, y_0), y(t; x_0, y_0))$ be a solution of (S) with initial data $(x(0), y(0)) = (x_0, y_0)$.

We note that $x(t)$, $y(t)$ are both strictly decreasing in t since $x'(t) < 0$ and $y'(t) < 0$ for all $t > 0$. Furthermore $(x(t), y(t)) \equiv (0, 0)$ is a solution of (S). Therefore, from the uniqueness of solutions of the initial value problem of (S) and (A2), we see that one of the followings holds:

$$\begin{aligned}
&\lim_{t\to T} x(t) > 0 \text{ and } \lim_{t\to T} y(t) = 0 \quad \text{for some } T > 0;\\
&\lim_{t\to T} x(t) = 0 \text{ and } \lim_{t\to T} y(t) = 0 \quad \text{for } T = \infty;\\
&\lim_{t\to T} x(t) = 0 \text{ and } \lim_{t\to T} y(t) > 0 \quad \text{for some } T > 0.
\end{aligned} \tag{2}$$

For an arbitrarily fixed $x_0 > 0$, define a subset S_{x_0} of $\mathbb{R}^2$ by

$$S_{x_0} = \big([0, x_0) \times \{0\}\big) \cup \big(\{0\} \times [0, \infty)\big)$$

and a map $\omega = \omega_{x_0} \colon (0, \infty) \to S_{x_0}$ by

$$\omega(y_0) = \lim_{t\to T} (x(t; x_0, y_0), y(t; x_0, y_0)),$$

where T is the maximal existence time of the solution $(x(t), y(t))$ obtained in (2).

We obtain the following:

Theorem 2.2. *Let* (A1) *and* (A2) *hold and fix* $x_0 > 0$ *arbitrarily. Then, the mapping* $\omega = \omega_{x_0}$ *is a continuous bijection from* $(0, \infty)$ *to* S_{x_0}. *Therefore, for any* $(C_1, C_2) \in S_{x_0}$, *there is one and only one solution* $(x(t), y(t))$ *of* (S) satisfying $x(0) = x_0$ and $\lim_{t\to T} (x(t), y(t)) = (C_1, C_2)$, where T is some positive number or $T = \infty$.

The following corollaries are immediate consequences of Theorem 2.2 and Lemma 3.1 in Section 3.

Corollary 2.3. *Assume* (A1) *and* (A2). *Then, for any* $x_0 > 0$, *there exists some* $\beta_0 > 0$ *satisfying the following*:

(i) *if* $0 < y_0 < \beta_0$, *then* $\lim_{t\to T} x(t; x_0, y_0) > 0$, $\lim_{t\to T} y(t; x_0, y_0) = 0$ *for some* $T > 0$;

(ii) *if* $y_0 = \beta_0$, *then* $\lim_{t\to\infty} x(t; x_0, y_0) = 0$, $\lim_{t\to\infty} y(t; x_0, y_0) = 0$;

(iii) *if* $\beta_0 < y_0$, *then* $\lim_{t\to T} x(t; x_0, y_0) = 0$, $\lim_{t\to T} y(t; x_0, y_0) > 0$ *for some* $T > 0$.

Now, for an arbitrarily fixed $x_0 > 0$, denote by $T(y_0)$ the maximal existence time of the solution $(x(t; x_0, y_0), y(t; x_0, y_0))$, namely,

$$T(y_0) := \sup\{t > 0 \mid x(t; x_0, y_0) > 0 \text{ and } y(t; x_0, y_0) > 0\}.$$

The following holds:

Corollary 2.4. *Assume* (A1) *and* (A2) *and fix* $x_0 > 0$ *arbitrarily. Then, for the constant* $\beta_0 > 0$ *obtained in Corollary* 2.3,

(i) *if* $0 < y_1 < y_2 < \beta_0$, *then* $\lim_{t\to T(y_1)} x(t; x_0, y_1) > \lim_{t\to T(y_2)} x(t; x_0, y_2)$;

(ii) *if* $\beta_0 < y_1 < y_2$, *then* $\lim_{t\to T(y_1)} y(t; x_0, y_1) < \lim_{t\to T(y_2)} y(t; x_0, y_2)$.

Finally, concerning the maximal existence time of the solution to (S), we obtain the following:

Theorem 2.5. *Assume* (A1) *and* (A2) *and fix* $x_0 > 0$ *arbitrarily. Then, for the constant* $\beta_0 > 0$ *obtained in Corollary* 2.3,

(i) *if* $0 < y_1 \le y_2 < \beta_0$, *then* $T(y_1) \le T(y_2)$;

(ii) *if* $y_0 = \beta_0$, *then* $T(y_0) = \infty$;

(iii) *if* $\beta_0 < y_1 \le y_2$, *then* $T(y_1) \ge T(y_2)$.

Remark 2.6. *There are related works. In* [5] *or* [13], *the authors of this paper have been concerned with another Lanchester-type model, originated from the so-called Lanchester's linear law model, of the form*

$$\text{(S}'\text{)} \qquad \begin{cases} x'(t) = -a(t)x(t)y(t), \\ y'(t) = -b(t)x(t)y(t). \end{cases}$$

Note that $(x(t), y(t)) \equiv (C_1, C_2)$ *is a solution of* (S′) *for any* $(C_1, C_2) \in S_{x_0}$. *Hence, one of the following conditions holds for solutions* $(x(t), y(t))$ *of* (S′) *satisfying* $x(0) > 0$ *and* $y(0) > 0$:

$$\begin{aligned}
&\lim_{t\to\infty} x(t) > 0 \ \textit{and} \ \lim_{t\to\infty} y(t) = 0;\\
&\lim_{t\to\infty} x(t) = 0 \ \textit{and} \ \lim_{t\to\infty} y(t) = 0;\\
&\lim_{t\to\infty} x(t) = 0 \ \textit{and} \ \lim_{t\to\infty} y(t) > 0.
\end{aligned}$$

Therefore, for (S′), $\omega\colon (0, \infty) \to S_{x_0}$ *is defined by*

$$\omega(y_0) = \lim_{t\to\infty} (x(t), y(t)).$$

In [5], *we showed that the statement of Theorem* 2.2 *and subsequent Corollaries* 2.3 *and* 2.4 *hold for* (S′) *with the definition of* ω *being replaced as above.*

3. Properties of solutions of (S)

To prove Theorems 2.2 and 2.5, the following lemmas play an important role.

Lemma 3.1. *Let* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *be solutions of system* (S).

(i) [Comparison theorem] *If* $x_1(0) \geq x_2(0)$ *and* $y_1(0) \leq y_2(0)$, *then* $x_1(t) \geq x_2(t)$ *and* $y_1(t) \leq y_2(t)$ *for all* $t > 0$.

(ii) [Strong comparison theorem] *If* $x_1(0) \geq x_2(0)$, $y_1(0) \leq y_2(0)$ *and* $(x_1(0), y_1(0)) \neq (x_2(0), y_2(0))$, *then* $x_1(t) > x_2(t)$ *and* $y_1(t) < y_2(t)$ *for all* $t > 0$.

Proof. First we consider the case $x_1(0) > x_2(0)$ and $y_1(0) < y_2(0)$ and show that

$$x_1(t) > x_2(t) \ \text{ and } \ y_1(t) < y_2(t) \qquad \text{for all } t > 0. \tag{3}$$

Suppose to the contrary that (3) does not hold. Without loss of generality, we may assume that

$$\begin{aligned}
&x_1(t_1) = x_2(t_1),\\
&x_1(t) > x_2(t) \text{ and } y_1(t) < y_2(t), \qquad 0 < t < t_1
\end{aligned}$$

for some $t_1 > 0$. Then, since

$$\frac{d}{dt}(x_1(t) - x_2(t)) = -b(t)\{y_1(t) - y_2(t)\} > 0, \qquad 0 < t < t_1,$$

$x_1(0) - x_2(0) > 0$ implies $x_1(t_1) - x_2(t_1) > 0$ and hence a contradiction. Thus (3) holds.

Now we prove statement (ii). Suppose $x_1(0) \geq x_2(0)$, $y_1(0) \leq y_2(0)$ and $(x_1(0), y_1(0)) \neq (x_2(0), y_2(0))$. Clearly $x_1(0) > x_2(0)$ or $y_1(0) < y_2(0)$ holds. We consider the case of $x_1(0) > x_2(0)$, since the case of $y_1(0) < y_2(0)$ can be treated similarly. Then there exists a sufficiently small $t_2 > 0$ satisfying

$$x_1(t) > x_2(t), \qquad 0 < t \leq t_2.$$

Hence

$$\frac{d}{dt}(y_1(t) - y_2(t)) = -a(t)\{x_1(t) - x_2(t)\} < 0, \qquad 0 < t < t_2,$$

which implies

$$y_1(t) < y_2(t), \qquad 0 < t \leq t_2.$$

Thus we have

$$x_1(t_2) > x_2(t_2) \quad \text{and} \quad y_1(t_2) < y_2(t_2).$$

Furthermore, by the same argument above, it follows from these inequalities that

$$x_1(t) > x_2(t) \text{ and } y_1(t) < y_2(t) \quad \text{for all } t > t_2.$$

Therefore (3) holds.

Finally we prove statement (i). Suppose $x_1(0) \geq x_2(0)$ and $y_1(0) \leq y_2(0)$ and let $(x_{1,n}(t), y_{1,n}(t))$, $(x_{2,n}(t), y_{2,n}(t))$ be solutions of (S) with initial data

$$\begin{aligned}(x_{1,n}(0), y_{1,n}(0)) &= (x_1(0) + 1/n, y_1(0) + 1/n),\\ (x_{2,n}(0), y_{2,n}(0)) &= (x_2(0) + 2/n, y_2(0) + 2/n),\end{aligned}$$

respectively. Then, by statement (ii) of this lemma,

$$x_{1,n}(t) > x_{2,n}(t) \text{ and } y_{1,n}(t) < y_{2,n}(t) \qquad \text{for all } t > 0.$$

Letting $t \to \infty$, we obtained the conclusion of statement (i). □

Lemma 3.2. *Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be solutions of system* (S) *satisfying $x_1(0) \geq x_2(0)$ and $y_1(0) \leq y_2(0)$.*

(i) *The function $x_2(t)/x_1(t)$ is nonincreasing, whereas the function $y_2(t)/y_1(t)$*

is nondecreasing in $t > 0$.

(ii) *Furthermore let either* $x_1(0) > x_2(0)$ *or* $y_1(0) < y_2(0)$ *hold. Then the function* $x_2(t)/x_1(t)$ *is strictly decreasing, whereas* $y_2(t)/y_1(t)$ *is strictly increasing in* $t > 0$.

Proof. By Lemma 3.1 (i),

$$\frac{d}{dt}\left(\frac{x_2(t)}{x_1(t)}\right) = -\frac{a(t)\{x_1(t)y_2(t) - y_1(t)x_2(t)\}}{x_1(t)^2} \leq 0, \quad t > 0,$$

$$\frac{d}{dt}\left(\frac{y_2(t)}{y_1(t)}\right) = -\frac{b(t)\{x_2(t)y_1(t) - x_1(t)y_2(t)\}}{y_1(t)^2} \geq 0, \quad t > 0.$$

Thus we obtain the conclusion of statement (i).

Furthermore, if $x_1(0) > x_2(0)$ or $y_1(0) < y_2(0)$ holds, then Lemma 3.1 (ii) and the above computation yield

$$\frac{d}{dt}\left(\frac{x_2(t)}{x_1(t)}\right) < 0 \text{ and } \frac{d}{dt}\left(\frac{y_2(t)}{y_1(t)}\right) > 0, \qquad t > 0.$$

Thus statement (ii) holds. We complete the proof. □

4. Proof of Theorems

[Proof of continuity of ω]
We show that ω is a continuous map. By assumption (A1), there exists positive constants m, M satisfying

$$m < \frac{b(t)}{a(t)} < M, \qquad t > 0. \tag{4}$$

Hence, for a solution $(x(t), y(t))$ of (S), we have

$$-Ma(t)x(t)y(t) < -b(t)x(t)y(t) < -ma(t)x(t)y(t), \qquad t > 0.$$

Therefore

$$Mx(t)x'(t) < y(t)y'(t) < mx(t)x'(t), \qquad t > 0,$$

which implies

(5) $m\big(x(t_1)^2 - x(t_2)^2\big) < y(t_1)^2 - y(t_2)^2 < M\big(x(t_1)^2 - x(t_2)^2\big) \quad \text{for } t_2 > t_1 > 0.$

First we consider the case where $\omega(y_0) = (C, 0)$ for some $C > 0$. For arbitrary $\varepsilon > 0$ satisfying $\varepsilon < C$, let γ_1 and γ_2 be the curves defined by

$$\gamma_1 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 = m\big(x^2 - (C - \varepsilon)^2\big)\},$$
$$\gamma_2 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 = M\big(x^2 - (C + \varepsilon)^2\big)\}.$$

Further let U be the open set in $\mathbf{R}^2$ surrounded by γ_1, γ_2, and the x-axis. For some T_0, we have $(x(T_0; x_0, y_0), y(T_0; x_0, y_0)) \in U$. Therefore, for sufficiently small $\delta > 0$ the property $|y(0) - y_0| < \delta$ implies that $(x(T_0; x_0, y(0)), y(T_0; x_0, y(0))) \in U$.

We can show that $\omega(y(0)) \in [C - \varepsilon, C + \varepsilon] \times \{0\}$. In fact, if this is not true, then there is a $T_1 > T_0$ satisfying

$$(x(t; x_0, y(0)), y(t; x_0, y(0))) \in U \quad \text{for } t \in [T_0, T_1),$$
$$(x(T_1; x_0, y(0)), y(T_1; x_0, y(0))) \text{ exists either on } \gamma_1 \text{ or on } \gamma_2.$$

Suppose that $(x(T_1; x_0, y(0)), y(T_1; x_0, y(0)))$ exists on γ_1. Then,

$$y(T_0)^2 - y(T_1)^2 < m\big(x(T_0)^2 - x(T_1)^2\big)$$

holds, which contradicts (5). Similarly we can get a contradiction for the case where $(x(T_1; x_0, y(0)), y(T_1; x_0, y(0)))$ exists on γ_2. Therefore, $\omega(y(0)) \in [C-\varepsilon, C+\varepsilon] \times \{0\}$ for y_0 satisfying $|y(0) - y_0| < \delta$. This shows the continuity of ω at $y = y_0$.

Next we consider the case where $\omega(y_0) = (0, 0)$. For arbitrary $\varepsilon > 0$, let γ_1 and γ_2 be the curves defined by

$$\gamma_1 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 - \varepsilon^2 = mx^2\},$$
$$\gamma_2 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 = M\big(x^2 - \varepsilon^2\big)\}.$$

Further let U be the open set in $\mathbf{R}^2$ surrounded by γ_1, γ_2, and the x-axis, y axis. Then, in the same way as above, there exists some $\delta > 0$ such that if $|y(0) - y_0| < \delta$ then $\omega(y(0)) \in ([0, \varepsilon] \times \{0\}) \cup (\{0\} \times [0, \varepsilon])$. This shows the continuity of ω at $y = y_0$.

Finally we consider the case where $\omega(\beta) = (0, C)$ for some $C > 0$. For arbitrary

$\varepsilon > 0$ satisfying $\varepsilon < C$, let γ_1 and γ_2 be the curves defined by

$$\gamma_1 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 - (C + \varepsilon)^2 = mx^2\},$$
$$\gamma_2 = \{(x, y) \mid x \geq 0,\ y \geq 0,\ y^2 - (C - \varepsilon)^2 = Mx^2\}.$$

Further let U be the open triangular set in $\mathbf{R}^2$ surrounded by γ_1, γ_2, and the y-axis. Then, as in the first case, there exists some $\delta > 0$ such that if $|y(0) - y_0| < \delta$ then $\omega(y(0)) \in \{0\} \times [C - \varepsilon, C + \varepsilon]$. This shows the continuity of ω at $y = y_0$.

[Proof of the surjectivity of ω]
Since ω is a continuous map from $(0, \infty)$ to $\big([0, x_0) \times \{0\}\big) \cup \big(\{0\} \times [0, \infty)\big)$ and since $(0, \infty)$ is connected, the image $\omega((0, \infty))$ is also connected. Fix $y_0 > 0$ and define $V \subset \mathbf{R}^2$ by

$$V = \{(x, y) \mid x > 0,\ y > 0,\ M(x^2 - x_0^2) < y^2 - {y_0}^2 < m(x^2 - {x_0}^2)\}.$$

Then, as in the proof of the continuity of ω, (5) implies $(x(t; x_0, y_0), y(t, x_0, y_0)) \in V$ for $t > 0$ satisfying $x(t; x_0, y_0) > 0$ and $y(t; x_0, y_0) > 0$, which shows

$$\lim_{y_0 \to +0} \omega(y_0) = (x_0, 0), \qquad \lim_{y_0 \to \infty} \omega(y_0) = (x_0, \infty)$$

So ω is surjective.

[Proof of the injectivity of ω]
By the uniqueness of solutions of initial value problem of (S), the restriction of ω on the set

$$\omega^{-1}((0, x_0) \times \{0\}) \cup \omega^{-1}(\{0\} \times (0, \infty))$$

is injective. Therefore we may show that $\omega^{-1}((0, 0))$ is a singleton. Suppose to the contrary that, for some y_{01} and y_{02} satisfying $y_{01} < y_{02}$, we have $\omega(y_{01}) = \omega(y_{02}) = (0, 0)$, that is,

$$\lim_{t \to \infty} (x(t; x_0, y_{01}), y(t; x_0, y_{01})) = \lim_{t \to \infty} (x(t; x_0, y_{02}), y(t; x_0, y_{02})) = (0, 0).$$

It follows from Lemma 3.2 (ii) that

$$\frac{x(t; x_0, y_{02})}{x(t; x_0, y_{01})} < \frac{x(t_0; x_0, y_{02})}{x(t_0; x_0, y_{01})} < \frac{x(0; x_0, y_{02})}{x(0; x_0, y_{01})} = \frac{x_0}{x_0} = 1 \quad \text{for} \quad t \geq t_0 > 0,$$

$$\frac{y(t;x_0,y_{02})}{y(t;x_0,y_{01})} > \frac{y(t_0;x_0,y_{02})}{y(t_0;x_0,y_{01})} > \frac{y(0;x_0,y_{02})}{y(0;x_0,y_{01})} = \frac{y_{02}}{y_{01}} > 1 \quad \text{for} \quad t \geq t_0 > 0.$$

Hence, l'Hospital's rule implies

$$1 > \lim_{t\to\infty} \frac{x(t;x_0,y_{02})}{x(t;x_0,y_{01})} = \lim_{t\to\infty} \frac{x'(t;x_0,y_{02})}{x'(t;x_0,y_{01})} = \lim_{t\to\infty} \frac{y(t;x_0,y_{02})}{y(t;x_0,y_{01})} > 1.$$

This contradiction proves the injectivity of ω.

Thus we complete the proof of Theorem 2.2. Theorem 2.5 follows from Lemma 3.1 immediately.

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