Josai Mathematical Monographs
Vol. 11 (2018) pp. 27–35

Measures of analytic type and semicharacters

Hiroshi YAMAGUCHI

Abstract. Helson and Lowdenslager extended the classical F. and M. Riesz theorem as follows: Let G be a compact abelian group with ordered dual. Let μ be a bounded regular measure on G which is of analytic type. Then

(I) μ_a and μ_s are of analytic type, and

(II) $\hat{\mu}_s(0) = 0$,

where μ_a and μ_s are the absolutely continuous part of μ and the singular part of μ, respectively. Forelli gave a generalization of the result of Helson and Lowdenslager for a compact abelian group with ordered dual. As for (I), Doss, Yamaguchi and Hewitt-Koshi-Takahashi gave its extensions for locally compact abelian groups. In this paper, we extend a result of Forelli, which is related to (II), to locally compact abelian groups.

1. Introduction

Let G be a LCA group (locally compact abelian group) with the dual group $\hat{G}$. We denote by m_G the Haar measure of G. Let $L^1(G)$ and $M(G)$ be the group algebra and the measure algebra, respectively. Let $M_a(G)$ be the set of all measures in $M(G)$ which are absolutely continuous with respect to m_G. Then we can identify $M_a(G)$ with $L^1(G)$. Let $M_s(G)$ be the closed subspace of $M(G)$ consisting of singular measures. We denote by $M^+(G)$ the subset of $M(G)$ consisting of positive measures. For $\mu \in M(G)$, let μ_a and μ_s be the absolutely continuous part of μ and the singular part of μ, respectively. $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ, i.e., $\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\,\mu(x) \quad$ for $\gamma \in \hat{G}$. For a subset E of $\hat{G}$, let $M_E(G)$ be the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E.

Helson and Lowdenslager extended the classical F. and M. Riesz theorem as follows.

Theorem A ([17, 8.2.3 Theorem]). Let G be a compact abelian group with ordered dual. Let μ be a measure in $M(G)$ such that $\hat{\mu}(\gamma) = 0 \quad$ for $\gamma < 0$. Then

---

2010 Mathematics Subject Classification. Primary 43A05; Secondary 43A10, 43A25..
Key Words and Phrases. LCA group, ordered group, measure, analytic type, Fourier transform..

(I) $\hat{\mu_a}(\gamma) = \hat{\mu_s}(\gamma) = 0 \quad \text{for } \gamma < 0,$

(II) $\hat{\mu_s}(0) = 0.$

For a compact abelian group G with ordered dual, let $A = \{f \in C(G) : \hat{f}(\gamma) = 0 \ \text{for} \ \gamma < 0\}$. Then A becomes a Dirichlet algebra. As for Theorem A (II), Forelli obtained the following.

Theorem B ([7, Theorem 2]). Let G be a compact abelian group with ordered dual. Let $\sigma \in M^+(G)$ be a representing measure for A. Let μ be a measure in $M_s(G)$ such that $\hat{\mu}(\gamma) = 0 \quad \text{for } \gamma < 0$. Then $\mu * \sigma$ belongs to $M_s(G)$.

Our purpose is to extend Theorem B to LCA groups. In section 2, we state definitions and our result. We give the proof in section 3.

2. Notation and results

Let G be a LCA group and P a proper closed semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. We note that $\overset{\circ}{P}$ (the interior of P) is dense in P. Set $\Lambda = P \cap (-P)$. Let $\overline{D}$ be the closed unit disc in the complex plane $\mathbb{C}$, i.e., $\overline{D} = \{z \in \mathbb{C} : |z| \leqq 1\}$. A function $\zeta : P \to \overline{D}$ is called a semicharacter (on P) if $\zeta(\gamma_1 + \gamma_2) = \zeta(\gamma_1)\zeta(\gamma_2)$ for $\gamma_1, \gamma_2 \in P$. We denote by Δ the set of all nonzero continuous semicharacters. Define a subset Δ^+ of Δ as follows:

$$\Delta^+ = \{\zeta \in \Delta : 0 \leqq \zeta(\gamma) \leqq 1 \text{ for all } \gamma \in P\}. \tag{2.1}$$

For each $\zeta \in \Delta$, $|\zeta|$ belongs to Δ^+. And, for $\rho \in \Delta^+$, we have $\rho(\gamma) = 1$ on Λ since Λ is a subgroup of $\hat{G}$. By [1, 3.1 Theorem], each $\zeta \in \Delta$ has a polar decomposition. That is, there exists $x \in G$ such that

$$\zeta = |\zeta|\, x. \tag{2.2}$$

For each $\zeta \in \Delta$, it follows from [1, 4.7 Theorem and 4.8 Corollary] that there exists a unique probability measure $m_\zeta \in M^+(G)$ such that

$$\hat{m_\zeta}(\gamma) = \zeta(\gamma) \ \text{for } \gamma \in P. \tag{2.3}$$

For $\zeta, \zeta' \in \Delta$, we have $m_{\zeta\,\zeta'} = m_\zeta * m_{\zeta'}$. Especially, (2.2) implies

$$m_\zeta = m_{|\zeta|} * \delta_{-x}, \tag{2.4}$$

where δ_{-x} is the point mass at $-x$.

Now we state our theorem.

Theorem 2.1. *For $\rho \in \Delta^+$ and $\mu \in M_P(G) \cap M_s(G)$, we have $\mu * m_\rho \in M_s(G)$.*

By (2.4) and Theorem 2.1, we get the following corollary.

Corollary 2.2. *For $\zeta \in \Delta$ and $\mu \in M_P(G) \cap M_s(G)$, we have $\mu * m_\zeta \in M_s(G)$.*

Remark 2.3. (i) *Let G be a compact abelian group with ordered dual. Let $P = \{\gamma \in \hat{G} : \gamma \geqq 0\}$ and $A = \{f \in C(G) : \hat{f}(\gamma) = 0$ for $\gamma < 0\}$. Let $\sigma \in M^+(G)$ be a representaing measure for A. Then $\hat{\sigma}$ becomes a nonzero semicharacter on P, and $m_{\hat{\sigma}} = \sigma$. Hence Theorem B follows from Corollary 2.2.*

(ii) *Let G be a compact abelian group, and suppose that there exists a nontrivial homomorphism ψ from $\hat{G}$ into $\mathbb{R}$. Put $P = \{\gamma \in \hat{G} : \psi(\gamma) \geqq 0\}$. Let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ, i.e., $(\phi(t), \gamma) = \exp(i\psi(\gamma)t)$ for $\gamma \in \hat{G}$, $t \in \mathbb{R}$. Let $\mu \in M_P(G) \cap M_s(G)$ and $\lambda \in M(\mathbb{R})$. Then, by [4, Theorem 3.1 and Lemma 5.1], we have $\mu * \phi(\lambda) \in M_s(G)$, where $\phi(\lambda)$ is the continuous image of λ under ϕ.*

3. Proof of Theorem

In this section we prove Theorem 2.1. First we state several lemmas. Let $\pi : \hat{G} \to \hat{G}/\Lambda$ be the natural homomorphism, and put $\tilde{P} = \pi(P)$. Then $\tilde{P}$ is a closed semigroup in $\hat{G}/\Lambda$ such that $\tilde{P} \cup (-\tilde{P}) = \hat{G}/\Lambda$ and $\tilde{P} \cap (-\tilde{P}) = \{0\}$. Thus $\tilde{P}$ induces a totally linear order on $\hat{G}/\Lambda$. For $\rho \in \Delta^+$, let $L_\rho^+ = \{\gamma \in P : \rho(\gamma) > 0\}$ and $\Gamma_\rho = L_\rho^+ - L_\rho^+$.

Lemma 3.1. *For $\rho \in \Delta^+$, the following hold.*

(i) Γ_ρ *is an open subgroup of* $\hat{G}$*;*

(ii) $\Lambda \subset L_\rho^+ \subset \Gamma_\rho$*;*

(iii) $P \cap \Gamma_\rho = L_\rho^+$*;*

(iv) *Let $\gamma \in P$ and $\xi \in \Gamma_\rho$. If $\pi(\gamma) \leqq \pi(\xi)$, then $\gamma \in L_\rho^+$;*

(v) $\Gamma_\rho = L_\rho^+ \cup (-L_\rho^+)$.

Proof. (i): Set $V_\rho = \{\gamma \in \mathring{P} : \rho(\gamma) > 0\}$. Since $\mathring{P}$ is dense in P, V_ρ is nonempty, and V_ρ is an open semigroup in $\hat{G}$. Hence Γ_ρ is an open subgroup of $\hat{G}$ since $V_\rho - V_\rho$

is included in Γ_ρ.
(ii): This is trivial since $\rho(\gamma) = 1$ on Λ.
(iii): For $\gamma \in P \cap \Gamma_\rho$, there exist $\xi_1, \xi_2 \in L_\rho^+$ such that $\gamma = \xi_1 - \xi_2$. Then $\xi_1 = \gamma + \xi_2$, and so $0 \neq \rho(\xi_1) = \rho(\gamma)\rho(\gamma_2)$. Hence $\rho(\gamma) \neq 0$, which yields $\gamma \in L_\rho^+$. Hence $P \cap \Gamma_\rho \subset L_\rho^+$. The reverse inclusion relation is trivial.
(iv): Since $0 \leqq \pi(\gamma) \leqq \pi(\xi)$, ξ belongs to P. Hence (iii) implies $\xi \in L_\rho^+$. On the other hand, since $\pi(\gamma) \leqq \pi(\xi)$, there exists $p \in P$ such that $\xi - \gamma = p$. Hence $0 \neq \rho(\xi) = \rho(\gamma)\rho(p)$, which yields $\gamma \in L_\rho^+$.
(v): We have $\Gamma_\rho = (P \cap \Gamma_\rho) \cup ((-P) \cap \Gamma_\rho) = L_\rho^+ \cup (-L_\rho^+)$, by (iii).
This completes the proof. □

LEMMA 3.2. *Let $\rho \in \Delta^+$, and put $\tilde{Q} = \{\pi(\gamma) \in \hat{G}/\Lambda : \pi(\gamma) \geqq \pi(\gamma')$ for some $\gamma' \in \Gamma_\rho\}$. Then the following hold.*

- *(i)* $\tilde{Q} \supset \tilde{P}$;
- *(ii)* $\tilde{Q}$ *is an open semigroup in* $\hat{G}/\Lambda$ *such that* $\tilde{Q} \cup (-\tilde{Q}) = \hat{G}/\Lambda$;
- *(iii)* $\tilde{Q} \cap (-\tilde{Q}) = \tilde{\Gamma}_\rho$, *where* $\tilde{\Gamma}_\rho = \pi(\Gamma_\rho)$.

PROOF. (i): This is trivial since $\pi(p) \geqq 0$ for all $p \in P$.
(ii): It is trivial that $\tilde{Q}$ is a semigroup in $\hat{G}/\Lambda$ such that $\tilde{Q} \cup (-\tilde{Q}) = \hat{G}/\Lambda$. Since $\tilde{Q} \supset \pi(\Gamma_\rho)$ and $\pi(\Gamma_\rho)$ is an open subgroup of $\hat{G}/\Lambda$, $\tilde{Q}$ is an open semigroup in $\hat{G}/\Lambda$.
(iii): Since $\tilde{Q} \supset \tilde{\Gamma}_\rho$, we may prove only $\tilde{Q} \cap (-\tilde{Q}) \subset \tilde{\Gamma}_\rho$. Let $\pi(\gamma) \in \tilde{Q} \cap (-\tilde{Q})$, where $\gamma \in \hat{G}$. We may assume $\gamma \in P$ because $P \cup (-P) = \hat{G}$. Since $\pi(\gamma) \in -\tilde{Q}$, $\pi(-\gamma)$ belongs to $\tilde{Q}$. Hence there exists $\xi \in \Gamma_\rho$ such that $\pi(-\gamma) \geqq \pi(\xi)$. Then $\pi(\gamma) \leqq \pi(-\xi)$ and $-\xi \in \Gamma_\rho$. It follows from Lemma 3.1 (iv) that $\gamma \in L_\rho^+$. Hence we have $\pi(\gamma) \in \pi(L_\rho^+) \subset \tilde{\Gamma}_\rho$.
This completes the proof. □

When G is a compact abelian group, the following lemma is obtained in [4, Theorem 3.1 and lemma 5.2]. We give its proof for completeness.

LEMMA 3.3. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbb{R}$. Let $\phi : \mathbb{R} \to G$ be the dual homomorphis of ψ. Let $\xi \in M(\mathbb{R})$, and let μ be a measure in $M_s(G)$ such that $\hat{\mu}$ vanishes on $\psi^{-1}((-\infty, 0))$. Then $\phi(\xi) * \mu$ belongs to $M_s(G)$.*

PROOF. Since $\hat{\mu}$ vanishes on $\psi^{-1}((-\infty, 0))$, we note that $\lim_{s \to 0} \|\mu - \mu * \delta_{\phi(s)}\| = 0$ (cf. [8, Theorem 4]). Since μ is singular, there exists a Borel set E in G such that

$m_G(E) = 0$ and $|\mu|(E) = \|\mu\|$. Set $E_0 = \bigcup_{r\in\mathbb{Q}} (\phi(r) + E)$, where $\mathbb{Q}$ is the rational numbers. Then

(3.1) $\quad m_G(E_0) = 0$, and

(3.2) $\quad |\mu|(-\phi(r) + E_0) = \|\mu\| \quad \text{for all} \quad r \in \mathbb{Q}$.

Since $s \to |\mu|(-\phi(s) + E_0)$ is a continuous function on $\mathbb{R}$, (3.2) implies

$$|\mu|(-\phi(s) + E_0) = \|\mu\|$$

for all $s \in \mathbb{R}$. Hence we get

$$\begin{aligned}
\phi(|\xi|) * |\mu|(E_0) &= \int_G |\mu|(-x + E_0) d\phi(|\xi|)(x) \\
&= \int_{-\infty}^{\infty} |\mu|(-\phi(s) + E_0) d|\xi|(s) \\
&= \|\mu\| \, \|\xi\| \\
&= \phi(|\xi|) * |\mu|(G).
\end{aligned}$$

This shows that $\phi(\xi) * \mu$ is concentrated on E_0. Thus we get $\phi(\xi) * \mu \in M_s(G)$, by (3.1), and the proof is complete. □

Lemma 3.4. *Let G, ψ and ϕ be as in the previous lemma. Let μ be a measure in $M_s(G)$ such that $\hat{\mu}$ vanishes on $\psi^{-1}((-\infty, 0))$. Let σ be a measure in $M(G)$ such that $\hat{\sigma}(\gamma) = \exp(-|\psi(\gamma)|)$ for all $\gamma \in \hat{G}$. Then $\sigma * \mu \in M_s(G)$.*

Proof. Define $\xi \in L^1(\mathbb{R})$ by $d\,\xi(t) = \frac{1}{\pi} \cdot \frac{dt}{1+t^2}$. Then

$$\hat{\xi}(x) = \exp(-|x|)$$

for all $x \in \mathbb{R}$, where $\hat{\xi}(x) = \int_{-\infty}^{\infty} \xi(t) e^{-ixt} dt$. Hence we have $\sigma = \phi(\xi)$. In fact,

$$\begin{aligned}
\phi(\xi)\hat{}(\gamma) &= \int_{-\infty}^{\infty} (-\phi(t), \gamma) \, d\xi(t) \\
&= \int_{-\infty}^{\infty} \exp(-i\psi(\gamma)t) d\xi(t) \\
&= \hat{\xi}(\psi(\gamma))
\end{aligned}$$

$$= \exp(-|\psi(\gamma)|)$$
$$= \hat{\sigma}(\gamma)$$

for all $\gamma \in \hat{G}$. Thus, by lemma 3.3, we get $\sigma * \mu = \phi(\xi) * \mu \in M_s(G)$. □

The following lemma is due to [22].

LEMMA 3.5. (cf. [22, Lemma 1.2]). *Let G be a LCA group, and let P be an open semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let H be the annihilator of $P \cap (-P)$. Let μ be a measure in $M_s(G) \cap M_P(G)$. Then $\mu * m_H$ belongs to $M_s(G)$.*

The following lemma follows from [5, Theorem 1].

LEMMA 3.6. (cf. [21, Lemmas (B) and (C)]). *Let G be a LCA group and F an open subgroup of $\hat{G}$. Let H be the annhilator of F, and let $\alpha : G \to G/H$ be the natural homomorphism. Then the following hold.*

(I) Let μ be a measure in $M_s(G)$ such that $supp(\hat{\mu}) \subset F$. Then $\alpha(\mu)$ belongs to $M_s(G/H)$.

(II) Let ν be a measure in $M_s(G/H)$. Then there exists a measure $\mu \in M_s(G)$ such that $\hat{\mu}(\gamma) = \hat{\nu}(\gamma)$ on F and $\hat{\mu}(\gamma) = 0$ on $\hat{G} \backslash F$.

Now we prove Theorem 2.1. Put $\Lambda = P \cap (-P)$ and $G_\rho = \Gamma_\rho^\perp$. Let $\pi : \hat{G} \to \hat{G}/\Lambda$ be the natural homomorphism. Set $\tilde{P} = \pi(P)$ and $\tilde{\Gamma}_\rho = \pi(\Gamma_\rho)$. It follows from Lemma 3.2 that there exists an open semigroup $\tilde{Q}$ in $\hat{G}/\Lambda$ such that

(3.3) $\tilde{Q} \supset \tilde{P}$, and

(3.4) $\tilde{Q} \cap (-\tilde{Q}) = \tilde{\Gamma}_\rho$.

Put $Q = \pi^{-1}(\tilde{Q})$. Then Q is an open semigroup in $\hat{G}$ such that $Q \cup (-Q) = \hat{G}$. Moreover, since $\Gamma_\rho \supset \Lambda$, it follows from (3.3) and (3.4) that

(3.5) $Q \supset P$, and

(3.6) $Q \cap (-Q) = \Gamma_\rho$.

Since $\mu \in M_P(G) \cap M_s(G)$, we have $\mu \in M_Q(G) \cap M_s(G)$, by (3.5); hence (3.6) and Lemma 3.5 imply

(3.7) $\mu * m_{G_\rho} \in M_s(G)$.

Let $\pi_{G_\rho} : G \to G/G_\rho$ be the natural homomorphism. Since $(\mu * m_{G_\rho})\hat{}\,(\gamma) = 0$ on $\hat{G}\backslash\Gamma_\rho$, it follows from (3.7) and Lemma 3.6 that

(3.8) $\pi_{G_\rho}(\mu * m_{G_\rho}) \in M_s(G/G_\rho)$.

Define $\tilde{\rho} : \Gamma_\rho \to \mathbb{R}^+\backslash\{0\}$ by $\tilde{\rho}(\gamma_1 - \gamma_2) = \rho(\gamma_1)\,\rho(\gamma_2)^{-1}$ $(\gamma_1, \gamma_2 \in L_\rho^+)$, where $\mathbb{R}^+$ is the set of nonnegative real numbers. Then $\tilde{\rho}$ is continuous and $\tilde{\rho}(\gamma_1 + \gamma_2) = \tilde{\rho}(\gamma_1)\,\tilde{\rho}(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma_\rho$. Moreover, $\tilde{\rho}|_{P\cap\Gamma_\rho} = \rho|_{P\cap\Gamma_\rho}$. Now we define a continuous homomorhism $\psi : \Gamma_\rho \to \mathbb{R}$ by

(3.9) $\psi(\gamma) = -\log\tilde{\rho}(\gamma)$.

The fact that $\tilde{\rho}|_{P\cap\Gamma_\rho} = \rho|_{P\cap\Gamma_\rho}$ implies

(3.10) $\psi^{-1}([0, \infty)) \supset P \cap \Gamma_\rho$,

which combined with the fact that $\hat{\mu}(\gamma) = 0$ on P^c yields

(3.11) $\pi_{G_\rho}(\mu * m_{G_\rho})\hat{}\,(\gamma) = 0$ on $\psi^{-1}((-\infty, 0))$.

On the other hand, since m_ρ is a positive measure and $\hat{m}_\rho = \rho(\gamma)$ for $\gamma \in P$, we have

$$\hat{m}_\rho(\gamma) = \begin{cases} \rho(\gamma) & \text{for } \gamma \in P \\ \rho(-\gamma) & \text{for } \gamma \in (-P)\backslash P. \end{cases}$$

Hence we get, by (3.9),

(3.12) $\hat{m}_\rho(\gamma) = \exp(-|\psi(\gamma)|)$ for $\gamma \in \Gamma_\rho$.

In fact, for $\gamma \in P \cap \Gamma_\rho$, we have

$$\begin{aligned} \hat{m}_\rho(\gamma) &= \rho(\gamma) = \tilde{\rho}(\gamma) = \exp(\log\tilde{\rho}(\gamma)) \\ &= \exp(-|\log\tilde{\rho}(\gamma)|) = \exp(-|\psi(\gamma)|). \end{aligned}$$

Let $\gamma \in \{(-P)\backslash P\} \cap \Gamma_\rho$. Then, since $-\gamma \in P \cap \Gamma_\rho$, we have

$$\hat{m}_\rho(\gamma) = \rho(-\gamma) = \exp(-|\psi(-\gamma)|)$$

$$= \exp(-|\psi(\gamma)|).$$

Thus (3.12) holds.

By (3.8), (3.11)-(3.12) and Lemma 3.4, we have

$$\pi_{G_\rho}(\mu * m_{G_\rho}) * \pi_{G_\rho}(m_\rho) \in M_s(G/G_\rho). \tag{3.13}$$

Define a map $S_{G_\rho} : M(G/G_\rho) \to M(G)$ by

$$S_{G_\rho}(\nu)\hat{}(\gamma) = \begin{cases} \hat{\nu}(\gamma) & \text{for } \gamma \in \Gamma_\rho \\ 0 & \text{for } \gamma \in \hat{G} \backslash \Gamma_\rho. \end{cases}$$

It follows from (3.13) and Lemma 3.6 that

$$S_{G_\rho}(\pi_{G_\rho}(\mu * m_{G_\rho}) * \pi_{G_\rho}(m_\rho)) \in M_s(G). \tag{3.14}$$

On the other hand, since $supp(\hat{m}_\rho) \subset \Gamma_\rho$ and $\hat{m}_{G_\rho} = 1$ on Γ_ρ, we have

$$\begin{aligned} S_{G_\rho}(\pi_{G_\rho}(\mu * m_{G_\rho}) * \pi_{G_\rho}(m_\rho)) &= S_{G_\rho}(\pi_{G_\rho}(\mu * m_{G_\rho} * m_\rho)) \\ &= \mu * m_\rho. \end{aligned}$$

Hence we have $\mu * m_\rho \in M_s(G)$, by (3.14), and the proof is complete.

References

[1] R. Arens and I. M. Singer, Generalized analytic functions, Trans. Amer. Math. Soc. **81** (1956), 379-393.

[2] C. Berg, J.P.R. Christensen and P. Ressel, Harmonic Analysis on Semigroups, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1982.

[3] S. Bochner, Boundary values of analytic functions in several variables and of almost periodic functions, Ann. of Math. **45** (1944), 708-722.

[4] K. DeLeeuw and I. Glicksberg, Quasi-invariance and analyticity of measures on compact groups, Acta Math. **109** (1963), 179-205.

[5] R. Doss, On the transform of a singular or absolutely continuous measure, Proc. Amer. Math. Soc. **19** (1968), 361-363.

[6] R. Doss, On measures with small transforms, Pacific J. Math. **26** (1968), 257-263.

[7] F. Forelli, Analytic measures, Pacific J. Math. **13** (1963), 571-578.

[8] F. Forelli, Analytic and quasi-invariant measures, Acta Math. **118** (1967), 33-57.

[9] T. Gamelin, Uniform Algebras, Printice-Hall, Englewood Cliffs, N.J., 1969.

[10] I. Glicksberg, The strong conclusion of the F. and M. Riesz theorem on groups, Trans. Amer. Math. Soc. **285** (1984), 235-240.

[11] H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, Acta Math. **99** (1958), 165-202.

[12] E. Hewitt and S. Koshi, Ordering in locally compact Abelian groups and the theorem of F. and M. Riesz, Math. Proc. Camb. Phil. Soc. **93** (1983), 441-457.
[13] E. Hewitt, S. Koshi and Y. Takahashi, The F. and M. Riesz theorem revisited, Math. Scand. **60** (1987), 63-76.
[14] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. I and II, Springer-Verlag, New York-Heidelberg-Berlin, 1963 and 1970.
[15] K. Hoffman, Boundary behavior of generalized analytic functions, Trans. Amer. Math. Soc. **87** (1958), 447-466.
[16] G.M. Leibowitz, Lectures on Complex Function Algebras, Scott, Foresman and Company, Glenview, 1970.
[17] W. Rudin, Fourier Analysis on Groups, New York, Interscience, 1962.
[18] S. Saeki, The F. and M. Riesz theorem and singular measures, Proc. Amer. Math. Soc. **90** (1984), 391-396.
[19] Y. Takahashi, On a theorem of S. Saeki, Hokkaido Math. J. **15** (1986), 157-161.
[20] T.V. Tonev, Big-Planes, Boundaries and Function Algebras, North-Holland, Amsterdam-London-New York-Tokyo, 1992.
[21] H. Yamaguchi, Some multipliers on the space consisting of measures of analytic type, Hokkaido. Math. J. **11** (1982), 173-200.
[22] H. Yamaguchi, Some multipliers on the space consisting of measures of analytic type, II, Hokkaido. Math. J. **12** (1983), 244-255.
[23] H. Yamaguchi, A property of some Fourier-Stieltjes transforms, Pacific J. Math. **108** (1983), 243-256.
[24] H. Yamaguchi, On a result of Saeki-Takahashi and a theorem of Bochner, Hokkaido Math. J. **38** (2009), 497-510.
[25] H. Yamaguchi, Quasi-invariance of measures of analytic type on locally compact abelian groups, Hokkaido Math. J. **43** (2014), 51-64.

Hiroshi YAMAGUCHI
Department of Mathematics, Faculty of Science, Josai University
Keyakidai 1-1, Sakado, Sakado, 350-0295, Japan
E-mail: hyama@josai.ac.jp