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A survey of a property of random walks on a cycle graph

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Abstract. This is a survey of a property of random walks on a cycle graph. We explain the Hunter vs. Rabbit game using the random walk framework. We consider a probability that the hunter catches the rabbit. By assuming the behavior of the characteristic function of rabbit's random walk near zero, we obtain a upper and lower bound of this probability.

1. Introduction

This is a survey of a property of random walks on a cycle graph and is based on [2].

We consider the Hunter vs. Rabbit game. A graph is given. This game is played by two players: the hunter and the rabbit. Each player occupies a vertex of the graph. At each unit time, the hunter can move an adjacent vertex or stay, and the rabbit can move a vertex of the graph. The hunter catches the rabbit when both of them occupy the same vertex at same time.

The Hunter vs. Rabbit game is a model of ad hoc network. Ad hoc network is the following. We can use a cellular phone, if there is a base station in the neighborhood, and a base station receives radio waves from a cellular phone. By this mechanism, it is necessary to install many base stations. To install many stations, it takes huge time and a lot of money. The idea of ad hoc network is to use a cellular phone or a mobile computer as the role of the base station. The place a radio wave occurs can move. An installed base station can not move. In the case of ad hoc network, a base station can move.

We consider the following problem. When a big accident happened and many basic stations broke down, how should a basic station car be operated?

We explain the setting of the hunter vs. rabbit game. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in the integer lattice $\mathbb{Z}$. A one-dimensional random walk

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$\{S_n\}_{n=1}^{\infty}$ is defined by

$$S_n = \sum_{j=1}^{n} X_j.$$

Let $Y_1, Y_2, \cdots$ be independent, identically distributed random variables defined on a probability space $(\Omega_H, \mathcal{F}_H, P_H)$ taking values in the integer lattice $\mathbb{Z}$ with

$$P_H(\{|Y_1| \le 1\}) = 1.$$

Let $N \in \mathbb{N}$ be fixed. We consider a cycle graph with N vertices as given graph. We denote by $X_0^{(N)}$ a random variable defined on a probability space $(\Omega_N, \mathcal{F}_N, \mu_N)$ taking values in $V_N := \{0, 1, 2, \cdots, N-1\}$ with

$$\mu_N(\{X_0^{(N)} = l\}) = \frac{1}{N} \quad (l \in V_N).$$

For $b \in \mathbb{Z}$, we denote by $(b \bmod N)$ the remainder of b divided by N.

A rabbit's strategy $\{R_n^{(N)}\}_{n=0}^{\infty}$ is defined by

$$R_0^{(N)} = X_0^{(N)} \text{ and } R_n^{(N)} = \left(X_0^{(N)} + S_n \bmod N\right).$$

$R_n^{(N)}$ indicates the position of the rabbit at time n on V_N. Hunter's strategy $\{H_n^{(N)}\}_{n=0}^{\infty}$ is defined by

$$H_0^{(N)} = 0 \text{ and } H_n^{(N)} = \left(\sum_{j=1}^{n} Y_j \mod N\right).$$

$H_n^{(N)}$ indicates the position of the hunter at time n on V_N. Put

$$P_R^{(N)} = \mu_N \times P \quad \text{and} \quad \tilde{P}^{(N)} = P_H \times P_R^{(N)}.$$

We discuss the probability that the hunter catches the rabbit by time N on V_N, that is,

$$\tilde{P}^{(N)}\left(\bigcup_{n=1}^{N} \{H_n^{(N)} = R_n^{(N)}\}\right).$$

We investigate the asymptotic estimate of this probability as $N \to \infty$.

2. Statements of results

DEFINITION 2.1. *We define conditions (A1),(A2) and (A3) as follows*

(A1) The random walk $\{S_n\}_{n=1}^{\infty}$ *is strongly aperiodic, i.e. for each* $y \in \mathbb{Z}$*, the smallest subgroup containing the set*

$$\{y + k \in \mathbb{Z} \mid P\{X_1 = k\} > 0\}$$

is $\mathbb{Z}$.

(A2) $P\{X_1 = k\} = P\{X_1 = -k\} \quad (k \in \mathbb{Z})$.

(A3) There exist $\beta \in (0, 2], c_* > 0$ *and* $\varepsilon > 0$ *such that*

$$\phi(\theta) := \sum_{k \in \mathbb{Z}} e^{i\theta k} P\{X_1 = k\} = 1 - c_* |\theta|^{\beta} + O(|\theta|^{\beta+\varepsilon}).$$

The next remark indicates a relation between the condition $(A3)$ (the behavior of the characteristic function ϕ near zero) and the tail of distribution of X_1.

Remark 2.2. *For* $\beta \in (0, 2)$*, let*

$$P\{X_1 = k\} = \begin{cases} \frac{1}{2a|k|^{\beta+1}} & (k \in \mathbb{Z} \setminus \{0\}) \\ 1 - \frac{1}{a} \sum_{k=1}^{\infty} \frac{1}{k^{\beta+1}} & (k = 0) \end{cases}$$

with a constant a *satisfying* $a > \sum_{k=1}^{\infty} (1/k^{\beta+1})$. *Then* $\phi(\theta)$ *in* $(A3)$ *is*

$$\phi(\theta) = 1 - \frac{\pi}{2a} \frac{|\theta|^{\beta}}{\Gamma(\beta + 1) \sin(\beta\pi/2)} + O\left(|\theta|^{\beta + (2-\beta)/2}\right),$$

where Γ *is the gamma function.*

If X_1 *satisfies*

$$E[X_1] = 0, \ E[|X_1|^{2+\varepsilon}] < \infty \text{ for some } \varepsilon \in (0, 1),$$

then

$$\phi(\theta) = 1 - \frac{1}{2} E[X_1^2] |\theta|^2 + O(|\theta|^{2+\varepsilon}).$$

THEOREM 2.3. *Assume that* X_1 *satisfies (A1) – (A3).*

If $\beta \in (0, 1)$*, then there exists a constant* $c_1 > 0$ *such that for* $N \in \mathbb{N} \setminus \{1\}$ *and*

$y_1, y_2, \cdots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \le 1$ $(n = 1, 2, \cdots, N-1)$,

$$c_1 \le P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)} = (y_n \mod N)\right\}\right).$$

If $\beta = 1$, then there exist constants $c_2 > 0$ and $c_3 > 0$ such that for $N \in \mathbb{N}\setminus\{1\}$ and $y_1, y_2, \cdots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \le 1$ $(n = 1, 2, \cdots, N-1)$,

$$\frac{c_2}{\log N} \le P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)} = (y_n \mod N)\right\}\right) \le \frac{c_3}{\log N}.$$

If $\beta \in (1, 2]$, then there exists a constant $c_4 > 0$ such that for $N \in \mathbb{N}\setminus\{1\}$ and $y_1, y_2, \cdots, y_N \in \mathbb{Z}$ with $|y_n - y_{n+1}| \le 1$ $(n = 1, 2, \cdots, N-1)$,

$$\frac{c_4}{N^{(\beta-1)/\beta}} \le P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)} = (y_n \mod N)\right\}\right) \le 1. \tag{1}$$

The following bounds are obtained as a corollary of Theorem 2.3.

COROLLARY 2.4. *Assume that X_1 satisfies (A1) – (A3).*

(I) If $\beta \in (0, 1)$, then there exists a constant $c_1 > 0$ such that for $N \in \mathbb{N}\setminus\{1\}$,

$$c_1 \le \tilde{P}^{(N)}\left(\bigcup_{n=1}^{N}\left\{H_n^{(N)} = R_n^{(N)}\right\}\right) \le 1.$$

(II) If $\beta = 1$, then there exist constants $c_2 > 0$ and $c_3 > 0$ such that for $N \in \mathbb{N}\setminus\{1\}$,

$$\frac{c_2}{\log N} \le \tilde{P}^{(N)}\left(\bigcup_{n=1}^{N}\left\{H_n^{(N)} = R_n^{(N)}\right\}\right) \le \frac{c_3}{\log N}. \tag{2}$$

(III) If $\beta \in (1, 2]$, then there exists a constant $c_4 > 0$ such that for $N \in \mathbb{N}\setminus\{1\}$,

$$\frac{c_4}{N^{(\beta-1)/\beta}} \le \tilde{P}^{(N)}\left(\bigcup_{n=1}^{N}\left\{H_n^{(N)} = R_n^{(N)}\right\}\right) \le 1.$$

Remark 2.5. *Adler et al. considered $\tilde{P}^{(N)}\left(\bigcup_{n=1}^{N}\left\{H_n^{(N)} = R_n^{(N)}\right\}\right)$ in the case* *of*

$$P\{X_1 = k\} = \begin{cases} \frac{1}{2(|k|+1)(|k|+2)} & (k \in \mathbb{Z} \setminus \{0\}) \\ \frac{1}{2} & (k = 0). \end{cases}$$

In this case, X_1 *satisfies* $(A1), (A2)$ *and*

$$\phi(\theta) = 1 - \frac{\pi}{2}|\theta| + 0(|\theta|^{3/2})$$

((A3) with $\beta = 1$*), and we have (2) in Corollary 2.4 which coincides with the result of Lemma 3 in [1].*

The inequality (1) seems to be sharp, because the powers of upper and lower bound appearing in (1) can not be improved. Indeed, we have the following estimates.

Proposition 2.6. *Assume that* X_1 *satisfies (A1) – (A3).*

If $\beta \in (1, 2]$*, then there exist constants* $c_5, c_6, c_7 > 0$ *such that for* $N \in \mathbb{N} \setminus \{1\}$*,*

$$\frac{c_5}{N^{(\beta-1)/\beta}} \leq P_R^{(N)} \left(\bigcup_{n=1}^{N} \left\{ R_n^{(N)} = 0 \right\} \right) \leq \frac{c_6}{N^{(\beta-1)/\beta}}, \tag{3}$$

$$c_7 \leq P_R^{(N)} \left(\bigcup_{n=1}^{N} \left\{ R_n^{(N)} = (n \mod N) \right\} \right). \tag{4}$$

3. Proof of Proposition 2.6(4)

In this section we prove Proposition 2.6(4) and the other proofs are given in [2, Theorem 1, Corollary 1 and Proposition 1(6)].

To prove Proposition 2.6(4), we introduce the following Lemma.

Lemma 3.1. *Assume* $(A1), (A2)$ *and* $(A3)$*. If* $\beta \in (1, 2]$*, then there exists a constant* $c_8 > 0$ *such that*

$$1 + \sum_{l=1}^{N-1} P(\{S_l \in [l]_N\}) \leq c_8, \tag{5}$$

where

$$[y]_N = \{y + kN \mid k \in \mathbb{Z}\}.$$

Proof. There exist C_* and $r \in (0, \pi/2)$ such that for $|\theta| < r$,

$$|\phi(\theta) - (1 - c_*|\theta|^\beta)| \leq C_*|\theta|^{\beta+\varepsilon}$$

by $(A3)$. We can choose $r_* \in (0, r]$ small enough so that

$$C_* r_*^\varepsilon \leq \frac{1}{2}c_* \quad \text{and} \quad c_* r_*^\beta \leq \frac{1}{3}.$$

Then for $|\theta| \leq r_*$,

$$|1 - \phi(\theta)| \leq \frac{3}{2}c_*|\theta|^\beta \leq \frac{1}{2}. \tag{6}$$

A strongly aperiodic random walk $(A1)$ has the property that $|\phi(\theta)| = 1$ only when θ is a multiple of 2π (see §7 Proposition 8 of [3]). By the definition of $\phi(\theta)$, $|\phi(\theta)|$ is a continuous function on the bounded closed set $[-\pi, -r_*] \cup [r_*, \pi]$, and $|\phi(\theta)| \leq 1$ ($\theta \in [-\pi, \pi]$). Hence, there exists a $\rho_* < 1$, depending on $r_* \in (0, \pi]$, such that

$$\max_{r_* \leq |\theta| \leq \pi} |\phi(\theta)| \leq \rho_*. \tag{7}$$

To show (5), we use the following relation (which is given in Proposition 5 of [2]): for $l \in \{1, 2, \cdots, N-1\}$,

$$P(\{S_l \in [l]_N\}) = \frac{1}{N} + \frac{2}{N} \sum_{1 \leq j \leq (N-1)/2} \phi^l\left(\frac{2j\pi}{N}\right) \cos\left(\frac{2j\pi}{N} l\right) + J_N(l, l),$$

where

$$J_N(l, l) = \begin{cases} (1/N)\phi^l(\pi)\cos(\pi l) & \text{(if } N \text{ is even)} \\ 0 & \text{(if } N \text{ is odd).} \end{cases}$$

Form this relation and (7), we obtain that for $l \in \{1, 2, \cdots, N-1\}$,

$$P(\{S_l \in [l]_N\}) \leq \frac{1}{N} + \frac{2}{N} \sum_{1 \leq j \leq (r_*/(2\pi))N} \phi^l\left(\frac{2j\pi}{N}\right) \cos\left(\frac{2j\pi}{N} l\right) + \rho_*^l.$$

Therefore

$$1+\sum_{l=1}^{N-1} P(\{S_l \in [l]_N\}) \le 1+\frac{2}{N}\sum_{1\le j\le (r_*/(2\pi))N} |\Phi(j,N)| + \frac{1}{1-\rho_*}, \tag{8}$$

where

$$\Phi(j,N)=\sum_{l=0}^{N-1}\phi^l\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi}{N}l\right).$$

Because of $(A2)$, $\phi(\theta)$ takes a real number. Then (6) and $(A1)$ mean that

$$\frac{1}{2}\le \phi(\theta)=|\phi(\theta)|<1 \quad (\theta\in(-r_*,0)\cup(0,r_*)) \tag{9}$$

and for $j\in[1,(r_*/(2\pi))N]\cap\mathbb{N}$,

$$\Phi(j,N)=\Re\left[\sum_{l=0}^{N-1}\phi^l\left(\frac{2j\pi}{N}\right)(e^{-2j\pi i/N})^l\right]$$

$$=\frac{\left(1-\phi^N\left(\frac{2j\pi}{N}\right)\right)\left(1-\phi\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi}{N}\right)\right)}{\left|1-\phi\left(\frac{2j\pi}{N}\right)e^{-2j\pi i/N}\right|^2}.$$

To estimate $|\Phi(j,N)|$, we use the last inequality of (9) and (6) which imply that for $j\in[1,(r_*/(2\pi))N]\cap\mathbb{N}$,

$$\left|\left(1-\phi^N\left(\frac{2j\pi}{N}\right)\right)\left(1-\phi\left(\frac{2j\pi}{N}\right)\cos\left(\frac{2j\pi}{N}\right)\right)\right|$$

$$\le 2\left(\left|1-\phi\left(\frac{2j\pi}{N}\right)\right|+\left|1-\cos\left(\frac{2j\pi}{N}\right)\right|\right)\le c_9\left(\frac{j}{N}\right)^{\beta},$$

where $c_9=(2\pi)^\beta(3c_*+r_*^{2-\beta})$. Here we note that $\beta\le 2$.

The inequality $\sin\theta\ge 2\theta/\pi$ $(\theta\in[0,\pi/2])$ and the first inequality of (9) show that for $j\in[1,(r_*/(2\pi))N]\cap\mathbb{N}$,

$$\left|1-\phi\left(\frac{2j\pi}{N}\right)e^{-2j\pi i/N}\right|^2\ge\left(\phi\left(\frac{2j\pi}{N}\right)\sin\left(\frac{2j\pi}{N}\right)\right)^2\ge 4\left(\frac{j}{N}\right)^2.$$

Thus

$$\frac{2}{N}\sum_{1\le j\le (r_*/(2\pi))N}|\Phi(j,N)| \le \frac{c_9}{2N^{\beta-1}}\sum_{1\le j\le (r_*/(2\pi))N} j^{\beta-2}. \tag{10}$$

By noticing that $\beta\in(1,2]$, it is easy to see that

(11)

$$\sum_{1\le j\le (r_*/(2\pi))N} j^{\beta-2} \le 1+\int_1^{(r_*/(2\pi))N} x^{\beta-2}\,dx \le \left(1+\frac{1}{\beta-1}\left(\frac{r_*}{2\pi}\right)^{\beta-1}\right)N^{\beta-1}.$$

Put the pieces ((8),(10),(11)) together, we have (5). □

To complete the proof of Proposition 2.6(4), we use the following inequality (which is given in Corollary 3 of [2]): for $N\in\mathbb{N}\setminus\{1\}$,

$$\frac{1}{1+\sum_{l=1}^{N-1}P(\{S_l\in[l]_N\})} \le P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)}=(n \mod N)\right\}\right).$$

By combining the above inequality with (5), we obtain Proposition 2.6(4).

4. Conclusion

We notice that the upper and lower bound which is appearing in Corollary 2.4 $(I)((II))$ have the same asymptotic behavior. Corollary 2.4 and Proposition 2.6(4) imply that for every $\beta\in(0,2]$, the asymptotic behavior of $P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)}=(n \mod N)\right\}\right)$ is same as that of the upper bound appearing in Corollary 2.4. This means that $H_n^{(N)}=n$ $(n=1,2,\cdots,N)$ is a better strategy of the hunter. Corollary 2.4 and Proposition 2.6(3) imply that for every $\beta\in(0,2]$, the asymptotic behavior of $P_R^{(N)}\left(\bigcup_{n=1}^{N}\left\{R_n^{(N)}=0\right\}\right)$ is same as that of the lower bound appearing in Corollary 2.4. This means that $H_n^{(N)}=0$ $(n=1,2,\cdots,N)$ is a worse strategy of the hunter.

We apply our result to practical use of a base station car. There is a area where towns are on a cycle. We assume that a big accident happened and many basic stations broke down in this area. It is a better use to make the base station car go around by uniform velocity along the cycle, and it is a worse use to park the base station car at a starting point.

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