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Operating functions in harmonic analysis

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Abstract. In this paper, we state about the Katznelson Theorem with respect to the operating functions of $A(\mathbb{T})$ and about the operating functions of modulation spaces in harmonic analysis.

1. Introduction

Study of operating function in harmonic analysis starts with N.Wiener. Wiener[17] studies the class of $A(\mathbb{T})$ of all continuous functions on the unit circle $\mathbb{T}$ with the absolutely convergent Fourier series, and shows $\frac{1}{f} \in A(\mathbb{T})$ for every $f \in A(\mathbb{T})$ with $f(x) \neq 0$ for all $x \in \mathbb{T}$. This means that the composition $F \circ f$ is in $A(\mathbb{T})$ when $F(z) = \frac{1}{z}$ and $f \in A(\mathbb{T})$ with $f(x) \neq 0$ for all $x \in \mathbb{T}$. Lèvy[12] gives an extension of this result which is called Wiener-Lèvy Theorem. After that, the converse of the Wiener-Lèvy Theorem is studied, and Katznelson[10] gives a result of it. Moreover, Helson-Kahane-Katznelson-Rudin[6] reveals the functions on $\mathbb{R}^2$ which operate on the Fourier transforms on non discrete locally compact abelian groups. In §2, we briefly state those results.

Thereafter, there are many papers about operating functions on some function spaces related to Fourier series or Fourier transforms([7],[8],[13],[18],[19], etc.).

In 1983, Feichtinger[4] first introduced the modulation spaces $M^{p,q}$ whose importance are indicated by the results related to Schrödinger probagator $e^{-it\Delta}$(cf.[2],[5], etc.).

Recently, Ruzhansky-Sugimoto-Wang[14] proposes the open problem with respect to general power type nonlinearity of the form $|u|^{\alpha}u$. Bhimani-Ratnakumar[3] gives a negative answer about the problem, and proposes an open problem related the problem in [3]. In §3, we state about this result and give an answer according to Kobayashi-Sato[11].

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2. The Katznelson Theorem

In this section, we briefly review some classical results with respect to operating functions. First we state the Wiener Theorem and the Wiener-Lèvy Theorem.

THEOREM 2.1. *(The Wiener Theorem) Let f be in $A(\mathbb{T})$ such that $f(x) \neq 0$ for all $x \in \mathbb{T}$. Then we hav $\frac{1}{f} \in A(\mathbb{T})$.*

THEOREM 2.2. *(The Wiener-Lèvy Theorem) Let D be a region in the complex plane, and $F(z)$ an analytic function on D. Then we have $F \circ f \in A(\mathbb{T})$ for every $f \in A(\mathbb{T})$ with $f(\mathbb{T}) \subset D$.*

After the Wiener-Lèvy Theorem is showed, there are many papers related to the converse of this theorem. Katznelson[10](cf.[9]) gives a solution of the converse problem which is called the Katznelson Theorem:

THEOREM 2.3. *(The Katznelson Theorem) Let $I = [-1, 1]$, and $F(t)$ be a complex-valued function on I. If F operates on $A(\mathbb{T})$, F extends to an analytic function on a neighborhood of I (We say that F operates on $A(\mathbb{T})$, if $F \circ f \in A(\mathbb{T})$ for every $f \in A(\mathbb{T})$ with $f(\mathbb{T}) \subset I$).*

A complex-valued function F on $\mathbb{R}^2$ is said to be real analytic (resp. real entire) on $\mathbb{R}^2$ if for each $(s_0, t_0) \in \mathbb{R}^2$, F has a power series expansion

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n \quad (resp.\ F(s,t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n)$$

which converges absolutely in a neighborhood of (s_0, t_0)(resp. in $\mathbb{R}^2$).

THEOREM 2.4. *(The Helson-Kahane-Katznelson-Rudin Theorem) Let Φ be a complex-valued function on $\mathbb{R}^2$. If $\Phi \circ f \in A(\mathbb{T})$ for every $f \in A(\mathbb{T})$, then Φ is real analytic on $\mathbb{R}^2$.*

Remark 2.5. *Let G be a non discrete locally compact abelian group with the dual $\widehat{G}$, and $A(G)$ the set of all Fourier transforms on $\widehat{G}$.*

(1) Let F be a function on $\mathbb{R}^2$. If we have $F(Re\ f, Im\ f) \in A(G)$ for every $f \in A(G)$, F is real analytic (cf.[13]).

*(2) Let $M(\widehat{G})$ be the set of all bounded regular Borel measures on $\widehat{G}$, and $B(G)$ the set of all Fourier-Stieltjes transforms on $\widehat{G}$. If we have $F(Re\ f, Im\ f) \in B(G)$ for every $f \in B(G)$, F is real entire (*cf.[13]*).*

3. Modulation spaces and the operating functions

First we introduce some notations for modualtion spaces. We write $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely partial differentiable functions on the d-dimensional Euclidean space $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ to denote the space of tempered distributions on $\mathbb{R}^d$, that is, the topological dual of $\mathcal{S}(\mathbb{R}^d)$. The Fourier transform of f is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$ and the inverse Fourier transform by $\check{f}(x) = \frac{1}{(2\pi)^d}\hat{f}(-x)$. We also write $C_c^\infty(\mathbb{R}^d)$ to denote the set of all complex-valued infinitely partial differentiable functions on $\mathbb{R}^d$ with compact support. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Then the short-time Fourier transform V_gf of f with respect to the window g is defined by

$$\begin{aligned} V_gf(x,\xi) &= < f(t), g(t-x)e^{it\xi} > \\ &= \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-it\xi}dt. \end{aligned}$$

Also let $1 \leq p, q \leq \infty$ and $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$. Then the modulation space $M^{p,q}(\mathbb{R}^d) = M^{p,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$||f||_{M^{p,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,\xi)|^p dx\right)^{\frac{q}{p}} d\xi\right)^{\frac{1}{q}}$$

is finite(with usual modification if $p = \infty$ or $q = \infty$). We note that since $V_g\bar{f}(x,\xi) = \overline{V_{\bar{g}}f(x,-\xi)}$, we have $||\bar{f}||_{M^{p,q}} = ||f||_{M^{p,q}}$, $||Re\ f||_{M^{p,q}} \leq ||f||_{Mp,q}$, $||Im\ f||_{M^{p,q}} \leq ||f||_{Mp,q}$. We collect basic properties of modulation spaces in the following lemma(cf.[5],[15],[16], etc.).

Lemma 3.1. *(1) The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space, whose definition is independent of the choice g. More precisely, we have*

$$||f||_{M^{p,q}_{[g_0]}(\mathbb{R}^d)} \leq C||g||_{M^{1,1}_{[g_0]}(\mathbb{R}^d)}||f||_{M^{p,q}_{[g]}(\mathbb{R}^d)}$$

for $f \in M^{p,q}(\mathbb{R}^d)$ and $g_0, g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$, where

$$||f||_{M^{p,q}_{[g]}(\mathbb{R}^d)} = ||||V_gf(x,\xi)||_{L^p(\mathbb{R}^d_x)}||_{L^q(\mathbb{R}^d_\xi)}.$$

(2)

$$M^{p,\min\{p,p'\}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,\max\{p,p'\}}(\mathbb{R}^d)$$

In particular, we have $M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$.

(3) $M^{p,1}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$, *that is,* f *is continuous on* $\mathbb{R}^d$ *if* $f \in M^{p,1}(\mathbb{R}^d)$.

(4) *If* $p_1 \le p_2$ *and* $q_1 \le q_2$, *then*

$$M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d).$$

(5) *(density and duality) If* $p, q < \infty$, *then* $\mathcal{S}(\mathbb{R}^d)$ *is dense in* $M^{p,q}(\mathbb{R}^d)$ *and* $(M^{p,q}(\mathbb{R}^d))' = M^{p',q'}(\mathbb{R}^d)$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$.

(6) *(Multiplication) If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} + 1$, *then*

$$||fg||_{M^{p,q}(\mathbb{R}^d)} \le C||f||_{M^{p_1,q_1}(\mathbb{R}^d)}||g||_{M^{p_2,q_2}(\mathbb{R}^d)} \quad (f, g \in \mathcal{S}(\mathbb{R}^d)).$$

(7) *(Dilation property) There exists a constant* $C > 0$ *such that*

$$||f_\lambda||_{M^{\infty,1}(\mathbb{R}^d)} \le C||f||_{M^{\infty,1}(\mathbb{R}^d)} \quad (f \in M^{\infty,1}(\mathbb{R}^d))$$

for $0 < \lambda \le 1$. *Here we denote* $f_\lambda(x) = f(\lambda x)$.

Let F be complex-valued function on $\mathbb{R}^2$. If $F(Re\ f, Im\ f) \in M^{p,1}(\mathbb{R}^d)$ for every $f \in M^{p,1}(\mathbb{R}^d)$, then we say that F operates on $M^{p,1}(\mathbb{R}^d)$.

M.Ruzhansky, M.Sugimoto and B.Wang pose the open problem in [14], namely the validity of the inequality $||f|f|^\alpha||_{M^{p,1}}^{\alpha+1}$ for all $f \in M^{p,1}(\mathbb{R}^d)$ and $\alpha \in (0,\infty)\backslash 2\,\mathbb{N}$. We have this inequality for $\alpha \in 2\,\mathbb{N}$ by $\bar{f} \in M^{p,1}(\mathbb{R}^d)$.

Bhimani-Ratnakumar[3] gives a negative answer to this problem by using the Helson-Kahane-Katznelson-Rudin Theorem with respect to operating functions in harmonic analysis:

Theorem 3.2. *([3]) Let* $1 \le p < \infty$ *and* F *be a complex-valued function on* $\mathbb{R}^2$. *If* F *operates on* $M^{p,1}(\mathbb{R}^d)$, *then* F *is a real analytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$. *Conversely, if* F *is a real anlytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$, *then* F *operates on* $M^{p,1}(\mathbb{R}^d)$.

Since $F(x, y) = (x + iy)|x + iy|^\alpha$ is non real analytic for $\alpha \in (0,\infty)\backslash 2\,\mathbb{N}$, Bhimani-Ratnakumar[3] obtains the following:

Corollary 3.3. *There exists* $f \in M^{p,1}(\mathbb{R}^d)$ *such that* $f|f|^\alpha \notin M^{p,1}(\mathbb{R}^d)$ *for every* $\alpha \in (0,\infty)\backslash 2\,\mathbb{N}$.

In [3], they propose an open problem: Is the condition Theorem 3.2 sufficient or not for $p > 1$? Kobayashi-Sato[11] gives an affirmative answer to this problem:

THEOREM 3.4. *([11, Theorem 1.1]) Let $1 < p < \infty$ and F be a real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. Then F operates on $M^{p,1}(\mathbb{R})$.*

In the same way as the above proof, we have the following:

THEOREM 3.5. *Let $1 < p < \infty$ and F be a real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. Then F operates on $M^{p,1}(\mathbb{R}^d)$.*

COROLLARY 3.6. *Let $1 \leq p < \infty$ and F be a complex-valued function on $\mathbb{R}^2$. Then F operates on $M^{p,1}(\mathbb{R}^d)$ if and only if F is a real analytic function with $F(0) = 0$.*

Theorem 3.5 is given by the slight modification of the proof in Kobayashi-Sato[11] as we said. In the remaining part, we state the outline of the proof of Theorem 3.5 which is almost same in the proof of Theorem 3.4.

DEFINITION 3.7. *Let $1 \leq p < \infty$ and f be a function on $\mathbb{R}^d$.*

(1) Let $x_0 \in \mathbb{R}^d$. If there exist a neighborhood V of x_0 and a function $g \in M^{p,1}(\mathbb{R}^d)$ satisfying $f(x) = g(x)$ for every $x \in V$, then we say f belongs to $M^{p,1}(\mathbb{R}^d)$ locally at a point $x_0 \in \mathbb{R}^d$.

(2) If there exist a compact set $K \subset \mathbb{R}^d$ and $h \in M^{p,1}(\mathbb{R}^d)$ satisfying $f(x) = g(x)$ for all $x \in \mathbb{R}^d \setminus K$, then we say f belongs to $M^{p,1}(\mathbb{R}^d)$ at ∞.

We denote by $M^{p,1}_{loc}(\mathbb{R}^d)$, the space of functions that are locally in $M^{p,1}(\mathbb{R}^d)$ at each point $x_0 \in \mathbb{R}^d$.

LEMMA 3.8. *([11, Lemma 2.4]) Let $1 \leq p < \infty$ and f be a function on $\mathbb{R}^d$.*

(1) f belongs to $M^{p,1}_{loc}(\mathbb{R}^d)$, if and only if $\phi f \in M^{p,1}(\mathbb{R}^d)$ for every $\phi \in C^\infty_c(\mathbb{R}^d)$.

(2) f belongs to $M^{p,1}_{loc}(\mathbb{R}^d)$ at ∞, if and only if there exists a function $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $(1 - \phi)f \in M^{p,1}(\mathbb{R}^d)$.

(3) If $f \in M^{p,1}_{loc}(\mathbb{R}^d)$ and f belongs to $M^{p,1}(\mathbb{R}^d)$ at ∞, then $f \in M^{p,1}(\mathbb{R}^d)$.

For the sake of the outline of the proof of Theorem 3.5, we use some notations in Fourier analysis. Let $C(\mathbb{T}^d)$ be the set of all continuous functions on the d-dimensional torus $\mathbb{T}^d$, and $A(\mathbb{T}^d)$ the set of all continuous functions having absolutely convergent Fourier series :

$$A(\mathbb{T}^d) = \{f \in C(\mathbb{T}^d) | \sum_{m \in \mathbb{Z}^d} |\hat{f}(m)| < \infty\},$$

where $\hat{f}(m) = \int_{[-\pi,\pi)^d} f(x)e^{-imx}dx$, the m-th Fourier coefficient of f. $A(\mathbb{T}^d)$ is a commutative Banach algebra under pointwise addition and multiplication with respect to the norm

$$||f||_{A(\mathbb{T}^d)} = \sum_{m \in \mathbb{Z}^d} |\hat{f}(m)|.$$

Here, we need the useful lemma for our theorem in Benyi-Oh[1].

Lemma 3.9. *(cf.[1,Proposition B.1]) Let $f \in M^{p,1}(\mathbb{R}^d)$, $1 \le p < \infty$ and ϕ a smooth function supported on $[0, 2\pi)^d$. Then $\phi f \in A(\mathbb{T}^d)$ and satisfies the inequality*

$$(1) \quad ||\phi f||_{A(\mathbb{T}^d)} \le C||f||_{M^{p,1}(\mathbb{R}^d)}.$$

Also let $f \in A(\mathbb{T}^d)$, $1 \le p < \infty$ and ϕ the above function. Then $\phi f \in M^{p,1}(\mathbb{R}^d)$ and satisfies the inequality

$$(2) \quad ||\phi f||_{M^{p,1}(\mathbb{R}^d)} \le C||f||_{A(\mathbb{T}^d)}.$$

Also we have the following:

Lemma 3.10. *Let $\lambda \in (0, 1)$ and define the 2π periodic function $V_\lambda \in C(\mathbb{T}^d)$ by*

$$V_\lambda(x) = V_\lambda^1(x_1) \cdots V_\lambda^d(x_d) \quad (x = (x_1, \cdots, x_d) \in [-\pi, \pi)^d),$$

where $V_\lambda^j(x_j) = 2\Delta_{2\lambda}(x_j) - \Delta_\lambda(x_j)$ with $\Delta_\lambda(x_j) = \max(0, 1 - \frac{|x_j|}{\lambda})$, $(j = 1, \cdots, d)$. Moreover, we define $V_\lambda^{x_0}(x) = V_\lambda(x - x_0)$ for $x_0 \in \mathbb{R}^d$. Then for each $g \in A(\mathbb{T}^d)$ with $g(x_0) = 0$, we have $||V_\lambda^{x_0} g||_{A(\mathbb{T}^d)} \to 0$ as $\lambda \to 0$.

This Lemma is obtained by applying a partial integration and the Parseval equality. But we omit the proof, since the proof is given by the slight modification in Kahane[9;pp.56-57].

By Lemma 3.10, we can show the following whose proof is same to [11].

PROPOSITION 3.11. *(cf.[11,Proposition 3.1]) Let* $1 \le p < \infty$ *and* F *be a complex-valued real analytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$*. If* $f \in M^{p,1}(\mathbb{R}^d)$*, then* $F(Re\ f, Im\ f)$ *belongs to* $M^{p,1}(\mathbb{R}^d)$ *locally at* x_0 *for all* $x_0 \in \mathbb{R}^d$*.*

Also the following is given:

PROPOSITION 3.12. *(cf.[11,Proposition 3.2]) Let* $1 \le p < \infty$ *and* $f \in M^{p,1}(\mathbb{R}^d)$*. For any* $\varepsilon > 0$*, there exists a real-valued function* $\Phi \in C_c^\infty(\mathbb{R}^d)$ *such that*

$$||(1-\Phi)f||_{M^{p,1}(\mathbb{R}^d)} < \varepsilon.$$

LEMMA 3.13. *(cf.[11,Corollary 3.3]) Let* $1 \le p < \infty$*, and* $f_1, \cdots, f_N \in M^{p,1}(\mathbb{R}^d)$*. For any* $\varepsilon > 0$*, there exists a real-valued function* $\Phi \in C_c^\infty(\mathbb{R}^d)$ *such that*

$$||(1-\Phi)f_j||_{M^{p,1}(\mathbb{R}^d)} < \varepsilon \quad (j = 1, \ldots, N).$$

PROPOSITION 3.14. *(cf.[11,Proposition 3.4]) Let* $1 \le p < \infty$ *and* F *be a real analytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$*. If* $f \in M^{p,1}(\mathbb{R}^d)$*, then there exists* $H \in M^{p,1}(\mathbb{R}^d)$ *such that*

$$H(x) = F(Re\ f(x), Im\ f(x))$$

except for some compact set in $\mathbb{R}^d$*.*

Theorem 3.4 is proved in Kobayashi-Sato[11], and Theorem 3.5 is proved by the slight change of the proof of Theorem 3.4 by those propositions and lemmas. We omit the details.

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