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Convergence theorems of the Choquet integral for three types of convergence of measurable functions

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Abstract. The purpose of this paper is to gather and rearrange some results known as convergence theorems of the Choquet integral in terms of three types of convergence of measurable functions, that is, pointwise convergence, almost everywhere convergence, and convergence in measure. Moreover, some characteristics of nonadditive measures are specified by the validity of those convergence theorems.

1. Introduction

A nonadditive measure is a monotonically increasing set function that is not necessarily additive and the Choquet integral is the nonlinear integral with respect to a nonadditive measure [4, 18, 25]. They are important in terms of expected utility theory, subjective evaluation problem, and the refinement of measure and integration theory, in which the σ-additivity of measures may be a strong constraint and the Lebesgue integral may not give a proper aggregation process [5, 6, 17, 26].

In order to put the Choquet integral into practical use and aim for application to various fields, it is indispensable to establish convergence theorems assuring that the limit of the integrals of a sequence of functions is the integral of the limit function. For this reason, many researchers have already studied several kinds of convergence theorems of the Choquet integral, such as the monotone convergence theorem, the bounded convergence theorem, and the dominated convergence theorem, but they are formulated under ones' own terms and settings. In addition, the outcomes are dispersed in numerous papers; see [3, 4, 10, 12, 13, 14, 15, 16, 18, 19, 24, 25] among others.

The purpose of this paper is to gather and rearrange some results known as convergence theorems of the Choquet integral in terms of three types of convergence of measurable functions, that is, pointwise convergence, almost everywhere convergence, and convergence in measure. Those theorems are refined in the sense

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that a counterexample can be found if any one of the assumptions of the theorem is not assumed. Moreover, with a few exceptions, some characteristics of nonadditive measures are specified by the validity of those convergence theorems.

The paper is organized as follows. In Section 2, some basic properties of nonadditive measures and the Choquet integral are recalled. In particular, the uniform Choquet integrability and some types of the uniform essential boundedness of functions are surveyed. In Section 3, several important convergence theorems, such as the monotone convergence theorem, the Fatou lemma, the dominated convergence theorem, the bounded convergence theorem, and the Vitali convergence theorem are gathered and rearranged in terms of pointwise convergence, almost everywhere convergence, and convergence in measure of measurable functions. The paper ends in Section 4 by giving the bounded convergence theorem and the dominated convergence theorem for the symmetric and asymmetric Choquet integrals of a sequence of not necessarily non-negative functions.

2. Preliminaries

In this paper, unless stated otherwise, X is a non-empty set and $\mathcal{A}$ is a field of subsets of X. Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real numbers and the set of all natural numbers. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ with usual total order. For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. For any functions $f, g \colon X \to \overline{\mathbb{R}}$, let $(f \vee g)(x) := f(x) \vee g(x)$ and $(f \wedge g)(x) := f(x) \wedge g(x)$ for every $x \in X$. We adopt the usual conventions for algebraic operations on $\overline{\mathbb{R}}$. We also adopt the convention $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ and $\inf \varnothing = \infty$. If a positive number c may take ∞, we explicitly write $c \in (0, \infty]$ instead of the ambiguous expression $c > 0$. In other words, $c > 0$ always means $c \in (0, \infty)$. This notational convention will be used for similar cases.

Let χ_A denote the characteristic function of a set A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. A function $f \colon X \to \overline{\mathbb{R}}$ is called *$\mathcal{A}$-measurable* if $\{f \geq t\} := \{x \in X \colon f(x) \geq t\} \in \mathcal{A}$ and $\{f > t\} := \{x \in X \colon f(x) > t\} \in \mathcal{A}$ for every $t \in \overline{\mathbb{R}}$. Any constant function and the characteristic function χ_A of any set $A \in \mathcal{A}$ are $\mathcal{A}$-measurable. If f and g are $\mathcal{A}$-measurable and $c \in \mathbb{R}$, then so are $f^+ := f \vee 0$, $f^- := (-f) \vee 0$, $|f| := f \vee (-f)$, cf, $f + c$, $f \vee g$, and $f \wedge g$. Note that $f = f \wedge c + (f - c)^+$. Let $\mathcal{F}(X)$ denote the set of all $\mathcal{A}$-measurable functions $f \colon X \to \overline{\mathbb{R}}$ and $\mathcal{F}^+(X) := \{f \in \mathcal{F}(X) \colon f \geq 0\}$.

2.1. Nonadditive measures

A *nonadditive measure* on X is a set function $\mu \colon \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. It is called *finite* if $\mu(X) < \infty$.

This type of set function is also called a monotone measure [25], a capacity [2], and a fuzzy measure [22] in the literature. Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu\colon \mathcal{A} \to [0,\infty]$. Let $\mathcal{M}_b(X) := \{\mu \in \mathcal{M}(X)\colon \mu(X) < \infty\}$. For $\mu \in \mathcal{M}_b(X)$, its *dual* $\bar{\mu} \in \mathcal{M}_b(X)$ is defined by $\bar{\mu}(A) := \mu(X) - \mu(A^c)$ for every $A \in \mathcal{A}$, where A^c denotes the complement of A. It is obvious that $\bar{\bar{\mu}} = \mu$. A nonadditive measure μ is called *subadditive* if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$ such that $A \cap B = \varnothing$ and called *additive* if the equality holds. If μ is additive, then $\bar{\mu} = \mu$.

2.2. The Choquet integral

The Choquet integral [2, 20] is a typical nonlinear integral. It is widely used in nonadditive measure theory and has applications in expected utility theory, subjective evaluation problem, economics with Knightian uncertainty, data mining, and others [5, 6, 17, 26].

Definition 2.1. Let $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{F}^+(X)$. The *Choquet integral* is defined by

$$\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f > t\})dt,$$

where the integral of the right hand side is the Lebesgue integral or the improper Riemann integral.

Remark 2.2. (1) Since the μ-distribution function $G_{\mu,f}(t) := \mu(\{f > t\})$ $(t \in \mathbb{R})$ is decreasing and non-negative, it has a well-defined improper Riemann integral and also defines the Lebesgue integral. Moreover, the function $\mu(\{f > t\})$ can be replaced with $\mu(\{f \ge t\})$ without changing the value of the integral.

(2) The Choquet integral is equal to the abstract Lebesgue integral if μ is σ-additive and $\mathcal{A}$ is a σ-field [21, Corollary 18]; see also [12, Propositions 8.1 and 8.2].

If $f \in \mathcal{F}^+(X)$ and $\mathrm{Ch}(\mu, f) < \infty$, then f is called *μ-integrable*. If $f \in \mathcal{F}(X)$ and $\mathrm{Ch}(\mu, |f|) < \infty$, then f is called *μ-absolutely integrable*, which will be called *μ-integrable* for short without any confusion. The Choquet integral has the following properties; see [4, 14, 18, 25].

Proposition 2.3 ([14, Propositions 2.6 and 2.7]). *Let $\mu \in \mathcal{M}(X)$, $A \in \mathcal{A}$, and $f, g, h \in \mathcal{F}^+(X)$. Let $a \ge 0$ be a constant.*

(1) $0 \le \mathrm{Ch}(\mu, f) \le \|f\|_\mu \cdot \mu(\{f > 0\})$, *where* $\|f\|_\mu := \inf\{c > 0\colon \mu(\{f > c\}) = 0\}$.

(2) *If* $f(x) \le g(x)$ *for every* $x \in X$, *then* $\mathrm{Ch}(\mu, f) \le \mathrm{Ch}(\mu, g)$.

(3) $\mathrm{Ch}(\mu, a\chi_A) = a\mu(A)$.

(4) $\mathrm{Ch}(\mu, af) = a\,\mathrm{Ch}(\mu, f)$.

(5) $\mathrm{Ch}(\mu, f + a) = \mathrm{Ch}(\mu, f) + a\mu(X)$.

(6) *Assume that either f or g is μ-integrable. If $|f(x) - g(x)| \leq a$ for every $x \in X$, then $|\mathrm{Ch}(\mu, f) - \mathrm{Ch}(\mu, g)| \leq a\mu(X)$.*

(7) *The following inequality holds.*

$$a\mu(A \cap \{f \geq a\}) \leq \mathrm{Ch}(\mu, \chi_A f) \leq a\mu(A \cap \{f > 0\}) + \mathrm{Ch}(\mu, \chi_{A\cap\{f>a\}} f).$$

Proposition 2.4 ([14, Proposition 2.8]). *Let $\mu \in \mathcal{M}(X)$ and $f \in \mathcal{F}(X)$. If f is μ-integrable, then the following conditions hold.*

(i) $\lim_{c\to\infty} \mu(\{|f| > c\}) = 0$.

(ii) $\lim_{c\to\infty} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) = 0$.

(iii) *f is μ-absolutely continuous, that is, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\mathrm{Ch}(\mu, \chi_A|f|) < \varepsilon$ whenever $A \in \mathcal{A}$ and $\mu(A) < \delta$.*

The Vitali theorem discussed in this paper needs the uniform integrability of functions, which takes the same form as the case of the Lebesgue integral.

Definition 2.5. Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$.

(1) $\mathcal{F}$ is called *uniformly μ-integral bounded* if $\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, |f|) < \infty$.

(2) $\mathcal{F}$ is called *uniformly μ-absolutely continuous* if, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_A|f|) < \varepsilon$ whenever $A \in \mathcal{A}$ and $\mu(A) < \delta$.

(3) $\mathcal{F}$ is called *uniformly μ-integrable* if $\lim_{c\to\infty} \sup_{f\in\mathcal{F}} \mathrm{Ch}(\mu, \chi_{\{|f|>c\}}|f|) = 0$.

For a non-empty $\mathcal{F} \subset \mathcal{F}(X)$ and $a \neq 0$, let $\mathcal{F}^+ := \{f^+ \colon f \in \mathcal{F}\}$, $\mathcal{F}^- := \{f^- \colon f \in \mathcal{F}\}$, $|\mathcal{F}| := \{|f| \colon f \in \mathcal{F}\}$, and $a\mathcal{F} := \{af \colon f \in \mathcal{F}\}$. The uniform integrability of functions for the Choquet integral has the same properties as that for the Lebesgue integral.

Proposition 2.6 ([14, Propositions 3.2, 3.3, 3.4, 6.4 and 6.5]). *Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$.*

(1) *If $\mathcal{F}$ is uniformly μ-integrable, then so are $a\mathcal{F}$, $\mathcal{F}^+$, $\mathcal{F}^-$, and $|\mathcal{F}|$, where $a \neq 0$ is a constant.*

(2) *If $\mathcal{F}$ is uniformly μ-integral bounded, then* $\lim_{c\to\infty}\sup_{f\in\mathcal{F}}\mu(\{|f|>c\})=0$.

(3) *Consider the following two conditions.*

 (i) *$\mathcal{F}$ is uniformly μ-integral bounded and uniformly μ-absolutely continuous.*

 (ii) *$\mathcal{F}$ is uniformly μ-integrable.*

Then (i) *implies* (ii). *Conversely,* (ii) *implies the uniform μ-absolute continuity of $\mathcal{F}$. If μ is finite, then* (ii) *also implies the uniform μ-integral boundedness of $\mathcal{F}$.*

(4) *$\mathcal{F}$ is uniformly μ-integrable if there is a μ-integrable function $g\in\mathcal{F}^+(X)$ such that $|f(x)|\le g(x)$ for every $x\in X$ and $f\in\mathcal{F}$.*

(5) *$\mathcal{F}$ is uniformly μ-integrable if there is $c>0$ such that $\mu(\{|f|>c\})=0$ for every $f\in\mathcal{F}$.*

(6) *If* $\sup_{f\in\mathcal{F}}\mathrm{Ch}(\mu,|f|^p)<\infty$ *for some $p>1$, then $\mathcal{F}$ is uniformly μ-integrable.*

When $\mu(X)=\infty$, the uniform μ-integral boundedness does not follow from the uniform μ-integrability; in particular, for each $f\in\mathcal{F}(X)$, $\mathrm{Ch}(\mu,|f|)<\infty$ does not follow from $\lim_{c\to\infty}\mathrm{Ch}(\mu,\chi_{\{|f|>c\}}|f|)=0$; see (2) of Remark 3.5 [14]. For this reason, the notion of the uniform integrability is more interesting for finite measures.

As for the bounded convergence theorem, the uniform essential (symmetric) boundedness of functions are needed for its formulation.

DEFINITION 2.7 ([14, Definition 6.1]). Let $\mu\in\mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$.

(1) $\mathcal{F}$ is called *uniformly μ-essentially bounded* if there is $c>0$ such that $\mu(\{f>c\})=0$ and $\mu(\{f>-c\})=\mu(X)$ for every $f\in\mathcal{F}$. In particular, $f\in\mathcal{F}(X)$ is called *μ-essentially bounded* if there is $c>0$ such that $\mu(\{f>c\})=0$ and $\mu(\{f>-c\})=\mu(X)$.

(2) $\mathcal{F}$ is called *uniformly μ-essentially symmetric bounded* if there is $c>0$ such that $\mu(\{f>c\})=\mu(\{f<-c\})=0$ for every $f\in\mathcal{F}$. In particular, $f\in\mathcal{F}(X)$ is called *μ-essentially symmetric bounded* if there is $c>0$ such that $\mu(\{f>c\})=\mu(\{f<-c\})=0$.

(3) $\mathcal{F}$ is called *uniformly μ-essentially absolute bounded* if there is $c > 0$ such that $\mu(\{|f| > c\}) = 0$ for every $f \in \mathcal{F}$. In particular, $f \in \mathcal{F}(X)$ is called *μ-essentially absolute bounded* if there is $c > 0$ such that $\mu(\{|f| > c\}) = 0$.

If μ is additive, then the above three notions of uniform boundedness coincide with each other. By (5) of Proposition 2.6, every uniformly μ-essentially absolute bounded subset of $\mathcal{F}(X)$ is uniformly μ-integrable. When $\mathcal{F} \subset \mathcal{F}^+(X)$, the uniform μ-essential boundedness, symmetric boundedness, and absolute boundedness coincide with each other and they are reduced to the condition that there is $c > 0$ such that $\mu(\{f > c\}) = 0$ for every $f \in \mathcal{F}$. For this reason, the terms uniform μ-essential symmetric boundedness and absolute boundedness will be avoided using for a family of non-negative functions.

Proposition 2.8 ([14, Propositions 6.3 and 6.5]). *Let $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$.*

(1) *$\mathcal{F}$ is uniformly μ-essentially symmetric bounded if and only if $\mathcal{F}^+$ and $\mathcal{F}^-$ are both uniformly μ-essentially bounded. In this case, $\mathcal{F}^+$ and $\mathcal{F}^-$ are both uniformly μ-integrable.*

(2) *Assume that μ is finite. $\mathcal{F}$ is uniformly μ-essentially bounded if and only if $\mathcal{F}^+$ is uniformly μ-essentially bounded and $\mathcal{F}^-$ is uniformly $\bar{\mu}$-essentially bounded. In this case, $\mathcal{F}^+$ is uniformly μ-integrable, while $\mathcal{F}^-$ is uniformly $\bar{\mu}$-integrable.*

See [4, 11, 18, 25] for more information on nonadditive measures and the Choquet integral.

3. Convergence theorems of the Choquet integral

In this section, some results known as convergence theorems of the Choquet integral are gathered and rearranged in terms of three types of convergence of measurable functions, that is, pointwise convergence, almost everywhere convergence, and convergence in measure; see [3, 4, 10, 12, 13, 14, 15, 16, 18, 19, 24, 25].

In what follows, those convergence theorems are refined in the sense that a counterexample can be found if any one of the assumptions of the theorem is not assumed. Moreover, with a few exceptions, some characteristics of nonadditive measures are specified by the validity of those convergence theorems.

3.1. Pointwise convergence theorems

Let $(X, \mathcal{A})$ be a measurable space. For $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(X)$ and $f \in \mathcal{F}(X)$, the symbol $f_n \to f$ denotes pointwise convergence, that is, $f_n(x) \to f(x)$ for every $x \in X$. It is written as $f_n \uparrow f$ if $f_n(x) \leq f_{n+1}(x)$ for every $n \in \mathbb{N}$ and $x \in X$ and as $f_n \downarrow f$ if $f_n(x) \geq f_{n+1}(x)$ for every $n \in \mathbb{N}$ and $x \in X$.

Let $\mathcal{F}$ be a non-empty subset of $\mathcal{F}(X)$. Then, $\mathcal{F}$ is called *uniformly bounded* if there is $c > 0$ such that $|f(x)| \leq c$ for every $f \in \mathcal{F}$ and $x \in X$. Every uniformly bounded subset of $\mathcal{F}(X)$ is uniformly μ-essentially bounded, symmetric bounded, and absolute bounded, and hence, uniformly μ-integrable. Moreover, $\mathcal{F}$ is called *having a dominating function* $g \in \mathcal{F}^+(X)$ if $|f(x)| \leq g(x)$ for every $f \in \mathcal{F}$ and $x \in X$. Every subset of $\mathcal{F}(X)$ having a μ-integrable dominating function is uniformly μ-integrable.

Pointwise convergence theorems are related to the following several types of the continuity of nonadditive measures. For $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$, the symbol $A_n \uparrow A$ denotes that $\{A_n\}_{n\in\mathbb{N}}$ is increasing and $A = \bigcup_{n=1}^\infty A_n$. Similarly, $A_n \downarrow A$ denotes that $\{A_n\}_{n\in\mathbb{N}}$ is decreasing and $A = \bigcap_{n=1}^\infty A_n$.

Definition 3.1. Let $\mu \in \mathcal{M}(X)$.

(1) μ is called *continuous from above* if $\mu(A_n) \to \mu(A)$ for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $A_n \downarrow A$.

(2) μ is called *conditionally continuous from above* if $\mu(A_n) \to \mu(A)$ for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $A_n \downarrow A$ and $\mu(A_1) < \infty$.

(3) μ is called *continuous from below* if $\mu(A_n) \to \mu(A)$ for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $A_n \uparrow A$.

(4) μ is called *conditionally continuous from below* if $\mu(A_n) \to \mu(A)$ for any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$ such that $A_n \uparrow A$ and $\mu(A) < \infty$.

(5) μ is called *continuous* if it is continuous from above and continuous from below.

(6) μ is called *conditionally continuous* if it is conditionally continuous from above and continuous from below.

(7) μ is called *conditionally continuous from above and below* if it is conditionally continuous from above and conditionally continuous from below.

The σ-additivity of measures implies not the continuity from above but the conditional continuity from above. The notion of the continuity from above is thus believed to be too strong, so that there is a tendency to avoid its use in ordinary measure theory. However, in nonadditive measure theory, where the additivity of measures is not a prerequisite, the infinite continuous nonadditive measure $\mu = \varphi \circ p$ on $\mathbb{R}$ is simply given by distorting a probability measure p on $\mathbb{R}$ by the function $\varphi\colon [0,1] \to [0,\infty]$ defined as

$$\varphi(t) := \begin{cases} \tan\left(\dfrac{\pi t}{2}\right) & \text{if } t \in [0,1) \\ \infty & \text{if } t = 1, \end{cases}$$

where $\varphi \circ p(A) := \varphi(p(A))$ for every Borel measurable subset A of $\mathbb{R}$. Therefore, the notion of the continuity of measures as well as the conditional continuity is often a subject of study in nonadditive measure theory.

The notions of the continuity, the conditional continuity, and the conditional continuity from above and below are equivalent to each other if μ is finite. For this reason, the terms continuity, continuity from above, and continuity from below will be used preferentially for finite nonadditive measures instead of the terms conditional continuity, conditional continuity from above, and conditional continuity from below. However, the nonadditive measure $\mu = \theta \circ \lambda$ on $\mathbb{R}$ given by distorting the Lebesgue measure λ on $\mathbb{R}$ by the function $\theta\colon [0,\infty] \to [0,\infty]$ defined as

$$\theta(t) := \begin{cases} \tan^{-1} t & \text{if } t \in [0,\infty) \\ \infty & \text{if } t = \infty \end{cases}$$

is conditional continuous from above and below, but neither continuous from above nor continuous from below.

THEOREM 3.2 ([3, 4, 9, 12, 14, 15, 18, 19, 21, 24, 25]). *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$.*

(1) *The following conditions are equivalent.*

  (i) *μ is continuous from below.*

  (ii) *The monotone increasing pointwise convergence theorem holds for μ, that is, $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ and any $f \in \mathcal{F}^+(X)$ such that $f_n \uparrow f$.*

  (iii) *The Fatou pointwise lemma holds for μ, that is,*

$$\mathrm{Ch}(\mu, \liminf_{n\to\infty} f_n) \le \liminf_{n\to\infty} \mathrm{Ch}(\mu, f_n)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$.

(2) *The following conditions are equivalent.*

(i) μ *is conditionally continuous from above.*

(ii) *The monotone decreasing pointwise convergence theorem holds for* μ, *that is,* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \downarrow f$ *and* $\mathrm{Ch}(\mu, f_1) < \infty$.

(iii) *The reverse Fatou pointwise lemma holds for* μ, *that is,*

$$\limsup_{n\to\infty} \mathrm{Ch}(\mu, f_n) \leq \mathrm{Ch}(\mu, \limsup_{n\to\infty} f_n)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *having a* μ*-integrable dominating function* $g \in \mathcal{F}^+(X)$.

(3) *The following conditions are equivalent.*

(i) μ *is conditionally continuous from above and below.*

(ii) *The dominated pointwise convergence theorem holds for* μ, *that is,* f *is* μ*-integrable and* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *having a* μ*-integrable dominating function* $g \in \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$.

(4) *Assume that* μ *is finite. The following conditions are equivalent.*

(i) μ *is continuous.*

(ii) *The bounded pointwise convergence theorem holds for* μ, *that is,* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any uniformly bounded* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$.

(iii) *The Vitali pointwise convergence theorem holds for* μ, *that is,* f *is* μ*-integrable and* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any uniformly* μ*-integrable* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$.

Remark 3.3. Let $X := [0, 1)$. Let λ be the Lebesgue measure on $\mathbb{R}$. Then, λ is a finite σ-additive measure on $[0, 1)$ and hence continuous. Let $f_n(x) := \frac{1}{n(1-x)}$ for every $x \in [0, 1)$ and $n \in \mathbb{N}$. Then, $f_n \downarrow 0$ but $\mathrm{Ch}(\lambda, f_n) = \infty$ for every $n \in \mathbb{N}$. This example shows that the convergence theorems in Theorem 3.2 does not established only by assuming pointwise convergence of f_n to f, except the monotone increasing pointwise theorem and the pointwise Fatou lemma.

3.2. Almost everywhere convergence theorems

Almost everywhere convergence theorems of the Choquet integral are related to the following characteristics of nonadditive measures.

DEFINITION 3.4 ([1, 25]). Let $\mu \in \mathcal{M}(X)$.

(1) μ is called *null-additive* if $\mu(A \cup B) = 0$ for any $A, B \in \mathcal{A}$ such that $\mu(B) = 0$.

(2) μ is called *null-continuous* if $\mu(A) = 0$ for any $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ and any $A \in \mathcal{A}$ such that $A_n \uparrow A$ and $\mu(A_n) = 0$ for every $n \in \mathbb{N}$.

Clearly, every subadditive nonadditive measure is null-additive. Every nonadditive measure that is continuous from below and every nonadditive measure that is continuous from above and null-additive is null-continuous [1, Proposition 9].

Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$. For $N \in \mathcal{A}$, if μ is not additive, then $\mu(N) = 0$ is not necessarily equivalent to $\mu(X \setminus N) = \mu(X)$. In addition, $\mu(N) = 0$ does not always imply $\mu(A \cup N) = \mu(A)$ for every $A \in \mathcal{A}$. Therefore, there are several types of definitions of "almost everywhere" for nonadditive measures. In this paper, standard definitions in ordinary measure theory are adopted when defining the notion of null sets and almost everywhere. In other words, a condition is said to hold *μ-almost everywhere* if there is $N \in \mathcal{A}$ with $\mu(N) = 0$ such that the condition holds for every $x \notin N$; see textbooks [7]. Thus, for $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(X)$ and $f \in \mathcal{F}(X)$, the symbol $f_n \to f$ μ-a.e. denotes μ-almost everywhere convergence, that is, there is $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $f_n(x) \to f(x)$ for every $x \notin N$. It is written as $f_n \uparrow f$ μ-a.e. if $f_n \leq f_{n+1}$ μ-a.e. for every $n \in \mathbb{N}$ and as $f_n \downarrow f$ μ-a.e. if $f_n \geq f_{n+1}$ μ-a.e. for every $n \in \mathbb{N}$.

Let $\mathcal{F} \subset \mathcal{F}(X)$. Then, $\mathcal{F}$ is called a *uniformly μ-a.e. bounded* if there is $c > 0$ such that $|f| \leq c$ μ-a.e. for every $f \in \mathcal{F}$. Every uniformly μ-a.e. bounded subset of $\mathcal{F}(X)$ is uniformly μ-essentially symmetric bounded and absolute bounded. It is uniformly μ-essentially bounded if μ is null-additive. Moreover, $\mathcal{F}$ is called *having a μ-a.e. dominating function* $g \in \mathcal{F}^+(X)$ if $|f| \leq g$ μ-a.e. for every $f \in \mathcal{F}$. Every non-empty subset of $\mathcal{F}(X)$ having a μ-integrable μ-a.e. dominating function is uniformly μ-integrable if μ is null-additive.

It should be mentioned that condition "$f_n \leq f_{n+1}$ μ-a.e. for every $n \in \mathbb{N}$" means that for each $n \in \mathbb{N}$, there is $N \in \mathcal{A}$ with $\mu(N) = 0$, which may depend on n, such that $f_n(x) \leq f_{n+1}(x)$ for every $x \notin N$. It does not imply that there is $N \in \mathcal{A}$ with $\mu(N) = 0$, which does not depend on n, such that $f_n(x) \leq f_{n+1}(x)$ for every $x \notin N$ and $n \in \mathbb{N}$. Both of them are equivalent if μ is null-continuous. The same is mentioned in other cases.

THEOREM 3.5 ([3, 14, 15, 19, 24, 25]). *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$.*

(1) *The following conditions are equivalent.*

(i) *μ is continuous from below and null-additive.*

(ii) *The monotone increasing a.e. convergence theorem holds for μ, that is, $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ and any $f \in \mathcal{F}^+(X)$ such that $f_n \uparrow f$ μ-a.e.*

(iii) *The Fatou a.e. lemma holds for μ, that is,*

$$\mathrm{Ch}(\mu, f) \leq \liminf_{n\to\infty} \mathrm{Ch}(\mu, f_n)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ and any $f \in \mathcal{F}^+(X)$ such that $f_n \to f$ μ-a.e.

(2) *The following conditions are equivalent.*

(i) *μ is conditionally continuous from above, null-additive, and null-continuous.*

(ii) *The monotone decreasing a.e. convergence theorem holds for μ, that is, $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ and any $f \in \mathcal{F}^+(X)$ such that $f_n \downarrow f$ μ-a.e. and $\mathrm{Ch}(\mu, f_1) < \infty$.*

In addition, the null-continuity of μ in assertion (i) *can be dropped if "$f_n \downarrow f$ μ-a.e." in assertion* (ii) *is replaced with "$f_n \to f$ μ-a.e. and $f_n(x) \geq f_{n+1}(x)$ for every $x \in X$ and $n \in \mathbb{N}$."*

(3) *Consider the following conditions.*

(i) *μ is continuous from above and null-additive.*

(ii) *μ is conditionally continuous from above, null-additive, and null-continuous.*

(iii) *The reverse Fatou a.e. lemma holds for μ, that is,*

$$\limsup_{n\to\infty} \mathrm{Ch}(\mu, f_n) \leq \mathrm{Ch}(\mu, f)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ having a μ-integrable μ-a.e. dominating function $g \in \mathcal{F}^+(X)$ and any $f \in \mathcal{F}^+(X)$ such that $f_n \to f$ μ-a.e.

(iv) *μ is conditionally continuous from above and null-additive.*

Then (i)⇒(ii)⇒(iii)⇒(iv) *holds. If* μ *is finite, then the above conditions are equivalent.*

(4) *The following conditions are equivalent.*

(i) μ *is conditionally continuous from above and below, null-additive, and null-continuous.*

(ii) *The dominated a.e. convergence theorem holds for* μ*, that is,* f *is* μ*-integrable and* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *having a* μ*-integrable* μ*-a.e. dominating function* $g \in \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

In addition, the null-continuity of μ *in assertion* (i) *can be dropped if "*μ*-a.e. dominating" in assertion* (ii) *is replaced with "dominating."*

(5) *Assume that* μ *is finite. The following conditions are equivalent.*

(i) μ *is continuous and null-additive.*

(ii) *The bounded a.e. convergence theorem holds for* μ*, that is,* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any uniformly* μ*-a.e. bounded* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

(iii) *The bounded a.e. convergence theorem holds for* $\bar{\mu}$*, that is,* $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ *for any uniformly* μ*-a.e. bounded* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

(iv) *The dominated a.e. convergence theorem holds for* μ*, that is,* f *is* μ*-integrable and* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *having a* μ*-integrable* μ*-a.e. dominating function* $g \in \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

(v) *The dominated a.e. convergence theorem holds for* $\bar{\mu}$*, that is,* f *is* $\bar{\mu}$*-integrable and* $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ *for any* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *having* $\bar{\mu}$*-integrable* μ*-a.e. dominating function* $g \in \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

(vi) *The Vitali a.e. convergence theorem holds for* μ*, that is,* f *is* μ*-integrable and* $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ *for any uniformly* μ*-integrable* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

(vii) *The Vitali a.e. convergence theorem holds for* $\bar{\mu}$*, that is,* f *is* $\bar{\mu}$*-integrable and* $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ *for any uniformly* $\bar{\mu}$*-integrable* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}^+(X)$ *and any* $f \in \mathcal{F}^+(X)$ *such that* $f_n \to f$ μ*-a.e.*

Remark 3.6. As for the dual measure forms in (5), the nonadditive measure that is assumed to be null-additive is not $\bar{\mu}$ but μ, and the convergence of functions is not $f_n \to f$ $\bar{\mu}$-a.e. but $f_n \to f$ μ-a.e. Moreover, $\{f_n\}_{n\in\mathbb{N}}$ is assumed to have not a $\bar{\mu}$-a.e. but μ-a.e. dominating function. In fact, if μ is null-additive, then every μ-null set is always $\bar{\mu}$-null, so that $f_n \to f$ μ-a.e. implies $f_n \to f$ $\bar{\mu}$-a.e. and every μ-a.e dominating function is also $\bar{\mu}$-a.e. dominating. However, since $\bar{\mu}$ is not always null-additive even if μ is null-additive [14, Example 2.2], those dual measure forms are not immediate consequences of the original ones.

3.3. Convergence in measure theorems

Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$. Let $\mathcal{F}_0(X) := \{f \in \mathcal{F}(X)\colon f \text{ is real-valued}\}$, $\mathcal{F}_0^+(X) := \{f \in \mathcal{F}_0(X)\colon f \geq 0\}$. For $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ and $f \in \mathcal{F}_0(X)$, the symbol $f_n \xrightarrow{\mu} f$ denotes convergence in μ-measure, that is, $\mu(\{|f_n - f| > \varepsilon\}) \to 0$ for every $\varepsilon > 0$. Obviously, if $f_n \xrightarrow{\mu} f$, then $|f_n| \xrightarrow{\mu} |f|$, $f_n^+ \xrightarrow{\mu} f^+$, and $f_n^- \xrightarrow{\mu} f^-$.

Convergence theorems of the Choquet integral for a sequence of measurable functions converging in measure need the following characteristics related to the quasi-additivity of nonadditive measures.

Definition 3.7 ([19, 25]). Let $\mu \in \mathcal{M}(X)$.

(1) μ is called *autocontinuous from above* if $\mu(A \cup B_n) \to \mu(A)$ for any $A \in \mathcal{A}$ and any $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $\mu(B_n) \to 0$.

(2) μ is called *autocontinuous from below* if $\mu(A \setminus B_n) \to \mu(A)$ for any $A \in \mathcal{A}$ and any $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $\mu(B_n) \to 0$.

(3) μ is called *autocontinuous* if it is autocontinuous from above and autocontinuous from below.

(4) μ is called *monotonely autocontinuous from above* if $\mu(A \cup B_n) \to \mu(A)$ for any $A \in \mathcal{A}$ and any decreasing $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $\mu(B_n) \to 0$.

(5) μ is called *monotonely autocontinuous from below* if $\mu(A \setminus B_n) \to \mu(A)$ for any $A \in \mathcal{A}$ and any decreasing $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ such that $\mu(B_n) \to 0$.

(6) μ is called *monotonely autocontinuous* if it is monotonely autocontinuous from above and monotonely autocontinuous from below.

The following implications are valid for several types of the quasi-additivity conditions defined above; see [8, 9, 19].

- subadditive $\Rightarrow$ autocontinuous from above $\Rightarrow$ monotonely autocontinuous from above $\Rightarrow$ null-additive

- subadditive $\Rightarrow$ autocontinuous from below $\Rightarrow$ monotonely autocontinuous from below $\Rightarrow$ null-additive

- μ is autocontinuous from above if it is continuous from above and autocontinuous from below, while μ is autocontinuous from below if it is continuous from below and autocontinuous from above [23].

- μ is monotonely autocontinuous from above if it is continuous from above and null-additive, while μ is monotonely autocontinuous from below if it is continuous from below and null-additive.

Remark 3.8. By definition, the combination of the continuity from above (below) with the null-additivity can be characterized in the following way [19, Proposition 3.1].

(1) μ is continuous from above and null-additive if and only if it is *strongly monotone autocontinuous from above*, that is, $\mu(A \cup B_n) \to \mu(A)$ for any $A, B \in \mathcal{A}$ and any $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $B_n \downarrow B$ and $\mu(B) = 0$.

(2) μ is continuous from below and null-additive if and only if it is *strongly monotone autocontinuous from below*, that is, $\mu(A \setminus B_n) \to \mu(A)$ for any $A, B \in \mathcal{A}$ and any $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $B_n \downarrow B$ and $\mu(B) = 0$.

A nonadditive measure μ is autocontinuous if it satisfies $\inf\{\mu(A) \colon A \in \mathcal{A}, A \neq \varnothing\} > 0$ and null-additive if it satisfies $\mu(A) > 0$ for every non-empty $A \in \mathcal{A}$ [25, Theorems 6.1 and 6.5]. Moreover, the distorted measure $\mu = \varphi \circ m$, where m is an additive measure on $(X, \mathcal{A})$ and $\varphi \colon [0, m(X)] \to [0, \infty]$ is an increasing function with $\varphi(0) = 0$, is

- subadditive if φ satisfies $\varphi(s + t) \leq \varphi(s) + \varphi(t)$ for every $s, t \in [0, m(X)]$,

- null-additive if $\varphi^{-1}(\{0\}) = \{0\}$,

- autocontinuous from above if $\varphi^{-1}(\{0\}) = \{0\}$ and φ is right continuous on $[0, m(X))$.

- autocontinuous from below if $\varphi^{-1}(\{0\}) = \{0\}$ and φ is left continuous on $(0, m(X)]$.

THEOREM 3.9 ([3, 4, 10, 14, 15, 16, 19, 24, 25]). *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$.*

(1) *The following conditions are equivalent.*

   (i) *μ is monotonely autocontinuous from below.*

   (ii) *The monotone increasing convergence in measure theorem holds for μ, that is, $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$ and $f_n(x) \le f_{n+1}(x) \le f(x)$ for every $x \in X$ and $n \in \mathbb{N}$.*

If μ is finite, then the above conditions are equivalent to the following.

   (iii) *The monotone decreasing convergence in measure theorem holds for $\bar{\mu}$, that is, $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$, $f(x) \le f_{n+1}(x) \le f_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$, and $\mathrm{Ch}(\bar{\mu}, f_1) < \infty$.*

(2) *Consider the following conditions.*

   (i) *μ is monotonely autocontinuous from above.*

   (ii) *The monotone decreasing convergence in measure theorem holds for μ, that is, $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$, $f(x) \le f_{n+1}(x) \le f_n(x)$ for every $x \in X$ and $n \in \mathbb{N}$, and $\mathrm{Ch}(\mu, f_1) < \infty$.*

Then (i)$\Rightarrow$(ii) *holds. If μ is finite,then* (i)$\Leftrightarrow$(ii) *holds and they are equivalent to the following.*

   (iii) *The monotone increasing convergence in measure theorem holds for $\bar{\mu}$, that is, $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$ and $f_n(x) \le f_{n+1}(x) \le f(x)$ for every $x \in X$ and $n \in \mathbb{N}$.*

(3) *The following conditions are equivalent.*

   (i) *μ is autocontinuous from below.*

   (ii) *The Fatou in measure lemma holds for μ, that is,*

$$\mathrm{Ch}(\mu, f) \le \liminf_{n\to\infty} \mathrm{Ch}(\mu, f_n)$$

   for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.

If μ is finite, then they are equivalent to the following.

(iii) *The reverse Fatou in measure lemma holds for $\bar{\mu}$, that is,*

$$\limsup_{n\to\infty} \mathrm{Ch}(\bar{\mu}, f_n) \leq \mathrm{Ch}(\bar{\mu}, f)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$ and $\mathrm{Ch}(\bar{\mu}, f_1) < \infty$.

(4) *Consider the following conditions.*

(i) *μ is autocontinuous from above.*

(ii) *The reverse Fatou in measure lemma holds for μ, that is,*

$$\limsup_{n\to\infty} \mathrm{Ch}(\mu, f_n) \leq \mathrm{Ch}(\mu, f)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$ and $\mathrm{Ch}(\mu, f_1) < \infty$.

Then (i)$\Rightarrow$(ii) *holds. If μ is finite, then* (i)$\Leftrightarrow$(ii) *holds and they are equivalent to the following.*

(iii) *The Fatou in measure lemma holds for $\bar{\mu}$, that is,*

$$\mathrm{Ch}(\bar{\mu}, f) \leq \liminf_{n\to\infty} \mathrm{Ch}(\bar{\mu}, f_n)$$

for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.

(5) *Consider the following conditions.*

(i) *μ is autocontinuous.*

(ii) *The dominated convergence in measure theorem holds for μ, that is, f is μ-integrable and $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ having a μ-integrable μ-a.e. dominating function $g \in \mathcal{F}^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.*

Then (i)$\Rightarrow$(ii) *holds. If μ is finite, then* (i)$\Leftrightarrow$(ii) *holds and they are equivalent to the following.*

(iii) *The dominated convergence in measure theorem holds for $\bar{\mu}$, that is, f is $\bar{\mu}$-integrable and $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$*

having a $\bar{\mu}$-integrable μ-a.e. dominating function $g \in \mathcal{F}^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.

(6) *Assume that μ is finite. The following conditions are equivalent.*

(i) *μ is autocontinuous.*

(ii) *The bounded convergence in measure theorem holds for μ, that is, f is μ-essentially bounded and $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any uniformly μ-essentially bounded $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.*

(iii) *The bounded convergence in measure theorem holds for $\bar{\mu}$, that is, f is $\bar{\mu}$-essentially bounded and $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ for any uniformly $\bar{\mu}$-essentially bounded $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.*

(iv) *The Vitali convergence in measure theorem holds for μ, that is, f is μ-integrable and $\mathrm{Ch}(\mu, f_n) \to \mathrm{Ch}(\mu, f)$ for any uniformly μ-integrable $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.*

(v) *The Vitali convergence in measure theorem holds for $\bar{\mu}$, that is, f is $\bar{\mu}$-integrable and $\mathrm{Ch}(\bar{\mu}, f_n) \to \mathrm{Ch}(\bar{\mu}, f)$ for any uniformly $\bar{\mu}$-integrable $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and any $f \in \mathcal{F}_0^+(X)$ such that $f_n \xrightarrow{\mu} f$.*

Remark 3.10. As for the dual measure forms in the above convergence theorems, the nonadditive measure that is assumed to be (monotonely) autocontinuous (from above or below) is not $\bar{\mu}$ but μ, and the convergence of functions is not $f_n \xrightarrow{\bar{\mu}} f$ but $f_n \xrightarrow{\mu} f$. Moreover, $\{f_n\}_{n\in\mathbb{N}}$ is assumed to have not a $\bar{\mu}$-a.e. but μ-a.e. dominating function. In fact, if μ is autocontinuous from below, then $f_n \xrightarrow{\mu} f$ implies $f_n \xrightarrow{\bar{\mu}} f$. In addition, if μ is null-additive, then every μ-a.e. dominating function is also $\bar{\mu}$-a.e. dominating and every uniformly μ-essentially bounded subset of $\mathcal{F}^+(X)$ is uniformly $\bar{\mu}$-essentially bounded; see Remark 3.6. However, those dual measure forms are not immediate consequences of the original ones since $\bar{\mu}$ is not always (monotonely) autocontinuous (from above or below) even if μ has the same property [14, Example 2.2].

4. Concluding remarks

The Choquet integral can be extended in the following two ways

$$\mathrm{Ch}^s(\mu, f) := \mathrm{Ch}(\mu, f^+) - \mathrm{Ch}(\mu, f^-), \quad (\mu, f) \in \mathcal{M}(X) \times \mathcal{F}(X)$$

$$\mathrm{Ch}^a(\mu, f) := \mathrm{Ch}(\mu, f^+) - \mathrm{Ch}(\bar{\mu}, f^-), \quad (\mu, f) \in \mathcal{M}_b(X) \times \mathcal{F}(X)$$

in order to consider the integrals of not necessarily non-negative functions [18, 20]. These are not defined if the right-hand side is of the form $\infty - \infty$. The Ch^s is called the *symmetric Choquet integral*, while Ch^a is called the *asymmetric Choquet integral*. The symmetric Choquet integral is symmetric in the sense that $\mathrm{Ch}^s(\mu, -f) = -\mathrm{Ch}^s(\mu, f)$ and the asymmetric Choquet integral is asymmetric in the sense that $\mathrm{Ch}^a(\mu, -f) = -\mathrm{Ch}^a(\bar{\mu}, f)$.

A function $f \in \mathcal{F}(X)$ is called *symmetrically μ-integrable* if $\mathrm{Ch}(\mu, f^+) < \infty$ and $\mathrm{Ch}(\mu, f^-) < \infty$, while it is called *asymmetrically μ-integrable* if $\mathrm{Ch}(\mu, f^+) < \infty$ and $\mathrm{Ch}(\bar{\mu}, f^-) < \infty$. Recall that f is μ-integrable if $\mathrm{Ch}(\mu, |f|) < \infty$. If f is μ-integrable, then so are f^+ and f^-, but the converse statement does not hold in general. In addition, f is not μ-integrable even if it is symmetrically and asymmetrically μ-integrable; see [14, Example 5.2]. Obviously, if f is μ-integrable, then f is symmetrically μ-integrable and $|\mathrm{Ch}^s(\mu, f)| \leq \mathrm{Ch}(\mu, |f|)$, but this is not the case for the asymmetric μ-integral [14, Example 5.3].

Although details are omitted, all results in Section 3 hold for the symmetric and asymmetric Choquet integrals with appropriate modifications. For instance, the bounded convergence theorem and the dominated convergence theorem can be formulated as follows.

THEOREM 4.1. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}_b(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(X)$ and $f \in \mathcal{F}(X)$. The following types of the bounded convergence theorem hold for the symmetric and asymmetric Choquet integrals.*

(1) *Assume that μ is continuous. Assume that $f_n \to f$ and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Then $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$ and $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$.*

(2) *Assume that μ is continuous and null-additive. Assume that $f_n \to f$ μ-a.e. and $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-a.e. bounded. Then $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$ and $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$.*

(3) *Assume that μ is autocontinuous. Assume that $f_n \xrightarrow{\mu} f$ and f_n and f are real-valued. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially symmetric bounded, then f is μ-essentially symmetric bounded and $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$. If $\{f_n\}_{n\in\mathbb{N}}$ is uniformly μ-essentially bounded, then f is μ-essentially bounded and $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$.*

THEOREM 4.2. *Let $(X, \mathcal{A})$ be a measurable space and $\mu \in \mathcal{M}(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(X)$ and $f \in \mathcal{F}(X)$. The following types of the dominated convergence theorem hold for the symmetric and asymmetric Choquet integrals.*

(1) *Assume that* μ *is conditionally continuous from above and below. Assume that* $f_n \to f$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *has a* μ*-integrable dominating function* $g \in \mathcal{F}^+(X)$*. Then* f *is symmetrically* μ*-integrable and* $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$*. Moreover, if* μ *is finite and* g *is simultaneously* μ*-integrable and* $\bar{\mu}$*-integrable, then* f *is asymmetrically* μ*-integrable and* $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$*.*

(2) *Assume that* μ *is conditionally continuous from above and below, null-additive, and null-continuous. Assume that* $f_n \to f$ μ*-a.e. and* $\{f_n\}_{n\in\mathbb{N}}$ *has a* μ*-integrable* μ*-a.e. dominating function* $g \in \mathcal{F}^+(X)$*. Then* f *is symmetrically* μ*-integrable and* $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$*. Moreover, if* μ *is finite and* g *is simultaneously* μ*-integrable and* $\bar{\mu}$*-integrable, then* f *is asymmetrically* μ*-integrable and* $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$*.*

(3) *Assume that* μ *is autocontinuous. Assume that* $f_n \xrightarrow{\mu} f$*,* $\{f_n\}_{n\in\mathbb{N}}$ *has a* μ*-integrable* μ*-a.e. dominating function* $g \in \mathcal{F}^+(X)$*, and* f_n *and* f *are real-valued. Then* f *is symmetrically* μ*-integrable and* $\mathrm{Ch}^s(\mu, f_n) \to \mathrm{Ch}^s(\mu, f)$*. Moreover, if* μ *is finite and* g *is simultaneously* μ*-integrable and* $\bar{\mu}$*-integrable, then* f *is asymmetrically* μ*-integrable and* $\mathrm{Ch}^a(\mu, f_n) \to \mathrm{Ch}^a(\mu, f)$*.*

Let $k \geq 1$ be a constant. A nonadditive measure μ is called *k-subadditive* if $\mu(A \cup B) \leq \mu(A) + k\mu(B)$ for any $A, B \in \mathcal{A}$ such that $A \cap B = \varnothing$ [18, Definition 11.9]. For instance, every subadditive nonadditive measure is 1-subadditive. Every k-subadditive nonadditive measure is autocontinuous. In general, the $\bar{\mu}$-integrability does not follow from the μ-integrability [14, Example 5.7]. However, if μ is finite and k-subadditive, then $\bar{\mu} \leq k\mu$, so that the (uniform) μ-integrability automatically implies the (uniform) $\bar{\mu}$-integrability.

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