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Tsallis relative operator entropy of negative order

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Abstract. In this paper, we show some fundamental properties of the quatum Tsallis relative entropy of negative order based on the properties of the α-quasi geometric mean for $\alpha \in [-1, 0)$ of positive semidefinite matrices. Moreover, we show matrix trace inequalities on the quantum Tsallis relative entropy of negative order, which includes the quasi geometric mean of positive definite matrices.

1. Introduction

As a quantum extension of the Shannon entropy [17], von Neumann [13] defined the entropy of the density matrix ρ by the formula

$$S(\rho) = \mathrm{Tr}[\eta(\rho)]$$

for the entropy function $\eta(t) = -t \log t$. As for the Shannon entropy, it is extremely useful to define a quantum version of the relative entropy. Suppose ρ and σ are density matrices. The quantum relative entropy of ρ with respect to σ is defined by

$$(1.1) \qquad S_U(\rho|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log \rho - \log \sigma)] & \text{if } \operatorname{supp} \rho \leq \operatorname{supp} \sigma, \\ +\infty & \text{otherwise,} \end{cases}$$

which was firstly introduced in the setting of von Neumann algebra by Umegaki [18] in 1962. This is a quantum generalization of the relative entropy due to Kullback and Leibler [12]. In [1], the Tsallis relative entropy of ρ to σ is defined by

$$(1.2) \qquad \mathbf{T}_\alpha(\rho|\sigma) = \frac{1 - \mathrm{Tr}[\rho^{1-\alpha}\sigma^\alpha]}{\alpha} = \mathrm{Tr}[\rho^{1-\alpha}(\ln_\alpha \rho - \ln_\alpha \sigma)]$$

for any $0 < \alpha < 1$, where $\ln_\alpha \rho = \frac{\rho^\alpha - 1}{\alpha}$ is the α-logarithmic function. The Tsallis relative entropy (1.2) is a one-parameter extension of the Umegaki relative entropy

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(1.1), and Ruskai and Stillinger [15] showed the following relation between the Tsallis relative entropy and the Umegaki relative entropy:
(1.3)
$$\mathbf{T}_\alpha(\rho|\sigma) = -\mathrm{Tr}\left[\frac{\rho^{1-\alpha}\sigma^\alpha - \rho}{\alpha}\right] \le S_U(\rho|\sigma) \le \mathrm{Tr}\left[\frac{\rho^{1+\alpha}\sigma^{-\alpha} - \rho}{\alpha}\right] = \mathbf{T}_{-\alpha}(\rho|\sigma)$$

for all $0 < \alpha < 1$ and positive definite ρ and σ.

On the other hand, there is another formulation of the quantum relative entropy: Fujii and Kamei [4] introduced the relative operator entropy which is a relative version of the operator entropy defined by Nakamura-Umegaki [16]: For positive definite matrices ρ and σ, the relative operator entropy is defined by

$$S(\rho|\sigma) = \rho^{1/2}(\log \rho^{-1/2}\sigma\rho^{-1/2})\rho^{1/2}.$$

By virtue of the relative operator entropy, we define the quantum relative entropy as

$$S_{FK}(\rho|\sigma) = -\mathrm{Tr}[S(\rho|\sigma)]. \tag{1.4}$$

The quantum quantity $\mathrm{Tr}[\rho(\log \rho^{1/2}\sigma^{-1}\rho^{1/2})]$ is firstly proposed by Belavkin and Staszewski [3]. Since we treat $S_{FK}(\rho|\sigma)$ as the minus of the trace of the relative operator entropy $S(\rho|\sigma)$, we call (1.4) the FK relative entropy, or the BS relative entropy in [14, pp125]. If ρ and σ commute, then we have $S_U(\rho|\sigma) = S_{FK}(\rho|\sigma)$. Generally, two quantum formulations of the relative entropy are different. In fact, Hiai and Petz [9] showed the following relation:

$$S_U(\rho|\sigma) \le S_{FK}(\rho|\sigma). \tag{1.5}$$

Moreover, Yanagi, Kuriyama and Furuichi [19] have been advancing research on the Tsallis relative operator entropy as an operator generalization of the Tsallis relative entropy, which is regarded as a parametric extension of the relative operator entropy by Fujii-Kamei: For positive definite matrices ρ and σ, the Tsallis relative operator entropy is defined by

$$T_\alpha(\rho|\sigma) = \frac{\rho \sharp_\alpha \sigma - \rho}{\alpha}$$

for $0 < \alpha \le 1$, where the matrix α-geometric mean is defined by

$$\rho \sharp_\alpha \sigma = \rho^{1/2}(\rho^{-1/2}\sigma\rho^{-1/2})^\alpha \rho^{1/2}.$$

Since $S(\rho|\sigma) \leq \frac{\rho \,\sharp_\alpha\, \sigma - \rho}{\alpha}$ for all $0 < \alpha \leq 1$, we have

$$-\mathrm{Tr}[T_\alpha(\rho|\sigma)] \leq S_{FK}(\rho|\sigma). \tag{1.6}$$

Compared (1.3) with (1.6), Furuichi, Yanagi and Kuriyama [6] showed the following relation which is an extension of the Hiai-Petz inequality (1.5):

$$\mathbf{T}_\alpha(\rho|\sigma) \leq -\mathrm{Tr}[T_\alpha(\rho|\sigma)]$$

for all $0 < \alpha \leq 1$. In fact, if we put $\alpha \to 0$, then we have (1.5).

Since

$$\mathrm{Tr}\left[\frac{\rho^{1+\alpha}\sigma^{-\alpha} - \rho}{\alpha}\right] = -\mathrm{Tr}\left[\frac{\rho^{1+\alpha}\sigma^{-\alpha} - \rho}{-\alpha}\right]$$

in (1.3) and $-1 \leq -\alpha < 0$, it suggests the formulation of the Tsallis relative operator entropy of negative order $\alpha \in [-1, 0)$: We use the notation $\natural_\alpha$ for the binary operation

$$\rho \,\natural_\alpha\, \sigma = \rho^{1/2}(\rho^{-1/2}\sigma\rho^{-1/2})^\alpha\rho^{1/2} \qquad \text{for } \alpha \in [-1, 0), \tag{1.7}$$

that have formula in common with $\sharp_\alpha$. Though $\rho \,\natural_\alpha\, \sigma$ for $\alpha \in [-1, 0)$ are not matrix means in the sense of Kubo-Ando theory [11], $\rho \,\natural_\alpha\, \sigma$ have mean-like properties for positive definite matrices ρ and σ. Thus we call (1.7) the α-quasi geometric mean for $\alpha \in [-1, 0)$. Then the Tsallis relative operator entropy of negative order is defined by

$$T_\alpha(\rho|\sigma) = \frac{\rho \,\natural_\alpha\, \sigma - \rho}{\alpha} \qquad \text{for } \alpha \in [-1, 0).$$

The α-quasi geometric mean and the Tsallis relative operator entropy of negative order are discussed in detail in [5].

In this paper, we show some fundamental properties of the quatum Tsallis relative entropy of negative order based on the properties of the α-quasi geometric mean for $\alpha \in [-1, 0)$ of positive semidefinite matrices. Moreover, we show matrix trace inequalities on the quantum Tsallis relative entropy of negative order, which includes the quasi geometric mean of positive definite matrices.

2. α-quasi geoemtric mean $\natural_\alpha$ for $-1 \le \alpha < 0$

First of all, we study the properties of the α-quasi geometric mean $\natural_\alpha$ for $\alpha \in [-1, 0)$ in non-invertible case.

Let ρ and σ be positive semidefinite matrices and $\alpha \in [-1, 0)$. Since $\rho \natural_\alpha (\sigma+\varepsilon)$ is monotone increasing on $\varepsilon \downarrow 0$, the α-quasi geometric mean $\rho \natural_\alpha \sigma$ for $\alpha \in [-1, 0)$ is defined as the following limit if it exists:

$$\rho \natural_\alpha \sigma = \lim_{\varepsilon \downarrow 0} \rho \natural_\alpha (\sigma + \varepsilon). \tag{2.1}$$

By the definition of (2.1), $\rho \natural_\alpha \sigma$ for $\alpha \in [-1, 0)$ exists if a set $\{\rho \natural_\alpha (\sigma+\varepsilon) : \varepsilon > 0\}$ is bounded above. For noninvertible case, we have the following properties of α-quasi geometric means $\rho \natural_\alpha \sigma$ for $\alpha \in [-1, 0)$:

Lemma 2.1. *Let ρ, σ, τ and ψ be positive semidefinite matrices. If $\rho \natural_\alpha \sigma$ and $\tau \natural_\alpha \psi$ exist for some $\alpha \in [-1, 0)$, then the following properties like matrix means hold:*

(1) **right reverse monotonicity:** *$\sigma \le \tau$ implies $\rho \natural_\alpha \sigma \ge \rho \natural_\alpha \tau$.*

(2) **super-additivity:** *$\rho \natural_\alpha \sigma + \tau \natural_\alpha \psi \ge (\rho + \tau) \natural_\alpha (\sigma + \psi)$.*

(3) **homogeneity:** *$(a\rho) \natural_\alpha (a\sigma) = a(\rho \natural_\alpha \sigma)$ for all $a > 0$.*

(4) **jointly convexity:** *For $a \in [0, 1]$*

$$((1-a)\rho + a\tau) \natural_\alpha ((1-a)\sigma + a\psi) \le (1-a)\rho \natural_\alpha \sigma + a\, \tau \natural_\alpha \psi.$$

Remark 2.2. (1) *in Lemma 2.1 means that if $\rho \natural_\alpha \sigma$ exists for some $\alpha \in [-1, 0)$ and $\rho \le \tau$, then $\rho \natural_\alpha \tau$ exists and $\rho \natural_\alpha \sigma \ge \rho \natural_\alpha \tau$. Similarly,* (2) *means that if $\rho \natural_\alpha \sigma$ and $\tau \natural_\alpha \psi$ exist for some $\alpha \in [-1, 0)$, then $(\rho + \tau) \natural_\alpha (\sigma + \psi)$ exists and $\rho \natural_\alpha \sigma + \tau \natural_\alpha \psi \ge (\rho + \tau) \natural_\alpha (\sigma + \psi)$.*

For noninvertible case, the α-quasi geometric mean $\rho \natural_\alpha \sigma$ for $\alpha \in [-1, 0)$ have the following information monotonicity:

Theorem 2.3. *Let ρ and σ be positive semidefinite matrices and Φ a positive linear map. If $\rho \natural_\alpha \sigma$ exists for some $\alpha \in [-1, 0)$, then $\Phi(\rho) \natural_\alpha \Phi(\sigma)$ exists and*

information monotonicity: $\Phi(\rho \natural_\alpha \sigma) \ge \Phi(\rho) \natural_\alpha \Phi(\sigma)$.

For $\alpha \in [-1, 0)$, since $1 \natural_\alpha \varepsilon$ is not bounded above for $\varepsilon > 0$, $1 \natural_\alpha 0$ does not make sense. Thus we consider an existence condition such that $\rho \natural_\alpha \sigma$ exists as a matrix, which is expressed by the boundedness of tangent lines: For $a > 0$ and $\alpha \in [-1, 0)$, we define the line tangent to the curve $y = x^{1-\alpha}$ at $x = a$ by

$$L_{a,\alpha}(\rho, \sigma) = (1-\alpha)a^{-\alpha}\rho + \alpha a^{1-\alpha}\sigma.$$

LEMMA 2.4. *Let ρ and σ be positive semidefinite matrices and $\alpha \in [-1, 0)$. Then $\rho \natural_\alpha \sigma$ exists as a matrix if and only if*

$$\sup_{a>0} L_{a,\alpha}(\rho, \sigma) = \sup_{a>0} \left[(1-\alpha)a^{-\alpha}\rho + \alpha a^{1-\alpha}\sigma\right] < +\infty. \tag{2.2}$$

The convention (2.2) means that there exists a scalar constant c with $\varphi(L_{a,\alpha}(\rho, \sigma)) \le c$ for all states φ and $a > 0$. In this case, it follows that $\rho \natural_\alpha \sigma \le c$.

In order to show one of equivalent conditions that $\rho \natural_\alpha \sigma$ for some $\alpha \in [-1, 0)$ exists, we need some preliminaries. The following lemma says that the α-quasi geometric mean for $\alpha \in [-1, 0)$ has normalization:

LEMMA 2.5. *Let ρ be a positive semidefinite matrix and $\alpha \in [-1, 0)$. Then $\rho \natural_\alpha \rho = \rho$.*

The following lemma shows that a kind of arithmetic-geometric mean inequality holds, also see [8, p129, Theorem 2]:

LEMMA 2.6. *Let ρ and σ be positive semidefinite matrices and $\alpha \in [-1, 0)$. If $\rho \natural_\alpha \sigma$ exists, then*

$$\rho \natural_\alpha \sigma \ge (1-\alpha)\rho + \alpha\sigma.$$

We have the following equivalent relations around existence conditions:

THEOREM 2.7. *The following three conditions are mutually equivalent for any positive semidefinite matrices ρ, σ and $\alpha \in [-1, 0)$:*

(1) **majorization** *or* **range inclusion**: *$\rho \le c\sigma$ for some $c > 0$, i.e.,* ran $\rho^{\frac{1}{2}} \subset$ ran $\sigma^{\frac{1}{2}}$.

(2) **existence condition**: *$\rho \natural_\alpha \sigma$ exists as a matrix, i.e.,*

$$\sup_{a>0} \left[(1-\alpha)a^{-\alpha}\rho + \alpha a^{1-\alpha}\sigma\right] < +\infty.$$

(3) **kernel inclusion**: ker $\rho \supset$ ker σ.

3. Quatum Tsallis relative entropy of negative order

For convenience, we denote another quantum Tsallis relative entropy of negative order $\alpha \in [-1,0)$ by

$$NT_\alpha(\rho|\sigma) = -\mathrm{Tr}[T_\alpha(\rho|\sigma)] = -\mathrm{Tr}\left[\frac{\rho \natural_\alpha \sigma - \rho}{\alpha}\right]$$

for positive semidefinite matrices ρ and σ. Since $\rho \natural_\alpha (\sigma+\varepsilon)$ is monotone increasing on $\varepsilon \searrow 0$, it follows that $T_\alpha(\rho|\sigma+\varepsilon)$ is monotone decreasing on $\varepsilon \searrow 0$ and thus for non-invertivle case we can define the quantum Tsallis relative entropy of negative order as

$$NT_\alpha(\rho|\sigma) = \lim_{\varepsilon\to 0} NT_\alpha(\rho|\sigma+\varepsilon)$$

if the limit exists. Let $\mathrm{supp}\,\rho$ and $\mathrm{supp}\,\sigma$ be the supports of ρ and σ, respectively. In the case when $\mathrm{supp}\,\rho \le \mathrm{supp}\,\sigma$, $NT_\alpha(\rho|\sigma+\varepsilon)$ has the well-defined limit as $\varepsilon \to 0$ for each $\alpha \in [-1,0)$ by Theorem 2.7.

We have the following properties of the quantum relative entropy NT_α of negative order $\alpha \in [-1,0)$ for non-invertible case, also see [5]:

Proposition 3.1. *Let ρ and σ be (non-invertible) density matrices. If $T_\alpha(\rho|\sigma)$ exists for some $\alpha \in [-1,0)$, then the following properties of the quantum Tsallis relative entropy NT_α hold:*

(1) (Non-negativity) $NT_\alpha(\rho|\sigma) \ge 0$.

(2) (Psedoadditivity)

$$NT_\alpha(\rho_1\otimes\rho_2|\sigma_1\otimes\sigma_2) = NT_\alpha(\rho_1|\sigma_1)+NT_\alpha(\rho_2|\sigma_2)+\alpha NT_\alpha(\rho_1|\sigma_1)NT_\alpha(\rho_2|\sigma_2).$$

(3) (Joint convexity) $NT_\alpha(\sum_j \lambda_j\rho_j|\sum_j \lambda_j\sigma_j) \le \sum_j \lambda_j NT_\alpha(\rho_j|\sigma_j)$.

(4) (Monotonicity) *For any trace-preserving positive linear map Φ*

$$NT_\alpha(\Phi(\rho)|\Phi(\sigma)) \le NT_\alpha(\rho|\sigma).$$

Proof. For (1), since $\rho \natural_\alpha \sigma$ exists for some $\alpha \in [-1,0)$ by assumption, it follows from Lemma 2.6 that $(1-\alpha)\rho + \alpha\sigma \le \rho \natural_\alpha \sigma$ and so

$$NT_\alpha(\rho|\sigma) = -\mathrm{Tr}\left[\frac{\rho \natural_\alpha \sigma - \rho}{\alpha}\right] \ge -\mathrm{Tr}\left[\frac{(1-\alpha)\rho+\alpha\sigma-\rho}{\alpha}\right] = -\mathrm{Tr}[\sigma-\rho] = 0.$$

For (2), since $\rho_1 \natural_\alpha \sigma_1$ and $\rho_2 \natural_\alpha \sigma_2$ exist for some $\alpha \in [-1, 0)$, it follows that

$$\begin{aligned}(\rho_1 \otimes \rho_2) \natural_\alpha (\sigma_1 + \varepsilon) \otimes (\sigma_2 + \varepsilon) &= [\rho_1 \natural_\alpha (\sigma_1 + \varepsilon)] \otimes [\rho_2 \natural_\alpha (\sigma_2 + \varepsilon)] \\ &\leq [\rho_1 \natural_\alpha \sigma_1] \otimes [\rho_2 \natural_\alpha \sigma_2]\end{aligned}$$

for all $\varepsilon > 0$ and so $T_\alpha(\rho_1 \otimes \rho_2 | \sigma_1 \otimes \sigma_2)$ exists. In a similar way that Furuichi, Yanagi and Kuriyama showed in [7, Theorem 4.2], we have

$$T_\alpha(\rho_1 \otimes \rho_2 | \sigma_1 \otimes \sigma_2) = \alpha T_\alpha(\rho_1 | \sigma_1) \otimes T_\alpha(\rho_2 | \sigma_2) + T_\alpha(\rho_1 | \sigma_1) \otimes \rho_2 + \rho_1 \otimes T_\alpha(\rho_2 | \sigma_2).$$

In fact,

$$\begin{aligned}
& T_\alpha(\rho_1 \otimes \rho_2 | \sigma_1 \otimes \sigma_2) \\
&= \frac{1}{\alpha} [(\rho_1 \otimes \rho_2) \natural_\alpha (\sigma_1 \otimes \sigma_2) - \rho_1 \otimes \rho_2] \\
&= \frac{1}{\alpha} [(\rho_1 \natural_\alpha \sigma_1) \otimes (\rho_2 \natural_\alpha \sigma_2) - \rho_1 \otimes \rho_2] \\
&= \frac{1}{\alpha} \Big[\frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes (\rho_2 \natural_\alpha \sigma_2) - \frac{1}{2}\rho_1 \otimes (\rho_2 \natural_\alpha \sigma_2) + \frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes (\rho_2 \natural_\alpha \sigma_2) \\
& \quad - \frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes \sigma_2 + \frac{1}{2}\rho_1 \otimes (\rho_2 \natural_\alpha \sigma_2) - \frac{1}{2}\rho_1 \otimes \rho_2 \\
& \quad + \frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes \sigma_2 - \frac{1}{2}\rho_1 \otimes \rho_2 \Big] \\
&= \frac{1}{2} T_\alpha(\rho_1 | \sigma_1) \otimes (\rho_2 \natural_\alpha \sigma_2) + \frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes T_\alpha(\rho_2 | \sigma_2) + \frac{1}{2}\rho_1 \otimes T_\alpha(\rho_2 | \sigma_2) \\
& \quad + \frac{1}{2} T_\alpha(\rho_1 | \sigma_1) \otimes \rho_2 \\
&= \frac{1}{2} T_\alpha(\rho_1 | \sigma_1) \otimes (\rho_2 \natural_\alpha \sigma_2) - \frac{1}{2} T_\alpha(\rho_1 | \sigma_1) \otimes \rho_2 + T_\alpha(\rho_1 | \sigma_1) \otimes \rho_2 \\
& \quad + \frac{1}{2}(\rho_1 \natural_\alpha \sigma_1) \otimes T_\alpha(\rho_2 | \sigma_2) - \frac{1}{2}\rho_1 \otimes T_\alpha(\rho_2 | \sigma_2) + \rho_1 \otimes T_\alpha(\rho_2 | \sigma_2) \\
&= \alpha T_\alpha(\rho_1 | \sigma_1) \otimes T_\alpha(\rho_2 | \sigma_2) + T_\alpha(\rho_1 | \sigma_1) \otimes \rho_2 + \rho_1 \otimes T_\alpha(\rho_2 | \sigma_2).
\end{aligned}$$

Hence we have the desired equality (2).

For (3), since $T_\alpha(\rho_j | \sigma_j)$ exist for each j, it follows from (4) jointly convexity of Lemma 2.1 that $T_\alpha(\sum_j \lambda_j \rho_j | \sum_j \lambda_j \sigma_j)$ exists and

$$NT_\alpha(\sum_j \lambda_j \rho_j | \sum_j \lambda_j \sigma_j) = -\mathrm{Tr}\left[\frac{(\sum \lambda_j \rho_j) \natural_\alpha (\sum \lambda_j \sigma_j) - \sum \lambda_j \rho_j}{\alpha} \right]$$

$$\leq -\mathrm{Tr}\left[\frac{\sum \lambda_j(\rho_j \natural_\alpha \sigma_j) - \sum \lambda_j \rho_j}{\alpha}\right]$$
$$= -\mathrm{Tr}\left[\sum_j \lambda_j T_\alpha(\rho_j|\sigma_j)\right] = \sum_j \lambda_j NT_\alpha(\rho_j|\sigma_j)$$

and thus we have (3).

For (4), since $T_\alpha(\rho|\sigma)$ exists for some $\alpha \in [-1, 0)$, it follows from the information monotonicity of Theorem 2.3 that $T_\alpha(\Phi(\rho)|\Phi(\sigma))$ exists and

$$\begin{aligned} NT_\alpha(\Phi(\rho)|\Phi(\sigma)) &= -\mathrm{Tr}\left[T_\alpha(\Phi(\rho)|\Phi(\sigma))\right] = -\mathrm{Tr}\left[\frac{\Phi(\rho) \natural_\alpha \Phi(\sigma) - \Phi(\rho)}{\alpha}\right] \\ &\leq -\mathrm{Tr}\left[\frac{\Phi(\rho \natural_\alpha \sigma) - \Phi(\rho)}{\alpha}\right] = -\mathrm{Tr}\left[\Phi(\frac{\rho \natural_\alpha \sigma - \rho}{\alpha})\right] \\ &= -\mathrm{Tr}\left[\frac{\rho \natural_\alpha \sigma - \rho}{\alpha}\right] \\ &= NT_\alpha(\rho|\sigma) \end{aligned}$$

and hence we have (4). □

4. Matrix trace inequalities related to Tsallis relative entropies

Let ρ and σ be positive definite matrices and $\alpha \in [-1, 0)$. Since $\frac{\rho \natural_\alpha \sigma - \rho}{\alpha} \leq S(\rho|\sigma)$, it follows that

$$S_{FK}(\rho|\sigma) \leq NT_\alpha(\rho|\sigma)$$

and $NT_\alpha(\rho|\sigma)$ converges to $S_{FK}(\rho|\sigma)$ as $\alpha \downarrow 0$.

On the other hand, we have $S_U(\rho|\sigma) \leq \mathbf{T}_\alpha(\rho|\sigma)$. Since $S_U(\rho|\sigma) \leq S_{FK}(\rho|\sigma)$ and $\mathbf{T}_\alpha(\rho|\sigma) \leq NT_\alpha(\rho|\sigma)$ holds for $\alpha \in (0, 1]$, we might expect that $\mathbf{T}_\alpha(\rho|\sigma) \leq NT_\alpha(\rho|\sigma)$ holds for all $\alpha \in [-1, 0)$. However, we do not know the relation between $NT_\alpha(\rho|\sigma)$ and $\mathbf{T}_\alpha(\rho|\sigma)$ for $\alpha \in [-1, 0)$. Thus, we consider another estimation of $NT_\alpha(\rho|\sigma)$. For this, we need some preliminaries. we recall the following inequality due to Araki [2]: Let ρ, σ be positive semidefinite matrices. Then

(4.1) $$\mathrm{Tr}\left[(\sigma^{1/2}\rho\sigma^{1/2})^{st}\right] \leq \mathrm{Tr}\left[(\sigma^{t/2}\rho^t\sigma^{t/2})^s\right] \qquad \text{for all } s > 0 \text{ and } t \geq 1,$$

or equivalently

(4.2) $$\mathrm{Tr}\left[(\sigma^{t/2}\rho^t\sigma^{t/2})^{s/t}\right] \leq \mathrm{Tr}\left[(\sigma^{1/2}\rho\sigma^{1/2})^s\right] \qquad \text{for all } s > 0 \text{ and } 0 < t \leq 1.$$

Moreover, in [10, Corollary 3.2], we showed that for $\alpha \in [-1, 0)$

$$\mathrm{Tr}((\rho^q \natural_\alpha \sigma^q)^{1/q} \leq \mathrm{Tr}((\rho^p \natural_\alpha \sigma^p)^{1/p}) \tag{4.3}$$

for all $0 < q \leq p$ and positive definite ρ and σ.

THEOREM 4.1. *Let ρ, σ be positive definite matrices. Then*

$$NT_\alpha(\rho|\sigma) \leq -\mathrm{Tr}\left[\frac{(\rho^{(1-\alpha)p/2}\sigma^{\alpha p}\rho^{(1-\alpha)p/2})^{1/p} - \rho}{\alpha}\right] \tag{4.4}$$

for all $-1 \leq \alpha \leq -1/2$ and $p \geq 2$. Also

$$NT_\alpha(\rho|\sigma) \geq -\mathrm{Tr}\left[\frac{(\sigma^{\alpha q/2}\rho^{(1-\alpha)q}\sigma^{\alpha q/2})^{1/q} - \rho}{\alpha}\right] \tag{4.5}$$

for all $-1 \leq \alpha < 0$ and $0 < q \leq 1/2$.

PROOF. For $-1 \leq \alpha \leq -1/2$ and $p \geq 1$, we have

$$\begin{aligned}
\mathrm{Tr}\,[\rho \natural_\alpha \sigma] &\leq \mathrm{Tr}\left[(\rho^p \natural_\alpha \sigma^p)^{1/p}\right] = \mathrm{Tr}\left[(\rho^{p/2}(\rho^{-p/2}\sigma^p\rho^{-p/2})^\alpha\rho^{p/2})^{1/p}\right] \quad \text{by (4.3)} \\
&= \mathrm{Tr}\left[(\rho^{p/2}(\rho^{p/2}\sigma^{-p}\rho^{p/2})^{-\alpha}\rho^{p/2})^{\frac{1}{-\alpha}\frac{-\alpha}{p}}\right] \\
&\leq \mathrm{Tr}\left[(\rho^{\frac{(\alpha-1)p}{2\alpha}}\sigma^{-p}\rho^{\frac{(\alpha-1)p}{2\alpha}})^{\frac{-2\alpha}{2p}}\right] \qquad \text{by (4.2) and } 1/2 \leq -\alpha \leq 1 \\
&\leq \mathrm{Tr}\left[(\rho^{(1-\alpha)p}\sigma^{2\alpha p}\sigma^{(1-\alpha)p})^{\frac{1}{2p}}\right] \qquad \text{by (4.1) and } 1 \leq -2\alpha \leq 2
\end{aligned}$$

and this implies the desired inequality (4.4) by replacing $2p(\geq 2)$ by p.

For $-1 \leq \alpha < 0$ and $0 < q \leq 1$, we have

$$\begin{aligned}
\mathrm{Tr}\,[\rho \natural_\alpha \sigma] &\geq \mathrm{Tr}\left[(\rho^q \natural_\alpha \sigma^q)^{1/q}\right] = \mathrm{Tr}\left[(\sigma^q \natural_{1-\alpha} \rho^q)^{1/q}\right] \quad \text{by (4.3)} \\
&= \mathrm{Tr}\left[(\sigma^{q/2}(\sigma^{-q/2}\rho^q\sigma^{-q/2})^{1-\alpha}\sigma^{q/2})^{1/q}\right] \\
&\geq \mathrm{Tr}\left[(\sigma^{\frac{q\alpha}{2(1-\alpha)}}\rho^q\sigma^{\frac{q\alpha}{2(1-\alpha)}})^{\frac{1-\alpha}{q}}\right] \qquad \text{by (4.1) and } 1 < 1-\alpha \leq 2 \\
&\geq \mathrm{Tr}\left[(\sigma^{\frac{q\alpha}{4}}\rho^{\frac{q(1-\alpha)}{2}}\sigma^{\frac{q\alpha}{4}})^{\frac{2}{q}}\right] \qquad \text{by (4.1) and } 1 \leq \tfrac{2}{1-\alpha} \leq 2
\end{aligned}$$

and this implies the desired inequality (4.5) by replacing $0 < \frac{q}{2} \leq \frac{1}{2}$ by q. □

Remark 4.2. *If we put $p = 2$ in* (4.4) *of Theorem 4.1, then we have*

$$NT_\alpha(\rho|\sigma) \leq -\mathrm{Tr}\left[\frac{|\sigma^\alpha \rho^{1-\alpha}| - \rho}{\alpha}\right]$$

for all $-1 \leq \alpha \leq -\frac{1}{2}$.

If we put $q = \frac{1}{2}$ in (4.5) *of Theorem 4.1, then we have*

$$NT_\alpha(\rho|\sigma) \geq -\mathrm{Tr}\left[\frac{(\rho^{\frac{1-\alpha}{2}}\sigma^{\frac{\alpha}{2}})^2 - \rho}{\alpha}\right]$$

for all $-1 \leq \alpha < 0$.

Since it follows from (4.2) *that*

$$\mathrm{Tr}[(\rho^{\frac{1-\alpha}{2}}\sigma^{\frac{\alpha}{2}})^2] = \mathrm{Tr}[(\rho^{\frac{1-\alpha}{4}}\sigma^{\frac{\alpha}{2}}\rho^{\frac{1-\alpha}{4}})^2] \leq \mathrm{Tr}[\rho^{\frac{1-\alpha}{2}}\sigma^\alpha\rho^{\frac{1-\alpha}{2}}] = \mathrm{Tr}[\rho^{1-\alpha}\sigma^\alpha],$$

unfortunately we have $\mathbf{T}_\alpha(\rho|\sigma) \geq -\mathrm{Tr}\left[\frac{(\rho^{\frac{1-\alpha}{2}}\sigma^{\frac{\alpha}{2}})^2-\rho}{\alpha}\right]$. Similarly, $\mathbf{T}_\alpha(\rho|\sigma) \leq -\mathrm{Tr}\left[\frac{|\sigma^\alpha\rho^{1-\alpha}|-\rho}{\alpha}\right]$ and hence we do not know the relation between $NT_\alpha(\rho|\sigma)$ and $\mathbf{T}_\alpha(\rho|\sigma)$ for $\alpha \in [-1, 0)$.

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