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Weak and strong convergence theorems for some classes of nonlinear mappings

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Abstract. In this paper, we study the concept of attractive points of nonlinear mappings and the concept of common attractive points of a nonexpansive semigroup. Then, we study weak convergence theorems of Baillon's type [5] for nonexpansive semigroups without convexity. We also study the concept of acute points for nonlinear mappings, and the concept of common acute points of the families of nonlinear mappings. We give weak and strong convergence theorems for a semigroup.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty subset of H. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [15] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;

(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1975, Baillon [5] proved the following first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \dfrac{1}{n} \displaystyle\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T (see also [13]).

Recently, Kocourek, Takahashi and Yao [7] introduced a wide class of nonlinear mappings called generalized hybrid. The class covers nonexpansive mappings, nonspreading mappings ([9]), and hybrid mappings ([14]). They proved a mean

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convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem. Motivated by Baillon [5], and Kocourek, Takahashi and Yao [7], Takahashi and Takeuchi [15] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for a generalized hybrid mapping.

In this paper, we study the concept of attractive points of nonlinear mappings and the concept of common attractive points of a nonexpansive semigroup. Then, we study weak convergence theorems of Baillon's type [5] for nonexpansive semigroups without convexity. We also study the concept of acute points for nonlinear mappings, and the concept of common acute points of the families of nonlinear mappings. We give weak and strong convergence theorems for a semigroup by using these concepts.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the metric projection of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [13] for more details. The following result is well-known; see also [13].

Lemma 2.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. A semitopological semigroup S is called right (resp. left) reversible if any two closed left (resp. right) ideals of S have nonvoid intersection. If S is right reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $s \leq t$

if and only if $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{St},\ s,t \in S$, where $\overline{A}$ is the closure of A.

Let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(st)$ for all $t \in S$. We also denote by ℓ_s^* the conjugate operator of ℓ_s. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. A linear functional μ on X is called a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t)d\mu(t)$ instead of $\mu(g)$ for $\mu \in X^*$ and $g \in X$. Furthermore, assume that X is invariant under every $\ell_s, s \in S$, i.e., $\ell_s X \subset X$ for each $s \in S$. Then, a mean μ on X is called invariant if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in X$. The following definition which was introduced by Takahashi [12] is crucial in the fixed point theory for abstract semigroups. Let h be a bounded function of S into H. Let X be a subspace of $B(S)$ containing 1 and invariant under every ℓ_s, $s \in S$. Assume that for each $z \in H$, the function $t \mapsto \langle h(t), z\rangle$ is an element of X. Then, for any $\mu \in X^*$ there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z\rangle = (\mu)_t \langle h(t), z\rangle = \int \langle h(t), z\rangle \, d\mu(t), \quad \forall z \in H.$$

If μ is a mean on X, then h_μ is contained in $\overline{\text{co}}\{h(t) : t \in S\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A (for example, see [12, 13]). Sometimes, h_μ will be denoted by $\int h(t)d\mu(t)$.

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a continuous representation of S as mappings on C if it satisfies the following conditions:

(i) $s \mapsto T(s)x$ is continuous;

(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$.

3. Attractive points and nonlinear ergodic theorems

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;

(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that for each $x \in C$ and $z \in H$, the weak closure of $\{T(t)x : t \in S\}$ is weakly compact and the mapping $t \mapsto \langle T(t)x, z\rangle$ is an element of X. Let μ be a mean on X. Following [11], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for all $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$.

Motivated by Takahashi and Takeuchi [15], we introduce the set $A(\mathcal{S})$ of all common attractive points of the family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings on C, i.e.,

$$A(\mathcal{S}) = \{x \in H : \|T(t)y - x\| \le \|y - x\|, \forall y \in C, t \in S\}.$$

In this section, we study a nonlinear mean ergodic theorem without convexity by using the concept of common attractive points of a nonexpansive semigroup $\mathcal{S}$ on C (see also [5, 15, 13]).

Theorem 3.1 ([3]). *Let H be a Hilbert space, let C be a nonempty subset of H. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\{T(t)x : t \in S\}$ is bounded for some $x \in C$. Let X be a subspace of $B(S)$ such that $1 \in X$, it is ℓ_s-invariant for each $s \in S$, and the function $t \mapsto \langle T(t)z, y\rangle$ is an element of X for each $z \in C$ and $y \in H$. Let $\{\mu_\alpha\}$ be a net of means on X such that $\lim_\alpha \|\mu_\alpha - \ell_s^* \mu_\alpha\| = 0$ for each $s \in S$. Let $P_{A(\mathcal{S})}$ be a metric projection of H onto $A(\mathcal{S})$. Then, the following hold:*

(1) *$A(\mathcal{S})$ is nonempty, closed and convex;*

(2) *for any $u \in C$, $\{T_{\mu_\alpha} u\}$ converges weakly to $u_0 \in A(\mathcal{S})$, where $u_0 = \lim_{t \in S} P_{A(\mathcal{S})} T(t)u$.*

Using Theorems 3.1, we get some nonliear mean ergodic theorems as in [13] (see [3]). For instance, we get the following theorem.

Theorem 3.2 ([3]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let T and U be nonexpansive mappings of C into itself such that $\{T^i U^j x : i, j = 0, 1, 2 \ldots\}$ is bounded for some $x \in C$. Let $P_{A(T) \cap A(U)}$ be the metric projection of H onto $A(T) \cap A(U)$. Then, the following hold:*

(1) *$A(T) \cap A(U)$ is nonempty, closed and convex;*

(2) *for any $u \in C$, $\{1/n^2 \sum_{i,j=0}^{n-1} T^i U^j u\}$ converges weakly to $u_0 \in A(T) \cap A(S)$.*

4. Acute points and convergence theorems

In this section, we study convergence theorems by using the concept of k-acute points of a mapping for $k \in [0,1]$. Let C be a subset of a Hilbert space H and let T be a mapping of C into H. A mapping T is said to be L-Lipschitzian if $\|Tx - Ty\| \le L\|x - y\|$ for any $x, y \in C$, where $L \in [0,\infty)$. A mapping T is said to be quasi-nonexpansive if

$$(1)\, F(T) \neq \varnothing, \quad (2) \quad \|Tx - v\| \le \|x - v\| \quad \text{for} \quad x \in C, v \in F(T).$$

Let I be the identity mapping on C. T is said to be k-pseudo-contractive if, for any $x, y \in C$,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,$$

where $k \in (0,1)$. T is said to be hemi-contractive if
(1) $F(T) \neq \varnothing$, (2) $\|Tx - v\|^2 \le \|x - v\|^2 + \|x - Tx\|^2$ for $x \in C,\ v \in F(T)$. T is said to be k-demi-contractive if

$$(1) \quad F(T) \neq \varnothing, \quad (2) \quad \|Tx-v\|^2 \le \|x-v\|^2 + k\|x-Tx\|^2 \quad \text{for} \quad x \in C,\ v \in F(T).$$

These concepts depend on the condition $F(T) \neq \varnothing$. We also call T a demi-contraction if T is a k-demi-contraction for some $k \in [0,1)$.

Let $k \in [0,1]$. We define the set of k-acute points $\mathcal{A}_k(T)$ of T by

$$\mathcal{A}_k(T) = \{\ v \in H :\ \|Tx - v\|^2 \le \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for all} \ \ x \in C\ \}.$$

We denote $\mathcal{A}_0(T)$ by $A(T)$ because $\mathcal{A}_0(T)$ and attractive points set of T are the same. We denote $\mathcal{A}_1(T)$ by $\mathcal{A}(T)$, that is,

$$\mathcal{A}(T) = \{\ v \in H :\ \|Tx - v\|^2 \le \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for all} \ \ x \in C\ \}.$$

We can get weak convergence theorems in the case $A(S) \neq \varnothing$ and $F(S) \subset \mathcal{A}(S)$ (see [4]). To have the following results, we have to assume demiclosedness at 0 of $I - S$.

THEOREM 4.1 ([4]). *Let $a, b \in (0,1)$ with $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let C be a weakly closed subset of a Hilbert space H. Let S be a self-mapping on C such that $F(S) \subset \mathcal{A}(S)$, $A(S) \neq \varnothing$, and $I - S$ is demiclosed at* 0. *Suppose* *there is a sequence* $\{u_n\}$ *in* C *such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges weakly to some* $u \in F(S)$.

Theorem 4.2 ([4]). *Let* $a, b \in (0,1)$ *with* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a,b]$. *Let* C *be a weakly closed subset of a Hilbert space* H *and* T *be a self-mapping on* C *such that* $I - T$ *is demiclosed at* 0. *Let* $k \in [0,1)$. *Assume that one of the followings hold.*

1. T *is hemi-contractive with* $A(T) \neq \varnothing$. S *is the mapping defined by* $S = T$.
2. T *is* k*-demi-contractive.* S *is the mapping defined by* $S = kI + (1-k)T$.
3. T *is quasi-nonexpansive.* S *is the mapping defined by* $S = T$.

Suppose S *is a self-mapping on* C *and there is a sequence* $\{u_n\}$ *in* C *such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges weakly to some* $u \in F(T)$.

Theorem 4.3 ([4]). *Let* $a, b \in (0,1)$ *with* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a,b]$. *Let* C *be a compact subset of a Hilbert space* H *and* T *be a continuous self-mapping on* C. *Let* $k \in [0,1)$. *Assume that one of the following holds.*

1. T *is hemi-contractive with* $A(T) \neq \varnothing$. S *is the mapping defined by* $S = T$.
2. T *is* k*-demi-contractive.* S *is the mapping defined by* $S = kI + (1-k)T$.
3. T *is quasi-nonexpansive.* S *is the mapping defined by* $S = T$.

Suppose S *is a self-mapping on* C *and there is a sequence* $\{u_n\}$ *in* C *such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } \ n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges strongly to some* $u \in F(T)$.

Now, we get a nonlinear mean ergodic theorem (see also [5]).

Theorem 4.4 ([4]). *Let* $k \in [0,1)$. *Let* C *be a bounded subset of a Hilbert space* H. *Let* T *be a* k*-strictly pseudo-contractive self-mapping on* C. *Let* S *be* *the mapping defined by* $Sx = (kI + (1-k)T)x$ *for* $x \in C$*. Assume that* S *is a self-mapping on* C*. Let* $\{v_n\}$ *and* $\{b_n\}$ *be sequences defined by* $v_1 \in C$ *and*

$$v_{n+1} = Sv_n, \quad b_n = \frac{1}{n}\sum_{t=1}^{n} v_t \qquad \text{for} \quad n \in \mathbb{N}.$$

Then the following hold.

1. $\mathcal{A}_k(T)$ *is nonempty, closed and convex.*
2. $\{b_n\}$ *converges weakly to some* $u \in \mathcal{A}_k(T)$*.*

Furthermore, if C *is closed and convex then the following hold.*

3. $F(T)$ *is nonempty, closed and convex.*
4. $\{b_n\}$ *converges weakly to* $u \in F(T)$*.*

5. Convergence theorems for a semigroup of mappings

Using the ideas in Sections 3 and 4, we get a nonlinear mean ergodic theorem for a semigroup. In this section, we study the case of nonexpansive semigroup (see also [2, 6, 13]).

THEOREM 5.1 ([2]). *Let* C *be a nonempty bounded closed subset of a Hilbert space* H *and let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a nonexpansive semigroup. Let* $x \in C$*. Then,* $\{\frac{1}{t}\int_0^t T(t)xdt\}$ *converges weakly to some* $z_0 \in \mathcal{A}_k(\mathcal{S})$*. Further, if* C *is closed and convex, then* $\{\frac{1}{t}\int_0^t T(t)xdt\}$ *converges weakly to* $z_0 \in F(\mathcal{S})$*, where* $z_0 = \lim_{t\to\infty} P_{F(\mathcal{S})}T(\mathcal{S})x$*.*

THEOREM 5.2 ([2]). *Let* $a, b \in (0,1)$ *with* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a,b]$*. Let* C *be a weakly closed convex subset of a Hilbert space* H *and let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a nonexpansive semigroup such that* $F(\mathcal{S}) \neq \emptyset$*. Let* $\{t_n\}$ *be a sequence in* $\mathbb{R}^+$ *with* $t_n \to \infty$*. Let* $\{u_n\}$ *be a sequence in* C *defined by* $u_1 \in C$ *and*

$$u_{n+1} = a_n u_n + (1-a_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds \qquad \text{for}\, n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges weakly to some* $u \in F(\mathcal{S})$*.*

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References

[1] S. Atsushiba, *Strong convergence to common attractive points of uniformly asymptotically regular nonexpansive semigroups*, J. Nonlinear Convex Anal. **16** (2015), 69-78.

[2] S. Atsushiba, *Weak and strong convergence theorems for some semigroups of nonlinear mappings*, submitted.

[3] S. Atsushiba, and W. Takahashi, *Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 209-219.

[4] S. Atsushiba, S. Iemoto, R. Kubota and Y. Takeuchi *Convergence theorems for some classes of nonlinear mappings in Hilbert spaces*, Linear and Nonlinear Analysis, **2** (2016), 125-153.

[5] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sei. Paris Ser. A-B **280** (1975), 1511 - 1514.

[6] A. Hester, C. H. Morales, *Semigroups generated by pseudo-contractive mappings under the Nagumo condition*, J. Differential Equations 245 (2008), 994–1013.

[7] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[8] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.

[9] F. Kohsaka & W. Takahashi *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[10] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[11] G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172–178.

[12] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space,* Proc. Amer. Math. Soc. **81** (1981), 253-256.

[13] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[14] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal., **11** (2010), 79-88.

[15] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

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