Josai Mathematical Monographs
Vol. 11 (2018) pp. 105–120

A shrinking projection method for generalized firmly nonexpansive mappings with nonsummable errors

Takanori IBARAKI and Shunsuke KAJIBA

Abstract. In this paper, we study an approximation method for mappings of type (P) [2], (Q) [2, 20], and (R) [2, 12] in a Banach space. Using the technique developed by Kimura and Takahashi [18] and Kimura [14], we prove strong convergence of iterative schemes generated by the shrinking projection method with errors by using the generalized projection. Moreover, using our results, we consider the problem of finding a zero of a maximal monotone operators in a Banach space.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. We say a mapping $T : C \to E$ is firmly nonexpansive [4] if

$$\|t(x-y) + (1-t)(Tx - Ty)\| \geq \|Tx - Ty\|$$

for every $x, y \in C$ and $t \geq 0$. If E is a Hilbert space, then one can show that T is firmly nonexpansive if and only if

$$\langle (x - Tx) - (y - Ty), Tx - Ty \rangle \geq 0$$

for every $x, y \in C$. One of the most important examples of this class of mappings in a real Hilbert space H is a resolvent operator $J_\lambda : H \to H$ of a maximal monotone operator $A \subset H \times H$ for $\lambda > 0$ defined by $J_\lambda = (I + \lambda A)^{-1}$. Moreover, the metric projection P_C of H onto a nonempty closed convex subset C of H is also an example of firmly nonexpansive mappings since P_C is a resolvent operator of the subdifferential of i_C, the indicator function with respect to C.

As a generalization of the resolvent operator defined on a Hilbert space, some different types of resolvents defined on a Banach space have been proposed and

2010 Mathematics Subject Classification. Primary 47H09; Secondary 47H05, 47J25.
Key Words and Phrases. fixed point, type (P), type (Q), type (R), shrinking projection method, generalized projection, error term.

studied. These notions correspond to variations of nonlinear mappings on a Banach space including firmly nonexpansive mappings. Following [2], we call them mappings of type (P) [2], (Q) [2,20], and (R) [2,12]; see the next section.

In the metric fixed point theory, approximation method of a fixed point of a nonlinear mapping is one of the most important topics and it has been rapidly developed in the recent research. In particular, the shrinking projection method proposed by Takahashi, Takeuchi, and Kubota [27] is a remarkable result (see also [23,24]).

Theorem 1.1 (Takahashi-Takeuchi-Kubota [27]). *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) = \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a[$, where $0 < a < 1$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} &= \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)} x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

We note that the original result of this theorem is a convergence theorem to a common fixed point of a family of nonexpansive mappings. Later, many researchers proposed various types of generalized result of this method (see [3, 6–9, 14–18] and others). In 2009, Kimura and Takahashi [18] obtain a strong convergence theorem for finding a common fixed point of relatively nonexpansive mappings in a Banach space by using the shrinking projection method. The method for its proof is different from the original one and it shows that the type of projection used in the iterative method is independent of the properties of the mappings.

On the other hand, in the original shrinking projection method, we needs to obtain the exact value of metric projection to generate sequence in every step and it is a task of difficulty. In 2012, Kimura [14] considers an error for obtaining the value of metric projections and prove that the sequence still has a nice property for approximating a fixed point of a mapping. Recently, Ibaraki and Kimura [9] prove strong convergence of iterative schemes for mappings of type (P), (Q), and (R) by using the technique developed by Kimura [14].

In this paper, we study an approximation method for mappings of type(P), (Q), and (R) in a Banach space. Using the technique developed by Kimura and Takahashi [18] and Kimura [14], we prove strong convergence of iterative schemes generated by the shrinking projection method with errors by using the generalized projection. The generalized projection is different type of projection used in Ibaraki and Kimura [9]. Moreover, using our results, we consider the problem of finding a zero of a maximal monotone operators in a Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. The normalized duality mapping $J : E \to E^*$ defined by

$$Jx = \{y^* \in E^* : \|x\|^2 = \langle x, y^* \rangle = \|y^*\|^2\}$$

for $x \in E$. If E is smooth, strictly convex and reflexive, then J is a single-valued bijection. Let C be a nonempty subset of a smooth Banach space E. A mapping $T : C \to E$ is said to be of type (P) [2] if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty) \rangle \geq 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (Q) [2, 20] if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy) \rangle \geq 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (R) [2, 12] if

$$\langle JTx - JTy, (x - Tx) - (y - Ty) \rangle \geq 0$$

for each $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of T is denoted by $\hat{F}(T)$. It is clear that if $T : C \to E$ is of type (P) and $F(T)$ is nonempty, then

$$\langle Tx - p, J(x - Tx) \rangle \geq 0$$

for each $x \in C$ and $p \in F(T)$. We also know that if $T : C \to E$ is of type (Q) and $F(T)$ is nonempty, then

$$\langle Tx - p, Jx - JTx \rangle \geq 0$$

for each $x \in C$ and $p \in F(T)$. If $T : C \to E$ is of type (R) and $F(T)$ is nonempty, then

$$\langle JTx - Jp, x - Tx \rangle \geq 0$$

for each $x \in C$ and $p \in F(T)$.

The following results describe the relation between the set of fixed points and that of asymptotic fixed points for each type of mapping.

Lemma 2.1 (Aoyama-Kohsaka-Takahashi [3]). *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (P). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

Lemma 2.2 (Kohsaka-Takahashi [20]). *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q). If $F(T)$ is nonempty, then $F(T)$ is closed and convex and $F(T) = \hat{F}(T)$.*

The mappings of types (Q) and (R) are strongly related to each other; it is a kind of duality in the following sense. Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty subset of E and, let T be a mapping form C into E. Define a mapping T^* as follows:

$$T^* x^* := JTJ^{-1} x^* \tag{2.1}$$

for each $x^* \in JC$, where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. We know that $JF(T) = F(T^*)$ (see [5,28]). Further, we have the following result.

Lemma 2.3 (Aoyama-Kohsaka-Takahashi [2]). *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty subset of E and let $T : C \to E$ be a mapping of type (R). Let $T^* : JC \to E^*$ be a mapping defined by* (2.1)*. Then T^* is of type (Q) in E^*.*

Let E be a smooth Banach space and consider the function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties (see [1,11,13,21]);

1. $(\|x\| - \|y\|)^2 \le V(x, y) \le (\|x\| + \|y\|)^2$ for each $x, y \in E$;
2. $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$;
3. $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$;
4. if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that

$$V(z, x) = \min_{y \in C} V(y, x).$$

Such a point z is denoted by $\Pi_C x$ and Π_C is called the generalized projection of E onto C (see [1]). We know that the generalized projection is a mapping of type (Q) (see [2, 20]). For a nonempty subset $C \subset E$ and a point $u \in E$, define $V(C, u) = \inf\{V(y, u) : y \in C\}$ and $V(u, C) = \inf\{V(u, y) : y \in C\}$.

In 2003, Ibaraki, Kimura and Takahashi [10] prove the following result for the generalized projection in a Banach space. For the exact definition of Mosco limit $\text{M-lim}_n C_n$ (see [22]).

Theorem 2.4 (Ibaraki-Kimura-Takahashi [10]). *Let E be a smooth Banach space and Let E^* have Fréchat differentiable norm, let $\{C_n\}$ be a sequence of nonempty closed convex subset of E. If $C_0 = \text{M-lim}_n C_n$ exists and nonempty, then for each $x \in E$, $\{\Pi_{C_n} x\}$ converges strongly to $\Pi_{C_0} x$.*

One of the simplest examples of the sequence $\{C_n\}$ satisfying the condition in the theorem above is a decreasing sequence with respect to inclusion; $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. In this case, $\text{M-lim}_n C_n = \bigcap_{n=1}^{\infty} C_n$.

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $R : C \to E$ is said to be generalized nonexpansive if $F(R)$ is nonempty and $V(Rx, u) \le V(x, u)$ for all $x \in C$ and $u \in F(R)$. A mapping $R : E \to C$ is said to be a retraction if $R^2 = R$. A mapping $R : E \to C$ is said to be sunny if $R(Rx + t(x - Rx)) = Rx$ for each $x \in E$ and $t \ge 0$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely decided. Then, such a sunny generalized nonexpansive retraction of E onto C is denoted by R_C. We know that the sunny generalized nonexpansive retraction is a mapping of type (R) (see [2, 11] for more details). The following lemma is well-known.

LEMMA 2.5 (KOHSAKA-TAKAHASHI [19]). *Let E be a smooth, strictly convex and reflexive Banach space and let C^* be a nonempty closed convex subset of E^*. Let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

The following results show that the existence of mappings $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$, and $\overline{g}_r^*$, related to the convex structures of a Banach space E and its dual space. These mappings play important roles in our result. For $r > 0$, define $B_r = \{x \in E : \|x\| \leq r\}$.

THEOREM 2.6 (XU [29]). *Let E be a Banach space and let $r \in]0, \infty[$. Then,*

(i) if E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\underline{g}_r : [0, 2r] \to [0, \infty[$ with $\underline{g}_r(0) = 0$ such that

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$;

(ii) if E is uniformly smooth, then there exists a continuous, strictly increasing and convex function $\overline{g}_r : [0, 2r] \to [0, \infty[$ with $\overline{g}_r(0) = 0$ such that

$$\|\alpha x + (1-\alpha)y\|^2 \geq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$.

From this theorem, we can show the following result.

THEOREM 2.7 (KIMURA [16]). *Let E be a uniformly smooth and uniformly convex Banach space and let $r \in]0, \infty[$. Then the functions $\underline{g}_r$ and $\overline{g}_r$ in Theorem 2.6 satisfies*

$$\underline{g}_r(\|x-y\|) \leq V(x, y) \leq \overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$.

Similar results for the mappings $\underline{g}_r^*$ and $\overline{g}_r^*$ also hold as follows:

THEOREM 2.8 (IBARAKI-KIMURA [9]). *Let E be a uniformly smooth and uniformly convex Banach space and let $r \in]0, \infty[$. Then there exists continuous, strictly increasing and convex functions $\underline{g}_r^*$, $\overline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\underline{g}_r^*(0) =$*

$\overline{g}_r^*(0) = 0$ *such that*

$$\underline{g}_r^*(\|Jx - Jy\|) \leq V(x,y) \leq \overline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$.

3. Approximation theorem for the mappings of type (P)

In this section, we propose an approximation theorem for a mapping of type (P), which includes the metric projections onto nonempty closed convex subset of a uniformly convex Banach space.

THEOREM 3.1. *Let E be a smooth and uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E, and let $r \in \,]0, \infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (P) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_{n\to\infty} \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C : \langle Tx_n - z, J(x_n - Tx_n)\rangle \geq 0\} \cap C_n, \\ x_{n+1} &\in \{z \in C : V(z,u) \leq V(C_{n+1}, u) + \delta_{n+1}\} \cap C_{n+1}, \end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$.

PROOF. Since C_n includes $F(T) \neq \varnothing$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $\pi_n = \Pi_{C_n} u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.4, it follows that $\{\pi_n\}$ converges strongly to $\pi_0 = \Pi_{C_0} u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. For the sake of simplicity, we may assume that δ_1 is so large that $V(x_1, u) \leq V(\pi_1, u) + \delta_1$. Then, since $V(C_n, u) = V(\pi_n, u)$, it follows that

$$V(x_n, u) \leq V(\pi_n, u) + \delta_n$$

for every $n \in \mathbb{N}$. Since C_n is closed and convex, from Theorem 2.6 (i), we have for $\alpha \in \,]0, 1[$,

$$V(\pi_n, u) \leq V(\alpha \pi_n + (1-\alpha)x_n, u)$$

$$\begin{aligned}
&= \|\alpha\pi_n + (1-\alpha)x_n\|^2 - 2\langle \alpha\pi_n + (1-\alpha)x_n, Ju\rangle + \|u\|^2 \\
&\le \alpha\|\pi_n\|^2 + (1-\alpha)\|x_n\|^2 - \alpha(1-\alpha)\underline{g}_r(\|\pi_n - x_n\|) \\
&\quad - 2\alpha\langle \pi_n, Ju\rangle - 2(1-\alpha)\langle x_n, Ju\rangle + \|u\|^2 \\
&= \alpha V(\pi_n, u) + (1-\alpha)V(x_n, u) - \alpha(1-\alpha)\underline{g}_r(\|\pi_n - x_n\|)
\end{aligned}$$

and thus

$$\alpha\underline{g}_r(\|\pi_n - x_n\|) \le V(x_n, u) - V(\pi_n, u) \le \delta_n.$$

Tending $\alpha \to 1$, we have $\underline{g}_r(\|\pi_n - x_n\|) \le \delta_n$ and thus

$$\|\pi_n - x_n\| \le \underline{g}_r^{-1}(\delta_n).$$

Using the definition of π_n, we have $\pi_{n+1} \in C_{n+1}$ and thus

$$\langle Tx_n - \pi_{n+1}, J(x_n - Tx_n)\rangle \ge 0,$$

or equivalently,

$$\langle x_n - \pi_{n+1}, J(x_n - Tx_n)\rangle \ge \|x_n - Tx_n\|^2.$$

Hence it follows that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - \pi_{n+1}\| \\
&\le \|x_n - \pi_n\| + \|\pi_n - \pi_{n+1}\| \\
&\le \underline{g}_r^{-1}(\delta_n) + \|\pi_n - \pi_{n+1}\|
\end{aligned}$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \pi_n = \pi_0$ and $\limsup_{n\to\infty} \delta_n = \delta_0$, we have

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\delta_0).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then, it follows that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(0) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r(\|x_n - \pi_n\|) \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - \pi_n\| = 0.$$

Then, by Lemma 2.1 and since $\pi_n \to \pi_0$, we have $x_n \to \pi_0 \in \hat{F}(T) = F(T)$. Since $F(T) \subset C_0$, we get that $\pi_0 = \mathit{\Pi}_{C_0} u = \mathit{\Pi}_{F(T)} u$, which completes the proof. □

4. Approximation theorem for the mappings of type (Q)

We next consider an approximation theorem for a mapping of type (Q). This type of mappings includes the generalized projections onto nonempty closed convex subset of a uniformly convex Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E, and let $r \in\,]0,\infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (Q) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_{n\to\infty} \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned}
&C_{n+1} = \{z \in C : \langle Tx_n - z, Jx_n - JTx_n\rangle \geq 0\} \cap C_n, \\
&x_{n+1} \in \{z \in C : V(z,u) \leq V(C_{n+1}, u) + \delta_{n+1}\} \cap C_{n+1},
\end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $\mathit{\Pi}_{F(T)} u$.

Proof. Since C_n includes $F(T) \neq \varnothing$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $\pi_n = \mathit{\Pi}_{C_n} u$ for all $n \in \mathbb{N}$. Then, by Theorem 2.4, it follows that $\{\pi_n\}$ converges strongly to $\pi_0 = \mathit{\Pi}_{C_0} u$, where $C_0 = \bigcap_{n=1}^{\infty} C_n$. For the sake of simplicity, we may assume that δ_1 is so large that $V(x_1, u) \leq V(\pi_1, u) + \delta_1$. Then, since $V(C_n, u) = V(\pi_n, u)$, it follows that

$$V(x_n, u) \leq V(\pi_n, u) + \delta_n$$

for every $n \in \mathbb{N}$. Since C_n is closed and convex, from Theorem 2.6 (i), we have for

$\alpha \in]0,1[$,

$$\begin{aligned}
V(\pi_n, u) &\le V(\alpha\pi_n + (1-\alpha)x_n, u) \\
&= \|\alpha\pi_n + (1-\alpha)x_n\|^2 - 2\langle \alpha\pi_n + (1-\alpha)x_n, Ju\rangle + \|u\|^2 \\
&\le \alpha\|\pi_n\|^2 + (1-\alpha)\|x_n\|^2 - \alpha(1-\alpha)\underline{g}_r(\|\pi_n - x_n\|) \\
&\quad - 2\alpha\langle \pi_n, Ju\rangle - 2(1-\alpha)\langle x_n, Ju\rangle + \|u\|^2 \\
&= \alpha V(\pi_n, u) + (1-\alpha)V(x_n, u) - \alpha(1-\alpha)\underline{g}_r(\|\pi_n - x_n\|)
\end{aligned}$$

and thus

$$\alpha\underline{g}_r(\|\pi_n - x_n\|) \le V(x_n, u) - V(\pi_n, u) \le \delta_n.$$

Tending $\alpha \to 1$, we have $\underline{g}_r(\|\pi_n - x_n\|) \le \delta_n$ and thus $\|\pi_n - x_n\| \le \underline{g}_r^{-1}(\delta_n)$. Using the definition of π_n, we have $\pi_{n+1} \in C_{n+1}$ and thus

$$\langle Tx_n - \pi_{n+1}, Jx_n - JTx_n\rangle \ge 0$$

From the property of the function V, we have

$$\begin{aligned}
0 &\le 2\langle Tx_n - \pi_{n+1}, Jx_n - JTx_n\rangle \\
&= 2\langle \pi_{n+1} - Tx_n, JTx_n - Jx_n\rangle \\
&= V(\pi_{n+1}, x_n) - V(\pi_{n+1}, Tx_n) - V(Tx_n, x_n) \\
&\le V(\pi_{n+1}, x_n) - V(Tx_n, x_n).
\end{aligned}$$

By Theorem 2.7, it follows that

$$\begin{aligned}
V(Tx_n, x_n) &\le V(\pi_{n+1}, x_n) \\
&= V(\pi_{n+1}, \pi_n) + V(\pi_n, x_n) + 2\langle \pi_{n+1} - \pi_n, J\pi_n - Jx_n\rangle \\
&\le V(\pi_{n+1}, \pi_n) + \overline{g}_r(\|\pi_n - x_n\|) + 2\|\pi_{n+1} - \pi_n\|\big(\|J\pi_n\| + \|Jx_n\|\big) \\
&\le \overline{g}_r(\|\pi_{n+1} - \pi_n\|) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 4r\|\pi_{n+1} - \pi_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty} \pi_n = \pi_0$ and $\limsup_{n\to\infty} \delta_n = \delta_0$, we have

$$\limsup_{n\to\infty} V(Tx_n, x_n) \le \overline{g}_r(\underline{g}_r^{-1}(\delta_0)).$$

Therefore, by Theorem 2.7, we have

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \limsup_{n\to\infty} \underline{g}_r^{-1}(V(Tx_n, x_n)) \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then, it follows that

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(0))) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r(\|x_n - \pi_n\|) \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - \pi_n\| = 0.$$

Then, by Lemma 2.2 and $\pi_n \to \pi_0$, we have $x_n \to \pi_0 \in \hat{F}(T) = F(T)$. Since $F(T) \subset C_0$, we get that $\pi_0 = \mathit{\Pi}_{C_0} u = \mathit{\Pi}_{F(T)} u$, which completes the proof. □

5. Approximation theorem for the mappings of type (R)

The mappings of type (R) is, in a sense, the dual of the mappings of type (Q). By using this fact, we obtain the following an approximation theorem for this mapping.

THEOREM 5.1. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty bounded subset of E with JC is closed and convex and $r \in]0, \infty[$ such that $C \subset B_r$. Let $T : C \to E$ be a mapping of type (R) such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_{n\to\infty} \delta_n$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C : \langle JTx_n - Jz, x_n - Tx_n \rangle \ge 0\} \cap C_n, \\ x_{n+1} &\in \{z \in C : V(u, z) \le V(u, C_{n+1}) + \delta_{n+1}\} \cap C_{n+1}, \end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{F(T)} u$.

PROOF. From Lemma 2.3, it follows that $T^* : JC \to E^*$ is of type (Q) in E^* with $F(T^*) \ne \varnothing$, where T^* is defined by (2.1). Put $x_n^* = Jx_n$ and $C_n^* = JC_n$ for each $n \in \mathbb{N}$. Then T^* and $\{x_n^*\}$ satisfy the conditions of Theorem 4.1 in E^*. Therefore, it follows that

$$\limsup_{n\to\infty} \|x_n^* - T^* x_n^*\| \le \underline{g}_r^{*\,-1}(\overline{g}_r^*(\underline{g}_r^{*\,-1}(\delta_0))) \tag{5.1}$$

where the functions $\underline{g}_r^*$ and $\overline{g}_r^*$ in Theorem 2.8. Moreover, if $\delta_0 = 0$, then $\{x_n^*\}$ converge strongly to $\Pi_{F(T^*)} Ju$, where $\Pi_{F(T^*)}$ is the generalized projection of E^* onto $F(T^*) = JF(T)$. From Theorems 2.7 and 2.8, we have

$$\underline{g}_r(\|Tx_n - x_n\|) \le V(Tx_n, x_n) \le \overline{g}_r^*(\|Jx_n - JTx_n\|). \tag{5.2}$$

From (5.1) and (5.2), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|Tx_n - x_n\| &\le \limsup_{n\to\infty} \underline{g}_r^{-1}(\overline{g}_r^*(\|Jx_n - JTx_n\|)) \\ &\le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*\,-1}(\overline{g}_r^*(\underline{g}_r^{*\,-1}(\delta_0))))).\end{aligned}$$

Finally, we show that $\{x_n\}$ converges strongly to $R_{F(T)}u$. Since E is uniformly smooth and uniformly convex, it follows that the duality mapping J^{-1} on E^* is continuous and $x_n = J^{-1}x_n^*$ for each $n \in \mathbb{N}$. Since $x_n^* \to \Pi_{F(T^*)}Ju$, it follows that

$$x_n = J^{-1}x_n^* \to J^{-1}\Pi_{F(T^*)}Ju = R_{F(T)}u.$$

This completes the proof. □

6. Deduced results

In the case where E is a Hilbert space, the functions $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$ and $\overline{g}_r^*$ become $\underline{g}_r = \overline{g}_r = \underline{g}_r^* = \overline{g}_r^* = |\cdot|^2$ for every $r \in \,]0,\infty[$. Therefore, as a direct consequence of Theorems 3.1, 4.1 and 5.1, we obtain the following result.

Corollary 6.1. *Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to E$ be a firmly nonexpansive mapping such that $F(T)$ is nonempty. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_{n\to\infty} \delta_n$. For a given point $u \in H$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned}&C_{n+1} = \{z \in C : \langle Tx_n - z, x_n - Tx_n\rangle \ge 0\} \cap C_n, \\ &x_{n+1} \in \{z \in C : \|u - z\|^2 \le d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1},\end{aligned}$$

for all $n \in \mathbb{N}$. *Then,*

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \le \sqrt{\delta_0}.$$

Moreover, if $\delta_0 = 0$*, then* $\{x_n\}$ *converges strongly to* $P_{F(T)}u$*, where* $P_{F(T)}$ *is the metric projection of* H *onto* $F(T)$*.*

We next consider the problem of finding a zero of a maximal monotone operator; Let E be a smooth, strictly convex, and reflexive Banach space. Then a operator $A \subset E \times E^*$ with domain $D(A) = \{z \in E : Az \neq \varnothing\}$ is said to be a monotone if $\langle x - y, x^* - y^*\rangle \ge 0$ for all $(x, x^*), (y, y^*) \in A$. A point $u \in E$ is satisfying that $0 \in Au$ is called a zero of A and the set of such points is denoted by $A^{-1}0$. A monotone operator $A \subset E \times E^*$ is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is monotone operator such that $A \subset A'$. Let $A \subset E \times E^*$, $B \subset E^* \times E$ be maximal monotone operators. Then, it is known that

$$R(I + \lambda J^{-1}A) = J^{-1}(R(J + \lambda A)) = R(I + \lambda BJ) = E$$

for all $\lambda > 0$. We also know that $\overline{D(A)}$ and $\overline{D(B)}$ are convex, where $\overline{K}$ is the closure of K. The following three single-valued mappings are well-defined for all $\lambda > 0$:

$$\begin{aligned} P_\lambda &= (I + \lambda J^{-1}A)^{-1} : \overline{D(A)} \to E; \\ Q_\lambda &= (J + \lambda A)^{-1}J : \overline{D(A)} \to E; \\ R_\lambda &= (I + \lambda BJ)^{-1} : J^{-1}\overline{D(B)} \to E. \end{aligned}$$

These mappings are called the resolvents of A or B. It is known that $F(P_\lambda) = F(Q_\lambda) = A^{-1}0$ and $F(R_\lambda) = (BJ)^{-1}0$. We also know that P_λ, Q_λ and R_λ for $\lambda > 0$ are mappings of type (P), (Q) and (R), respectively (see [2, 6, 8, 11, 25, 26] and others). Therefore, as a direct consequence of Theorems 3.1, we obtain the following result;

THEOREM 6.2. *Let* E *be a smooth and uniformly convex Banach space and let* $A \subset E \times E^*$ *be a maximal monotone operator with* $D(A)$ *being bounded. Let* $\lambda \in]0, \infty[$ *and let* $r \in]0, \infty[$ *such that* $D(A) \subset B_r$*. Let* $\{\delta_n\}$ *be a nonnegative real sequence and let* $\delta_0 = \limsup_{n\to\infty} \delta_n$*. For a given point* $u \in E$*, generate a sequence* $\{x_n\}$ *by* $x_1 = x \in \overline{D(A)}$*,* $C_1 = \overline{D(A)}$*, and*

$$C_{n+1} = \{z \in \overline{D(A)} : \langle P_\lambda x_n - z, J(x_n - P_\lambda x_n)\rangle \ge 0\} \cap C_n,$$

$$x_{n+1} \in \{z \in \overline{D(A)} : V(z,u) \le V(C_{n+1},u) + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. *If* $A^{-1}0$ *is nonempty, then*

$$\limsup_{n\to\infty} \|x_n - P_\lambda x_n\| \le \underline{g}_r^{-1}(\delta_0).$$

Moreover, if $\delta_0 = 0$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{A^{-1}0}u$.

Simlarly, the following resut is a direct consequence of Theorem 4.1.

Theorem 6.3. *Let* E *be a uniformly smooth and uniformly convex Banach space and let* $A \subset E \times E^*$ *be a maximal monotone operator with* $D(A)$ *being bounded. Let* $\lambda \in \,]0,\infty[$ *and let* $r \in \,]0,\infty[$ *such that* $D(A) \subset B_r$. *Let* $\{\delta_n\}$ *be a nonnegative real sequence and let* $\delta_0 = \limsup_{n\to\infty} \delta_n$. *For a given point* $u \in E$, *generate a sequence* $\{x_n\}$ *by* $x_1 = x \in \overline{D(A)}$, $C_1 = \overline{D(A)}$, *and*

$$C_{n+1} = \{z \in \overline{D(A)} : \langle Q_\lambda x_n - z, Jx_n - JQ_\lambda x_n\rangle \ge 0\} \cap C_n,$$
$$x_{n+1} \in \{z \in \overline{D(A)} : V(z,u) \le V(C_{n+1},u) + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. *If* $A^{-1}0$ *is nonempty, then*

$$\limsup_{n\to\infty} \|x_n - Q_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{A^{-1}0}u$.

Finally, we can show the following result from Theorem 5.1.

Theorem 6.4. *Let* E *be a uniformly smooth and uniformly convex Banach space and let* $B \subset E^* \times E$ *be a maximal monotone operator with* $D(B)$ *being bounded. Let* $\lambda \in \,]0,\infty[$ *and let* $r \in \,]0,\infty[$ *such that* $D(BJ) \subset B_r$. *Let* $\{\delta_n\}$ *be a nonnegative real sequence and let* $\delta_0 = \limsup_{n\to\infty} \delta_n$. *For a given point* $u \in E$, *generate a sequence* $\{x_n\}$ *by* $x_1 = x \in \overline{D(BJ)}$, $C_1 = \overline{D(BJ)}$, *and*

$$C_{n+1} = \{z \in \overline{D(BJ)} : \langle JR_\lambda x_n - Jz, x_n - R_\lambda x_n\rangle \ge 0\} \cap C_n,$$
$$x_{n+1} \in \{z \in \overline{D(BJ)} : V(u,z) \le V(u,C_{n+1}) + \delta_{n+1}\} \cap C_{n+1},$$

for all $n \in \mathbb{N}$. *If* $B^{-1}0$ *is nonempty, then*

$$\limsup_{n\to\infty} \|x_n - R_\lambda x_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}u$.

Proof. In the setting of Theorem 6.4, it is easy to see that $J^{-1}D(B) = D(BJ)$ and $J^{-1}\overline{D(B)} = \overline{D(BJ)}$ (see, for example, [8]). So, we obtain the desired result by Theorem 5.1. $\square$

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.

[2] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties*, J. Nonlinear Convex Anal., **10** (2009), 131–147.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Strong convergence theorems for a family of mappings of type (P) and applications*, Proceedings of Asian Conference on Nonlinear Analysis and Optimization, Yokohama Publishers, 2009, 1–17.

[4] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math., **47** (1973), 341–355.

[5] T. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications*, Int. J. Math. Stat., **6** (2010), 46–64.

[6] T. Ibaraki, *Approximation of a zero point of monotone operators with nonsummable errors*, Fixed Point Theory Appl., **2016**:48 (2016), 14 pages.

[7] T. Ibaraki, *Iterative approximation with errors of zero points of maximal monotone operators in a Hilbert space*, Linear Nonlinear Anal., **3** (2017), 171–178.

[8] T. Ibaraki, *Iterative approximation with errors of zero points of monotone operators in a Banach space*, Yokohama Math. J., to appear.

[9] T. Ibaraki and Y. Kimura, *Approximation of a fixed point of generalized firmly nonexpansive mappings with nonsummable errors*, Linear Nonlinear Anal., **2** (2016), 301–310.

[10] T. Ibaraki, Y. Kimura and W. Takahashi, *Convergence theorems for generalized projections and maximal monotone operaters in Banach spaces*, Abstra. appl. Anal., **2003** (2003), 621–629.

[11] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory, **149** (2007), 1–14.

[12] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal., **10** (2009), 21–32.

[13] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13** (2002), 938–945.

[14] Y. Kimura, *Approximation of a fixed point of nonexpansive mapping with nonsummable errors in a geodesic space*, Proceedings of the 10th International Conference on Fixed Point Theory and Its Applications, 2012, pp. 157–164.

[15] Y. Kimura, *Approximation of a fixed point of nonlinear mappings with nonsummable errors in a Banach space*, Proceedings of the International Symposium on Banach and Function Spaces IV (Kitakyushu, Japan), (L. Maligranda, M. Kato, and T. Suzuki eds.), 2014, pp. 303–311.

[16] Y. Kimura, *Approximation of a common fixed point of a finite family of nonexpansive mappings with nonsummable errors in a Hilbert space*, J. Nonlinear Convex Anal., **15** (2014), 429–436.

[17] Y. Kimura, K. Nakajo, and W. Takahashi, *Strongly convergent iterative schemes for a sequence of nonlinear mappings*, J. Nonlinear Convex Anal., **9** (2008), 407–416.

[18] Y. Kimura and W. Takahashi, *On a hybrid method for a family of relatively nonexpansive mappings in a Banach space*, J. Math. Anal. Appl., **357** (2009), 356–363.

[19] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal type algorithm in Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 197–209.

[20] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly non-expansive type mappings in Banach spaces*, SIAM J. Optim., **19** (2008), 824–835.

[21] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory, **134** (2005), 257–266.

[22] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. in Math., **3** (1969), 510–585.

[23] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[24] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. Ser. A. **87** (2000), 189–202.

[25] W. Takahashi, *Nonlinear Functional Analysis – Fixed Point Theory and Its Applications*, Yokohama Publishers, 2000.

[26] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000. (in Japanese)

[27] W. Takahashi, Y Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **341** (2008), 276–286.

[28] W. Takahashi and J.-C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math., **15** (2011), 787–818.

[29] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138.

Takanori IBARAKI

Department of Mathematics Education, Yokohama National University

Tokiwadai, Hodogaya-ku, Yokohama, 240-8501, Japan

E-mail: ibaraki@ynu.ac.jp

Shunsuke KAJIBA

Graduate School of Education, Yokohama National University

Tokiwadai, Hodogaya-ku, Yokohama, 240-8501, Japan

E-mail: kajiba-shunsuke-zg@ynu.jp