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On classification of commutative Banach algebras and Banach modules

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Abstract. In this paper, we explain the problem of classifying semisimple commutative Banach algebras from the viewpoint of the BSE and BED properties, and give a prospect for the same type of classification of Banach modules over a semisimple commutative Banach algebra.

1. Explanation on classification of commutative Banach algebras

1.1. Commutative Banach algebras

Commutative Banach algebras are a broad world. I learned this world when I was a fourth-year or graduate student at Niigata University. The motivation was that I was told to read Rickart's book "General Theory of Banach Algebras" at the seminar of my honorable supervisor Prof. Eiichiro Honma. After that, I was instructed to read Dixmier's book "Les C*-algèbres et leurs représentations" and had been staying long in this world.

1.2. BSE-algebras

Time passed to the early 1980s, and for a commutative Banach algebra A I came to vaguely consider the problem of characterizing the image $\widehat{A}$ of the Gelfand transform of A and the image $\widehat{M}(A)$ of the Helgason-Wang transform of multiplier algebra $M(A)$. Afterwards, I learned the Bochner-Schoenberg-Eberlein theorem which characterizes the image of the Fourier-Stieltjes transform of the measure algebra appearing in harmonic analysis. Based on the belief that

"A good theorem can be a definition in itself",

I modified this theorem so that it fits into the world of commutative Banach algebras, and referred to a commutative Banach algebra as a *BSE-algebra* if it satisfies a natural property called *BSE* extracted from the theorem.

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Research on the BSE-algebras will provide a solution to the problem of characterizing the above-mentioned image $\widehat{M}(A)$ of the Helgason-Wang transform. Also, the Bochner-Schoenberg-Eberlein theorem states, in a word, that every group algebra is a BSE-algebra. Of course, every commutative C*-algebra is also a BSE-algebra, but at that time I was worried about whether there is such an algebra other than those. I then asked Prof. Osamu Hatori whether any disc algebra is a BSE-algebra or not and he finely answered "Yes". This encouraged me to submit these results to Proceedings of the AMS as a joint paper with him ([9]). Later on, Kaniuth and Ülger showed that every uniform algebra is a BSE-algebra ([8]). Therefore, it is anticipated that the world of BSE-algebras is broader than was expected, and indeed nowadays a number of BSE-algebras have been found.

However, not surprisingly, there was a pioneer named Birtel, who had already published a paper ([1]) 28 years earlier than us, suggesting that the BSE property also makes a sense for semisimple commutative Banach algebras, despite that he did not use the terminology BSE. As far as I know, since the appearance of the BSE, nobody has pointed out this fact so far. The other day I have learned it from Prof. Jyunji Inoue (Professor emeritus at Hokkaido University).

1.3. BED-algebras

Time further passed to the end of 1992, and Prof. Inoue invited me to eat the Genghis Khan dish at the Sapporo Beer Garden after the first seminar on function space at Hokkaido University. I had planned to return home at Funabashi by a night sleeper train "Hokutosei" departing around 22:30 and so I had enough time. I remember that I drank draft beer very much because the menu was "all-you-can-drink" and I was so young. When I was very drunk, Prof. Inoue told me that there was a paper that characterizes the image of the Fourier transform of a group algebra. I took an interest in this and then asked him to send the paper to my home at Funabashi. At that time e-mails by a computer were not so popular as nowadays. The paper arrived at my home on New Year's day and it turned out that the paper was authored by R. Doss ([2]) and characterizes the image of the Fourier transform of a group algebra in terms of a kind of continuity. I tried hard to modify this theorem so that it can be applied to general commutative Banach algebras. Then, based on the principle à la Kepler

"Truth likes incompleteness",

I reached the concept of *quasi-topology.* I do not go into the details here, but it can be said that if we find a quasi-topology $\mathcal{Q}$ such that $\widehat{A} = C(\Phi_A; \mathcal{Q})$ and $\mathcal{Q}_0 < \mathcal{Q}$, then it is a more precise result which characterizes $\widehat{A}$. This is a fundamental idea, but research on quasi-topology has not progressed so much, although there are a lot of problems to be solved.

From the subsequent paper of Doss ([3]), a new quasi-topology $\mathcal{Q}^0_{BSE}$ was discovered by Prof. Inoue's excellent view. We then called an algebra induced from the new quasi-topology a *BED-algebra*. It follows from the main theorem in [3] that every group algebra is a BED-algebra. Obviously, study on the BED-algebras will give a solution to the problem of characterizing the above-mentioned image $\widehat{A}$ of the Gelfand transform.

1.4. Classification problems

The problem is to elucidate the relationship between BSE-algebras and BED-algebras. In fact, we can see that they are equivalent in some class of semisimple commutative Banach algebras. On the other hand, some opposite cases are found. I then felt interested in classifying the semisimple commutative Banach algebras from the viewpoint of the BSE and BED properties. Let us denote by $\mathcal{B}_{sc}$ the class of all semisimple commutative Banach algebras. Furthermore, in this class, let us denote by $\mathcal{B}^1_{sc}, \mathcal{B}^2_{sc}, \mathcal{B}^3_{sc}$, and $\mathcal{B}^0_{sc}$ the subclasses of semisimple commutative Banach algebras which are

BSE and BED,

BSE and not BED,

not BSE and BED, and

not BSE and not BED,

respectively.

Then we have the disjoint union

$$\mathcal{B}_{sc} = \mathcal{B}^1_{sc} \cup \mathcal{B}^2_{sc} \cup \mathcal{B}^3_{sc} \cup \mathcal{B}^0_{sc}.$$

Of course, two algebras in different subclasses are not isomorphic to each other as Banach algebras. I introduced the concept of general Segal algebras and investigated specific Banach algebras in each subclass, but the current situation is far from complete determination.

By the way, if A and B are semisimple commutative Banach algebras, then so is the direct product $A \times B$. The property of $A \times B$ is shown in Table 1.

We can see that this table corresponds to a special semilattice of order 4 where $\mathcal{B}^1_{sc}$ is the identity element and $\mathcal{B}^0_{sc}$ is the zero element. I believe that the table will play a role in solving classification problems.

Now a semisimple commutative Banach algebra A is of *type I* if A satisfies $\widehat{M(A)} = C^b(\Phi_A)$, otherwise A is of *type II*, where Φ_A denotes the Gelfand space of A and $C^b(\Phi_A)$ the set of all complex-valued bounded continuous functions over Φ_A.

Table 1. Property of $A \times B$

$B \backslash A$	$\mathcal{B}_{sc}^1$	$\mathcal{B}_{sc}^2$	$\mathcal{B}_{sc}^3$	$\mathcal{B}_{sc}^0$
$\mathcal{B}_{sc}^1$	$\mathcal{B}_{sc}^1$	$\mathcal{B}_{sc}^2$	$\mathcal{B}_{sc}^3$	$\mathcal{B}_{sc}^0$
$\mathcal{B}_{sc}^2$	$\mathcal{B}_{sc}^2$	$\mathcal{B}_{sc}^2$	$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^0$
$\mathcal{B}_{sc}^3$	$\mathcal{B}_{sc}^3$	$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^3$	$\mathcal{B}_{sc}^0$
$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^0$	$\mathcal{B}_{sc}^0$

Motivated by the discovery ([6]) of a BSE-algebra of type I which is not isomorphic to any C*-algebra, classification of Banach algebras of type I has been activated. It also turned out that these algebras have the same table as Table 1 for the direct product and there is no Banach algebra of type I in $\mathcal{B}_{sc}^1$ except for commutative C*-algebras. However, for the other class, although we succeeded in giving a subclass, it appears that the current status is far from complete determination.

We refer the reader to [4] for BSE-algebras and BED-algebras, [5] for general Segal algebras, and [7] for classification of semisimple commutative Banach algebras of type I.

2. Prospect for classification of Banach modules

In [10], we defined the BSE property for a Banach module over a semisimple commutative Banach algebra having a bounded approximate identity. However, we can define it even if the Banach algebra does not have a bounded approximate identity, and recently we have defined the BED property as in the case of semisimple commutative Banach algebras. Therefore, if we fix a semisimple commutative Banach algebra A, we apparently can classify Banach modules over A from the BSE and BED properties, but indeed it might be difficult. For example, if A is a commutative C*-algebra having a discrete Gelfand space, any Banach module over A is BSE (see Theorem 3.4 in [10]). Although I have not yet proved it, I think that it is probably BED too. What happens if the Gelfand space is not discrete? For example, is there any Banach module over $C_0(\mathbf{R})$ that is not BSE nor BED? I think that we should begin with tackling these questions.

Another difficulty exists in how we define a continuous vector field. The continuous vector field of a C*-algebra is defined in association with the Stone-Weierstrass problem of non-commutative C*-algebras, but it cannot be mimicked in our case. Thus, I gave perhaps the most natural definition in [10], which is expected to have a natural structure in the case where each fiber is one-dimensional. However, this is not clear and I am struggling to find a solution to the problem.

I have mentioned various things, but research on this topic has just begun and unfortunately I cannot even see a clear prospect for that.

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