The Josai Journal of Business Administration (2013), Vol. 10, No.1, 17-28


# Redundancy in the Optimization of Labor Disposal and Working Efficiency

**Toshitake Kohmura**
*Josai University*

**Meilong Le**
*Shanghai Jiaotong University and Shanghai Maritime University*

## Abstract

In order to meet the fluctuation of demand, enterprises dispose a number of superfluous employees for their operation. The superfluous disposal of employees is inevitable in actual labor management. As a result of the superfluity, redundancy arises in the optimization of labor disposal. With regard to the redundancy in the optimization of labor disposal, the origin of the redundant solutions of the coupled inequalities for a mathematical model of labor disposal is clarified. Some methods to remove the redundant solutions of the coupled inequalities are proposed.

**Key Words:** Redundancy in Optimization, Superfluous labor disposal, Solutions of coupled inequalities

## 1. Mathematical Analysis of Coupled Inequalities

The demand for product fluctuates. It is difficult to predict the future demand quantity. Enterprises endeavor to meet the fluctuating demand by disposing more employees than the needed for the operation. The superfluous disposal of employees creates expenses and makes the labor operation less efficient[1]. Therefore, the optimization of labor disposal management is a substantial task in corporate administration to save costs incurred in the superfluous disposal of employees.

Redundancy problems are often seen in the optimization of labor management. In this paper, we will clarify the mathematical origins of the redundancy arising in the optimization of labor disposal management and propose some methods to remove the redundancy in the optimization.

In this section for the purpose of elucidating the mathematical structure of the labor disposal problem, which will be discussed in Section 2, we use some simplified mathematical models in terms of coupled inequalities for non-negative integral variables. The solution of the coupled inequalities should satisfy the condition that the sum of the variables gets a minimum value. Analyzing simple mathematical problems, we will show that the sets of coupled inequalities yield redundant solutions. The mathematical origin of the redundancy is discussed. In Section 2, we introduce a more realistic mathematical model for the labor disposal problem and discuss redundancy in the optimization of labor disposal management.

### 1) *Mathematical Model free from Redundancy of Solution*

In this Section, we take some simplified mathematical models in terms of coupled inequalities for two non-negative integral variables $x_1$ and $x_2$. The solution of the coupled inequalities should satisfy the condition that the sum of $x_1$ and $x_2$ gets a minimum value. In Subsection 1), we take the following coupled inequalities which are lower bounded evenly. We will see that the solution of the coupled inequalities is free from redundancy.

$$2x_1 + x_2 \geq 3,$$
$$x_1 + 2x_2 \geq 3. \quad (1)$$

In order to obtain the minimum value of the sum of $x_1$ and $x_2$, adding the above two inequalities, we get the inequality,

$$3(x_1 + x_2) \geq 6, \quad (2)$$

which leads to the sum of the solution $x_1$ and $x_2$ to be lower bounded as

$$x_1 + x_2 \geq 2. \quad (3)$$

We see that the sum of $x_1$ and $x_2$ yields the minimum value equal to 2, when the left hand sides of the two inequalities (1) take the lowest boundary values. Therefore, replacing the inequalities (1) by the equations, we get the following coupled equations,

$$2x_1 + x_2 = 3,$$
$$x_1 + 2x_2 = 3, \quad (4)$$

which yield the solution $x_1 = 1$ and $x_2 = 1$ of the coupled inequalities (1) with the minimum value of the sum of $x_1$ and $x_2$ equal to 2. The non-negative integral solution $x_1 = 1$ and $x_2 = 1$ of the coupled equations (4) is the unique solution of the coupled inequalities (1) which satisfies the condition that the sum of non-negative integers $x_1$ and $x_2$ takes the minimum value. Therefore, it can be seen that the solution of the inequalities (1) is free from redundancy.

### 2) *Mathematical Model involved with Redundancy of Solution*

Now, we proceed to a more complicated set of coupled inequalities, which are unevenly lower bounded as

$$2x_1 + x_2 \geq 5,$$
$$x_1 + 2x_2 \geq 1. \quad (5)$$

Adding the above two inequalities, we get the inequality,

$$3(x_1 + x_2) \geq 6. \quad (6)$$

We see that the sum of the solution $x_1$ and $x_2$ for the coupled inequalities (5) is lower bounded as

$$x_1 + x_2 \geq 2. \tag{7}$$

Therefore, the solution of the coupled inequalities (5) with the minimum sum of $x_1$ and $x_2$ equal to 2 is obtained by solving the coupled equations,

$$\begin{aligned} 2x_1 + x_2 &= 5, \\ x_1 + 2x_2 &= 1. \end{aligned} \tag{8}$$

The $x_2$ of the solution $x_1 = 3$ and $x_2 = -1$ of the coupled equations (8) is, however, a negative integer. This solution does not satisfy the condition that the variables $x_1$ and $x_2$ are non-negative integers.

Therefore, we modify the coupled equations (8) so as the sum of solution $x_1$ and $x_2$ is increased to be 3, which is bigger by 1 than the sum of $x_1$ and $x_2$ equal to 2 for the solution of the coupled equations (8). This procedure to increase the sum $x_1 + x_2$ to be 3 leads to the relation,

$$3(x_1 + x_2) = 9 \tag{9}$$

in place of the inequality (6). The above relationship brings about the following four sets of coupled equations which satisfy the inequalities (5);

$$\begin{aligned} 2x_1 + x_2 &= 5, \\ x_1 + 2x_2 &= 4, \end{aligned} \tag{10}$$

$$\begin{aligned} 2x_1 + x_2 &= 6, \\ x_1 + 2x_2 &= 3, \end{aligned} \tag{11}$$

$$\begin{aligned} 2x_1 + x_2 &= 7, \\ x_1 + 2x_2 &= 2, \end{aligned} \tag{12}$$

and

$$\begin{aligned} 2x_1 + x_2 &= 8, \\ x_1 + 2x_2 &= 1. \end{aligned} \tag{13}$$

The solution of the coupled equations (10) is $x_1 = 2$ and $x_2 = 1$, that of (11) is $x_1 = 3$ and $x_2 = 0$, that of (12) is $x_1 = 4$ and $x_2 = -1$ and that of (13) is $x_1 = 5$ and $x_2 = -2$. These solutions $x_1$ and $x_2$ of the four sets of coupled equations from (10) to (13) have the same value of the sum of $x_1$ and $x_2$ equal to 3.

Thus we have seen that the coupled inequalities (5) which are lower bounded unevenly, yield some redundant solutions. The $x_2 = -1$ of the solution of the coupled equations (12) and the $x_2 = -2$ of the equations (13) are negative integers. Therefore the cases of the sets (12) and (13) of the coupled equations are not acceptable for the solution of the coupled inequalities ( 5 ) of non-negative integers $x_1$ and $x_2$. The solutions of the coupled equation sets (10) and (11) are the solutions of the coupled inequalities ( 5 ) which satisfy the conditions.

### 3) *Mathematical Origin of Redundancy*

In this subsection, we will analyze how redundant the solutions of coupled inequalities are. In order to clarify the origin of the redundancy of the solution of the coupled inequalities, we express the inequalities in a generalized way as

$$\begin{aligned} 2x_1+x_2 &\geq a_{10}, \\ x_1+2x_2 &\geq a_{20}. \end{aligned} \tag{14}$$

The sum of the two variables $x_1$ and $x_2$ is lower bounded as

$$x_1+x_2 \geq \frac{1}{3}(a_{10}+a_{20}). \tag{15}$$

By replacing the values of left hand side of inequalities (14) by the lowest boundary values, we obtain the coupled equations,

$$\begin{aligned} 2x_1+x_2 &= a_{10}, \\ x_1+2x_2 &= a_{20}, \end{aligned} \tag{16}$$

which yield the solution,

$$\begin{aligned} x_1 &= \frac{1}{3}(2a_{10}-a_{20}), \\ x_2 &= \frac{1}{3}(2a_{20}-a_{10}). \end{aligned} \tag{17}$$

This is the solution of the coupled inequalities (14) with the minimum value of the sum,

$$x_1+x_2 = \frac{1}{3}(a_{10}+a_{20}). \tag{18}$$

In the case where the inequalities (14) are lower bounded unevenly, i.e., the boundary values $a_{10}$ and $a_{20}$ of the inequalities are not equal to one another ($a_{20} < \frac{1}{2}a_{10}$, for example), either solution $x_1$ or $x_2$ of the coupled equations (16) may be a negative integer, which is not acceptable for a non-negative solution of the coupled inequalities (14).

In case that one of $x_1$ and $x_2$ is a negative integer, the coupled equations (16) are replaced by the coupled equations,

$$2x_1 + x_2 = a_1,$$
$$x_1 + 2x_2 = a_2 \qquad (19)$$

with the values modified as

$$a_1 = a_{10} + \Delta a_1,$$
$$a_2 = a_{20} + \Delta a_2,$$
$$\Delta a_1 + \Delta a_2 = 3n \qquad (20)$$

with non-negative integers $\Delta a_1$ and $\Delta a_2$, and a positive integer $n$. With the values $a_{10}$ and $a_{20}$ of the two coupled equations (16) increased by $\Delta a_1$ and $\Delta a_2$, respectively, the sum of $x_1$ and $x_2$ is increased by $n$ as

$$x_1 + x_2 = \frac{1}{3}(a_1 + a_2) = \frac{1}{3}(a_{10} + a_{20}) + n. \qquad (21)$$

In order to obtain non-negative solutions for the coupled equations (19), let us take $n = 1$ at first. Then, the number of the choices of two non-negative integers $\Delta a_1$ and $\Delta a_2$ satisfying the relation (20) is the combination with repetitions[2] ${}_2H_3 = {}_4C_3 = 4$ of two non-negative integers $\Delta a_1$ and $\Delta a_2$ for the divisions of the integer 3 into 2 groups, i.e., $(\Delta a_1, \Delta a_2) =$ (3, 0), (2, 1), (1, 2) and (0, 3). In Subsection 2), concerned with the coupled inequalities (5) with $a_{10} = 5$ and $a_{20} = 1$, we have presented the four sets of coupled equations from (10) to (13), which stand for the cases of $(\Delta a_1, \Delta a_2) =$ (0, 3), (1, 2), (2, 1) and (3, 0), respectively.

If all the sets of the coupled equations (19) with the values $a_1$ and $a_2$ modified from $a_{10}$ and $a_{20}$ by $n = 1$ still bring about only the solutions of the coupled inequalities of which either the solution $x_1$ or $x_2$ is a negative integer, we proceed to the manipulation with $n = 2$.

The plural choices of the $\Delta a_1$ and $\Delta a_2$ available for a same minimum value of the sum of $x_1$ and $x_2$ with $n \geq 1$ originate the redundancy of the solution of the coupled inequalities (14).

## 2. Labor Disposal Management

An enterprise may practice the following labor disposal management: the laborers work seven days a week including weekends. Each employee works for five continuous days a week. The employees are disposed into seven groups, each starting work on a given day of the week. The number of employees required for working on each day of week is given. The given numbers of employees required on days of the week are in an uneven distribution. The given uneven distribution of the number of employees required to work on each day of the week stands for a mathematical model for the fluctuation of demand. The uneven distribution of the required numbers of employees imposes a problem of how employees are divided into seven groups to satisfy the numbers of employees required on each day of the week. We will solve the problem of how to optimize this labor disposal.

In this section, we will show that the optimization for minimizing the total number of employees gives rise to a redundancy of the solution. We will present some examples of redundant solutions for a given case and propose some methods to optimize labor disposal.

We will solve the coupled inequalities, each of which indicates the number of employees required for working on a given day of the week,

$$\begin{aligned} x_1 \quad\quad\quad + x_4 + x_5 + x_6 + x_7 &\geq a_{10}, \\ x_1 + x_2 \quad\quad\quad + x_5 + x_6 + x_7 &\geq a_{20}, \\ x_1 + x_2 + x_3 \quad\quad\quad + x_6 + x_7 &\geq a_{30}, \\ x_1 + x_2 + x_3 + x_4 \quad\quad\quad + x_7 &\geq a_{40}, \\ x_1 + x_2 + x_3 + x_4 + x_5 &\geq a_{50}, \\ x_2 + x_3 + x_4 + x_5 + x_6 &\geq a_{60}, \\ x_3 + x_4 + x_5 + x_6 + x_7 &\geq a_{70}, \end{aligned} \tag{22}$$

where a non-negative integer $x_i$ stands for the number of employees in the group who start working on day $i$ of week. From the inequalities (22), we see that the accumulated number of employees is lower bounded as

$$\sum_{i=1}^{7} x_i \geq \frac{1}{5} \sum_{i=1}^{7} a_{i0}. \tag{23}$$

We will analyze the labor disposal problem in a case where the minimum numbers $a_{i0}$ of employees required for working on day $i$ are given by an uneven distribution as

$$\begin{aligned} a_{10} &= 17, \\ a_{20} &= 13, \\ a_{30} &= 15, \\ a_{40} &= 19, \\ a_{50} &= 14, \\ a_{60} &= 16, \\ a_{70} &= 11. \end{aligned} \tag{24}$$

This uneven distribution of the numbers of employees required on each day stands for a mathematical model for the fluctuation of demand.

If the number of employees who work on day $i$ is settled to be $a_i$, the coupled inequalities (22) are replaced by the coupled equations,

$$\begin{aligned} x_1 \quad\quad\quad + x_4 + x_5 + x_6 + x_7 &= a_1, \\ x_1 + x_2 \quad\quad\quad + x_5 + x_6 + x_7 &= a_2, \\ x_1 + x_2 + x_3 \quad\quad\quad + x_6 + x_7 &= a_3, \\ x_1 + x_2 + x_3 + x_4 \quad\quad\quad + x_7 &= a_4, \\ x_1 + x_2 + x_3 + x_4 + x_5 &= a_5, \\ x_2 + x_3 + x_4 + x_5 + x_6 &= a_6, \\ x_3 + x_4 + x_5 + x_6 + x_7 &= a_7, \end{aligned} \tag{25}$$

where the numbers $a_i$ of employees who work on day $i$ should satisfy the required conditions in (22),

$$
\begin{aligned}
a_1 &\geq a_{10},\\
a_2 &\geq a_{20},\\
a_3 &\geq a_{30},\\
a_4 &\geq a_{40},\\
a_5 &\geq a_{50},\\
a_6 &\geq a_{60},\\
a_7 &\geq a_{70}.
\end{aligned}
\tag{26}
$$

The equations (25) lead to the total number $\sum_i x_i$ of employees related to the sum of numbers $a_i$ of employees required on day $i$ as

$$5\times(x_1+x_2+x_3+x_4+x_5+x_6+x_7) = a_1+a_2+a_3+a_4+a_5+a_6+a_7. \tag{27}$$

The coupled equations (22) yield a unique solution,

$$
\begin{aligned}
x_1 &= \frac{3}{5}(a_1+a_3+a_5)-\frac{2}{5}(a_7+a_2+a_4+a_6),\\
x_2 &= \frac{3}{5}(a_2+a_4+a_6)-\frac{2}{5}(a_1+a_3+a_5+a_7),\\
x_3 &= \frac{3}{5}(a_3+a_5+a_7)-\frac{2}{5}(a_2+a_4+a_6+a_1),\\
x_4 &= \frac{3}{5}(a_4+a_5+a_1)-\frac{2}{5}(a_3+a_5+a_7+a_2),\\
x_5 &= \frac{3}{5}(a_5+a_7+a_2)-\frac{2}{5}(a_4+a_6+a_1+a_3),\\
x_6 &= \frac{3}{5}(a_6+a_1+a_3)-\frac{2}{5}(a_5+a_7+a_2+a_4),\\
x_7 &= \frac{3}{5}(a_7+a_2+a_4)-\frac{2}{5}(a_6+a_1+a_3+a_5).
\end{aligned}
\tag{28}
$$

First we solve the coupled equations (25) for the case where the numbers $a_i$ of employees who work on day $i$ are the lowest boundaries $a_{i0}$ required by the inequalities (26), i.e.,

$$
\begin{aligned}
a_1 &= a_{10} = 17,\\
a_2 &= a_{20} = 13,\\
a_3 &= a_{30} = 15,\\
a_4 &= a_{40} = 19,\\
a_5 &= a_{50} = 14,\\
a_6 &= a_{60} = 16,\\
a_7 &= a_{70} = 11.
\end{aligned}
\tag{29}
$$

The solution of the coupled equations (25) in this case is

$$
\begin{aligned}
x_1 &= 4, \\
x_2 &= 6, \\
x_3 &= -2, \\
x_4 &= 10, \\
x_5 &= -4, \\
x_6 &= 6, \\
x_7 &= 1,
\end{aligned} \tag{30}
$$

where the numbers $x_3$ and $x_5$ of employees in the groups 3 and 5 are negative integers, i.e., an unacceptable solution. The $x_i$'s of the solution in this unrealistic case are expressed as $x_{i0}$'s, i.e.,

$$
\begin{aligned}
x_{10} &= 4, \\
x_{20} &= 6, \\
x_{30} &= -2, \\
x_{40} &= 10, \\
x_{50} &= -4, \\
x_{60} &= 6, \\
x_{70} &= 1.
\end{aligned} \tag{31}
$$

The accumulated number of employees required in the case (29) is

$$a_{10}+a_{20}+a_{30}+a_{40}+a_{50}+a_{60}+a_{70} = 105, \tag{32}$$

which indicates that the sum of $x_{i0}$'s,

$$x_{10}+x_{20}+x_{30}+x_{40}+x_{50}+x_{60}+x_{70} = 21. \tag{33}$$

Taking away the unrealistic solution (30), we search for realistic cases in which the numbers $x_i$ of employees in the seven groups are non-negative integers. For the purpose of getting realistic coupled equations (25), we increase the numbers $a_i$ of employees working on Day $i$ by $\Delta a_i$ ($\geq 0$) from $a_{i0}$ in equations (24) so as

$$
\begin{aligned}
a_1 &= a_{10}+\Delta a_1, \\
a_2 &= a_{20}+\Delta a_2, \\
a_3 &= a_{30}+\Delta a_3, \\
a_4 &= a_{40}+\Delta a_4, \\
a_5 &= a_{50}+\Delta a_5, \\
a_6 &= a_{60}+\Delta a_6, \\
a_7 &= a_{70}+\Delta a_7.
\end{aligned} \tag{34}
$$

Owing to the equations (28), the $a_i$'s increased by $\Delta a_i$ modify the $x_i$'s from $x_{i0}$ by the variations $\Delta x_i = x_i - x_{i0}$ as

$$\begin{aligned}
\Delta x_1 &= \frac{3}{5}(\Delta a_1+\Delta a_3+\Delta a_5)-\frac{2}{5}(\Delta a_7+\Delta a_2+\Delta a_4+\Delta a_6),\\
\Delta x_2 &= \frac{3}{5}(\Delta a_2+\Delta a_4+\Delta a_6)-\frac{2}{5}(\Delta a_1+\Delta a_3+\Delta a_5+\Delta a_7),\\
\Delta x_3 &= \frac{3}{5}(\Delta a_3+\Delta a_5+\Delta a_7)-\frac{2}{5}(\Delta a_2+\Delta a_4+\Delta a_6+\Delta a_1),\\
\Delta x_4 &= \frac{3}{5}(\Delta a_4+\Delta a_6+\Delta a_1)-\frac{2}{5}(\Delta a_3+\Delta a_5+\Delta a_7+\Delta a_2),\\
\Delta x_5 &= \frac{3}{5}(\Delta a_5+\Delta a_7+\Delta a_2)-\frac{2}{5}(\Delta a_4+\Delta a_6+\Delta a_1+\Delta a_3),\\
\Delta x_6 &= \frac{3}{5}(\Delta a_6+\Delta a_1+\Delta a_3)-\frac{2}{5}(\Delta a_5+\Delta a_7+\Delta a_2+\Delta a_4),\\
\Delta x_7 &= \frac{3}{5}(\Delta a_7+\Delta a_2+\Delta a_4)-\frac{2}{5}(\Delta a_6+\Delta a_1+\Delta a_3+\Delta a_5).
\end{aligned} \tag{35}$$

The relation (27) in terms of integers $x_i$ indicates that the sum of $a_i$'s should be a multiple of 5, i.e., the variation of the sum of $a_i$'s,

$$\begin{aligned}
&\Delta a_1+\Delta a_2+\Delta a_3+\Delta a_4+\Delta a_5+\Delta a_6+\Delta a_7\\
&= 5\times(\Delta x_1+\Delta x_2+\Delta x_3+\Delta x_4+\Delta x_5+\Delta x_6+\Delta x_7)\\
&= 5\times n > 0
\end{aligned} \tag{36}$$

with a positive integer $n$.

The negative integer $x_5 = -4$ of the solution (30) is not acceptable for the number of employees who start working on day $i = 5$. Therefore, it is required that

$$\Delta x_5 = \frac{3}{5}(\Delta a_5+\Delta a_7+\Delta a_2)-\frac{2}{5}(\Delta a_4+\Delta a_6+\Delta a_1+\Delta a_3) \geq 4, \tag{37}$$

i.e.,

$$\Delta a_5+\Delta a_7+\Delta a_2 \geq \frac{4\times 5}{3}+\frac{2}{3}(\Delta a_4+\Delta a_6+\Delta a_1+\Delta a_3) \geq 6.66. \tag{38}$$

The optimization of labor distposal is performed with the condition that the total number of employees should be the minimum. Owing to the relationship of (36) and (38), the condition requires that the variation of the sum of $a_i$'s is

$$\Delta a_1+\Delta a_2+\Delta a_3+\Delta a_4+\Delta a_5+\Delta a_6+\Delta a_7 = 5\times n = 10 \tag{39}$$

with $n = 2$, i.e., the sum of $a_i$'s is

$$\sum_{i=1}^{7} a_i = \sum_{i=1}^{7}(a_{i0}+\Delta a_i) = 115. \tag{40}$$

Substituting the above relationship to the equation(27), we obtain the total number $\Sigma_i\, x_i$ of employees as 23.

The relation (35) leads to the variations $\Delta x_i$ of $x_i$ to be within the limits of $-4$ and 6, i.e.,

$$\begin{aligned}\Delta x_i &= \frac{3}{5}(\Delta a_i+\Delta a_{i+2}+\Delta a_{i+4})-\frac{2}{5}(\Delta a_{i-1}+\Delta a_{i+1}+\Delta a_{i+3}+\Delta a_{i+5})\\ &= \frac{3}{5}\{10-(\Delta a_{i-1}+\Delta a_{i+1}+\Delta a_{i+3}+\Delta a_{i+5})\}\\ &\quad -\frac{2}{5}(\Delta a_{i-1}+\Delta a_{i+1}+\Delta a_{i+3}+\Delta a_{i+5})\\ &= \frac{30}{5}-(\Delta a_{i-1}+\Delta a_{i+1}+\Delta a_{i+3}+\Delta a_{i+5}) \leq 6, \end{aligned} \tag{41}$$

and

$$\begin{aligned}\Delta x_i &= \frac{3}{5}(\Delta a_i+\Delta a_{i+2}+\Delta a_{i+4})-\frac{2}{5}(\Delta a_{i-1}+\Delta a_{i+1}+\Delta a_{i+3}+\Delta a_{i+5})\\ &= \frac{3}{5}(\Delta a_i+\Delta a_{i+2}+\Delta a_{i+4})-\frac{2}{5}\{10-(\Delta a_i+\Delta a_{i+2}+\Delta a_{i+4})\}\\ &= -\frac{20}{5}+(\Delta a_i+\Delta a_{i+2}+\Delta a_{i+4}) \geq -4. \end{aligned} \tag{42}$$

The relationship of (31) and (42) shows that $x_1$, $x_2$, $x_4$ and $x_6$ remain as non-negative integers by the variations $\Delta a_i$. However, $x_3$, $x_5$ and $x_7$ become non-negative integers, only when the following requirements for the variations $\Delta a_i$ are satisfied:

$$\begin{aligned}\Delta x_3 &= \frac{3}{5}(\Delta a_3+\Delta a_5+\Delta a_7)-\frac{2}{5}(\Delta a_2+\Delta a_4+\Delta a_6+\Delta a_1) \geq 2,\\ \Delta x_5 &= \frac{3}{5}(\Delta a_5+\Delta a_7+\Delta a_2)-\frac{2}{5}(\Delta a_4+\Delta a_6+\Delta a_1+\Delta a_3) \geq 4,\\ \Delta x_7 &= \frac{3}{5}(\Delta a_7+\Delta a_2+\Delta a_4)-\frac{2}{5}(\Delta a_6+\Delta a_1+\Delta a_3+\Delta a_5) \geq -1. \end{aligned} \tag{43}$$

The relationship of (39) and (43) for the variations $\Delta a_i$ are necessary and sufficient conditions to make the constants $a_i$ for the coupled equations(25) to yield realistic solutions. Under the condition (39), the relations (43) are changed to be

$$\begin{aligned}\Delta a_3+\Delta a_5+\Delta a_7 &\geq 6,\\ \Delta a_5+\Delta a_7+\Delta a_2 &\geq 8,\\ \Delta a_7+\Delta a_2+\Delta a_4 &\geq 3. \end{aligned} \tag{44}$$

The condition (39) for the $\Delta a_i$'s yields the combination with repetitions[2] ${}_7\mathrm{H}_{10} = {}_{16}C_{10} = 8008$ of seven non-negative integers $\Delta a_i$'s for the devisions of the integer 10 into 7 groups, i.e., 8008 sets of ($\Delta a_1$, $\Delta a_2$, $\Delta a_3$, $\Delta a_4$, $\Delta a_5$, $\Delta a_6$, $\Delta a_7$). By a handwork search for the realistic sets of $\Delta a_i$'s from the 8008 sets, we found that 282 sets of the constants $a_i$ modified by $\Delta a_i$ satisfy the relationship of (39) and (43) for the realistic cases. Each set of the constants $a_i$ yields a unique solution of the coupled equations(25). Therefore, the guiding principle of the optimization that the total number of employees should be the minimum, does not guarantee a definite solution, yielding redundant solutions of the optimization problem of labor disposal. The redundancy of the solution originates from the uneven distribution of the numbers of employees required for working on day $i$ of the week which is given in equations (24).

Some numerical examples of the solutions can now be shown by choosing the sets of the constants $a_i$ which satisfy the conditions (39) and (44). The following three methods are proposed to optimize labor disposal in order to remove the redundancy of the solution.

1. Optimize the numbers $a_i$ of employees who work on day $i$. We chose a set of $a_i$'s which are more or less equal to each other.
2. Optimize the variations $\Delta a_i$ of number $a_i$ of employees from the required numbers $a_{i0}$. We chose a set of $a_i$'s whose variations $\Delta a_i$ from $a_{i0}$ are more or less equal to each other.
3. Optimize the numbers $x_i$ of employees. We chose a set of $a_i$'s for which the $x_i$'s of the solution of coupled equations (25) are more or less equal to each other.

In Table 1, we show the solutions of the coupled equations (25) obtained in the above optimization methods. We chose the set of the constants $a_i$ which meet one of the above conditions 1 to 3 for the optimization. The set of the chosen constants $a_i$ and the $x_i$'s of the solution are shown in the Table. The numbers 1 to 3 in the first column in the Table show the optimization methods 1 to 3, respectively.

**Table 1 The number $x_j$ of employees in the group $j$ and the number $a_i$ of those working on day $i$ disposed in the three optimization methods**

| Optimization | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 5 | 1 | 6 | 1 | 3 | 4 | 17 | 16 | 16 | 19 | 16 | 16 | 15 |
| 2 | 3 | 6 | 0 | 7 | 0 | 4 | 3 | 17 | 16 | 16 | 19 | 16 | 17 | 14 |
| 3 | 4 | 3 | 3 | 6 | 2 | 2 | 3 | 17 | 14 | 15 | 19 | 18 | 16 | 16 |

In Optimization method 1, the following procedures are taken to dispose of the employees: 1) The average of the numbers $a_{i0}$ of employees required for working on day $i$ is 15. 2) On days $i = 1, 4$ and 6, we dispose the numbers $a_{10} = 17$, $a_{40} = 19$ and $a_{50} = 16$ of employees required in equations (24). 3) Onto the other days, we dispose the additional number $\sum_i \Delta a_i = 10$ of employees. In order to make a more or less even distribution of the numbers of employees, we dispose additional numbers $\Delta a_2 = 3$, $\Delta a_3 = 1$, $\Delta a_5 = 2$ and $\Delta a_7 = 4$ of employees, i.e., the numbers of employees are $a_2 = a_3 = a_5 = 16$ and $a_7 = 15$. This disposal of employees satisfies the conditions (43).

In Optimization method 2, due to the average $\frac{10}{7} = 1.47$ of additional numbers $\Delta a_i$ of employees in the seven groups and the conditions (44) for the additional numbers $\Delta a_i$, we put

$$\begin{aligned} \Delta a_3 + \Delta a_5 + \Delta a_7 &= 6, \\ \Delta a_5 + \Delta a_7 + \Delta a_2 &= 8, \\ \Delta a_7 + \Delta a_2 + \Delta a_4 &= 6. \end{aligned} \tag{45}$$

Therefore, we dispose additional numbers $\Delta a_2 = 3$, $\Delta a_3 = 1$, $\Delta a_4 = 0$, $\Delta a_5 = 2$ and $\Delta a_7 = 3$ of employees. One additional employee is disposed on day $i = 6$ as $\Delta a_6 = 1$.

In Optimization method 3, the average number $\frac{23}{7} = 3.28$ of employees in the seven groups is taken into account. In order to make the numbers of employees be $x_3 = 3$, $x_5 = 2$ (the maximum in the condition (41)) and $x_7 = 3$, it is assumed that

$$\begin{aligned} \Delta a_3 + \Delta a_5 + \Delta a_7 &= 9, \\ \Delta a_5 + \Delta a_7 + \Delta a_2 &= 10, \\ \Delta a_7 + \Delta a_2 + \Delta a_4 &= 6, \end{aligned} \tag{46}$$

i.e., we dispose the additional numbers $\Delta a_2 = 1$, $\Delta a_3 = \Delta a_4 = 0$, $\Delta a_5 = 4$ and $\Delta a_7 = 5$ of employees. Then, the number $x_6$ of employees is 2.

Under normal working condition where the employees work five continuous days a week, a superfluous disposal of employees is inevitable to meet the fluctuation of demand and makes the working operation inefficient on days when a fewer number of employees is required[1]. This superfluous disposal of employees brings about redundancy in the optimization of labor disposal. In Table 2, we show the number $a_i - a_{i0}$ of superfluous employees and the efficiency of the operation defined by $\dfrac{a_{i0}}{a_i}$ for the present case.

**Table 2 The number $a_i - a_{i0}$ of superfluous employees and the operation efficiency $a_{i0}/a_i$ for working on day $i$ in the three optimization methods**

| Optimization | $a_1 - a_{10}$ | $a_2 - a_{20}$ | $a_3 - a_{30}$ | $a_4 - a_{40}$ | $a_5 - a_{50}$ | $a_6 - a_{60}$ | $a_7 - a_{70}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 1 | 0 | 2 | 0 | 4 |
| 2 | 0 | 3 | 1 | 0 | 2 | 1 | 3 |
| 3 | 0 | 1 | 0 | 0 | 4 | 0 | 5 |
| **Optimization** | $a_{10}/a_1$ | $a_{20}/a_2$ | $a_{30}/a_3$ | $a_{40}/a_4$ | $a_{50}/a_5$ | $a_{60}/a_6$ | $a_{70}/a_7$ |
| 1 | 1.00 | 0.81 | 0.94 | 1.00 | 0.88 | 1.00 | 0.73 |
| 2 | 1.00 | 0.81 | 0.94 | 1.00 | 0.88 | 0.94 | 0.79 |
| 3 | 1.00 | 0.93 | 1.00 | 1.00 | 0.78 | 1.00 | 0.69 |

## 3. Concluding Remarks

We have discussed the situations where the coupled inequalities for labor disposal management yield redundant solutions. Methods to remove the redundancy are proposed.

It is interesting to see the results of Optimization methods 1 and 2 in Table 1. The $a_i$'s are equal to each other between the two methods except for $a_6$ and $a_7$, which are different from each other only by 1, while the $x_i$'s of the solution of coupled equations (25) are quite different from each other. This feature results from the redundancy of the solutions in the optimization.

**References**


[1] Li Xiao Jie, Analysis of Working Hour and Handling Efficiency in a Logistics Company, Master Thesis, Josai University, 2011.

[2] Yadolah Dodge, *The Concise Encyclopedia of Statistics,* Springer Reference.