The Embeddability of a Locally Compact Flow in Bebutov Flow

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1. Introduction

Let R be the real line. A flow on a metric space X is the triplet (X, R, π), where π is a map of $X\times R$ into the space X such that

1) $\pi(x, 0)=x$ for every $x\in X$,

2) $\pi(\pi(x, s), t)=\pi(x, s+t)$ for every $x\in X$ and $s, t\in R$,

and

3) π is continuous.

As general references for flows, consult [1] or [2].

The Bebutov space, denoted by X_u, is a metric space which consists of all continuous functions on R into R^n with metric d defined by

$$d(\varphi, \psi)=\sup\{\min\{\max_{|x|\leqq x_0}|\varphi(x)-\psi(x)|, x_0-1\} : x_0>0\}.$$

Define a map $f_u : X_u\times R\to X_u$ by setting

$$f_u(\phi, t)=\phi\circ g_t$$

for every $\phi\in X_u$ and $t\in R$, where g_t is a map from R onto R defined by the equality

$$g_t(x)=x+t$$

for every $x\in R$. The map f_u defines a flow (X_u, R, f_u), which is refered to as Bebutov flow.

Bebutov flow is important in the sense that a large class of compact flows, i. e., the flows on compact spaces, may be embedded in it by virtue of Bebutov-Kakutani theorem: a necessary and sufficient condition for a compact flow to be isomorphic to some subflow of Bebutov flow is that the set of all rest points of the compact flow is isomorphic to some subset of the real line $\boldsymbol{R}$ [2, p. 84].

The aim of the present paper is to show that a necessary condition for a locally compact flow, i. e., the flow on a locally compact space, to be isomorphic to some subsystem of Bebutov flow is that it is embeddable in some linear flow on a finite-dimensional Euclidean space (Theorem 4.1).

2. Definitions and Notations

Let (X, R, π) be a flow on a metric space X. A set $M\subset X$ is called invariant in (X, R, π) if $\pi(x, t)\in M$ for every $x\in M$ and every $t\in \boldsymbol{R}$. Let M be a closed subset of

X as well as an invariant set in (X, R, π). Then a flow (M, R, π) can be defined by restricting the map π on $M \times \boldsymbol{R}$. The flow (M, R, π) is refered to as a subflow (X, R, π).

A flow (X, R, π) is said to be embeddable in a flow $(Y, \boldsymbol{R}, \alpha)$, if (X, R, π) is isomorphic to some subflow of $(Y, \boldsymbol{R}, \alpha)$, i. e., if there exists a subflow, say (M, R, α), of $(Y, \boldsymbol{R}, \alpha)$ and a homeomorphism $h: X \to M$ such that

$$\forall x \in X, \ \forall t \in R : h(\pi(x, t)) = \alpha(h(x), t)$$

holds.

Definition 2.1 [3, p. 16] Let X be a linear topological space. A flow (X, R, g) is said to be a linear flow, if $g^t : X \to X$ is a linear automorphism for every $t \in R$, where $g^t(x) = g(x, t)$ for every $x \in X$.

The linear space spanned by a set Y is denoted by span$Y$.

3. Bebutov flow

Bebutov space X_u is a lnear space over the real field with the usual addition and scalar multiplication, as is easily verified. Moreover, both of these operations are continuous with respect to the topology defined by the metric d, so that Bebutov space X_u is a linear topological space with these operations. X_u is obviously infinite dimensional.

Propsition 3.1 Bebutov flow (X_u, R, f_u) is a linear flow.

Proof.
$$\begin{aligned} f_u(\varphi+\psi, t)(x) &= ((\varphi+\psi)\cdot g^t)(x) = (\varphi+\psi)(g^t(x)) \\ &= \varphi(g^t(x)) + \psi(g^t(x)) = f_u(\varphi, t)(x) + f_u(\psi, t)(x) \end{aligned}$$

for every $\varphi, \psi \in X_u$, $t, x \in R$. Hence f_u^t is additive. Homogeneity of f_u^t for every $t \in R$ can be proved as follows.

$$f_u(a\psi, t)(x) = ((a\psi)\cdot g^t)(x) = a\psi(g^t(x)) = af_u(\psi, t)(x)$$

for every $\psi \in X_u$, $t, a \in R$.

Q. E. D.

The linear topological subspace of Bebutov space is not necessarily invariant in Bebutov flow (X_u, R, f_u), as is shown by an example: span$\{\cos(\cdot)\}$ is a linear topological subspace of X_u, but

$$f_u\left(\cos(\cdot), \frac{\pi}{2}\right) = \sin(\cdot) \notin \text{span}\ \{\cos(\cdot)\}.$$

We will show a necessary and sufficient condition for a linear subspace of X_u to be invariant in (X_u, R, f_u).

Proposition 3.2 Let M be a linear subspace of X_u. M is invariant in Bebutov flow (X_u, R, f_u) if and only if

$$f_u(B_M, t) \subset M$$

holds for any $t \in R$, where B_M is a Hamel basis for M.

Proof. Let M be an invariant set of (X_u, R, f_u). Since $B_M \subset M$, we have

$$f_u(B_M, t) \subset f_u(M, t) \subset M \tag{1}$$

for any $t \in R$. Conversely, assume that (1) holds. Take any $\psi \in M$. Then there exists a finite suset of B_M, say C, such that ψ is represeted uniquely as a linear conbination

$$\psi = \sum_{\phi \in C} a_\phi \phi, \quad a_\phi \in R,$$

so that

$$f_u(\psi, t) = f_u(\sum_{\phi \in C} a_\phi \phi, t) = \sum_{\phi \in C} a_\phi f_u(\phi, t) \in M$$

for every $t \in R$. Hence M is invariant in (X_u, R, f_u).

Q. E. D.

4. Embeddability of a locally compact flow in Bebutov flow

Bebutov space is reguler, since it is a metric space. Hence the small inductive dimension may be assigned to every subspace of Bebutov space. Let M be a subspace of Bebutov space. The small inductive dimension of M is denoted by ind M [4, p. 3].

Theorem 4.1 Let X be a locally compact metric space. If a flow (X, R, π) is embeddable in Bebutov flow (X_u, R, f_u), then

(i) ind X is finite

and

(ii) there exists a linear flow on a real Euclidean space of finite dimension in which (X, R, π) is embeddable.

Proof. By virtue of the assumption Bebutov flow has a subflow, say (M, R, f_u), to which (X, R, π) is isomorphic. Here M is a locally compact subset of X_u. Let Y be the linear subspace of X_u spanned by M. Y is locally compact in X_u. To show this, choose $g \in M$ arbitrarily. There exists a neighborhood U of g in M such that cl(U), the closure of U in M, is compact. Now choose any $a \in Y$ and let $l = a - g$. Define a map $\alpha : M \to Y$ by setting $\alpha(x) = x + l$. Since α is a homeomorphism of M onto $\alpha(M)$, $\alpha(U) = U + l$ is a neghborhood of $a = \alpha(g) = g + l$, and $\alpha(\mathrm{cl}(U)) = \mathrm{cl}(\alpha(U))$ holds, whence $\mathrm{cl}(\alpha(U))$ is compact. Since a is an arbitrary point, Y is locally compact. On the other hand, Y is a linear topological subspace of X_u. Hence Y is finite dimensional, because every linear topological subspace which is locally compact is finite dimensional [5, Th. 45 bis, p. 100]. This implies the finiteness of ind M, since $M \subset Y$, so that

$$\mathrm{ind}\, M \leq \mathrm{ind}\, Y = \dim Y.$$

Thus (i) is proved.

The proof of (ii). Y is closed in X_u, since Y is a finite dimensional linear subspace of X_u, which is T_1 space [6, Problem 9. e, p. 130]. Now we show that Y

is invariant in (X_u, R, f_u). Since $Y=\text{span}\, M$, M contains a Hamel base for Y, which we denote by B_Y. M is invariant in (X_u, R, f_u), because (M, R, f_u) is a subflow of (X_u, R, f_u) by the assumption. Hence

$$f_u(B_Y, t)\subset f_u(M, t)\subset M\subset Y$$

for every $t\in R$. This implies that Y is invariant in (X_u, R, f_u), by virtue of proposition 3.2. Thus Y is closed in X_u and invariant in (X_u, R, f_u), so that we can define the subflow (Y, R, f_u) of (X_u, R, f_u). Let $\dim Y=m$. It is known that all m-dimensional linear topological T_1-spaces with the same scalar field are linearly homeomorphic [5, Theorem 3.3–H, p. 127]. Hence Y is linearly homeomorphic to R^m, i. e., there exists a linear homeomorphism $T: Y\rightarrow R^m$. Now we define a map α: $R^m\times R\rightarrow R^m$ by setting

$$\alpha(y, t)=T(f_u(T^{-1}y, t))$$

for any $y\in R^m$ and $t\in R$. Then we can show that α defines a flow (R^m, R, α) as follows. The proof of

i) $\alpha(y, 0)=y$ for any $y\in R^m$

and

ii) $\alpha(\alpha(y, s), t)=\alpha(y, s+t)$ for any $y\in R^m$ and all $t, s,\in R$

are straightforward. On the other hand, let I be the identity map of R onto R. Let $\beta=(T^{-1}, I)$. Then α is the composite of T, f_u and β, all of which are continuous on their respective domains. This proves the continuity of α on $R^m\times R$.

The flow (R^m, R, α) is in fact a linear flow, since $\alpha^t=T\cdot f_u^t\cdot T^{-1}$ for every $t\in R$, where all of T, f_u^t, and T^{-1} are linear maps on these respective domains. Here $T(M)$ is closed in R^m and invariant in (R^m, R, α), because M is closed in X_u and invariant in (X_u, R, f_u) by the assumption, so that $T(M)$ defines $(T(M), R, \alpha)$, a subflow of (R^m, R, α). On the other hand there exists a homeomorphism $h: X\rightarrow M$ (onto), such that

$$h(\pi(x, t))=f_u(h(y), t)$$

for every $x\in X$ and $t\in R$, so that

$$T\circ h(\pi(x, t))=T(f_u(h(x), t))=\alpha(T(h(x)), t)=\alpha(T\circ h(x), t)$$

holds for every $x\in X$ and $t\in R$, where $T_0 h$ is obviously a homeomorphism from X onto $T(M)$. Consequently (X, R, π) is isomorphic to $(T(M), R, \alpha)$. Thus we have proved that (X, R, π) is embeddable in (R^m, R, α).

Q. E. D.

References

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