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Olech-types lemma and Visintin-types theorem in Pettis integration and $L^1_{E'}[E]$

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Summary. We present several versions of Olech's lemma and Visintin's theorem in Pettis integration and $L^1_{E'}[E]$ in the same vein as Amrani-Castaing-Valadier [4] and Benabdellah-Castaing [12].

1. Introduction. In the framework of Pettis integration (see, for instance, [2, 17, 23, 25, 29]), Amrani-Castaing-Valadier [4] stated a Visintin-type theorem and presented a version of Olech's lemma as a consequence. Using a recent weak compactness result in the space $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ of scalarly integrable functions defined on a probability space $(\Omega, \mathcal{F}, \mu)$ taking values in the dual E' of a separable Banach space E, Benabdellah-Castaing [12] stated a version of Visintin's theorem and also gave a version of Olech's lemma in $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ via the subspace $L^{1,\rho}_{E'}[E](\Omega, \mathcal{F}, \mu)$ associated to a lifting ρ in $\mathcal{L}^\infty_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. In the present paper we aim to present some Olech-types lemma and Visintin-types theorem in $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ and Pettis integration.

This paper is divided in two parts. The first part is devoted to the study of Olech-types lemma and Visintin-types theorem under denting point condition in Pettis integration by exploiting some new properties of the denting points of the Pettis integral of a closed convex valued measurable multifunction. In the second part we present a Visintin-type theorem in $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ via the recent results in [12]. Our results shed a new light on the problem of "norm convergence is implied by the weak" in both $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ and Pettis integration setting. For more on Olech-types lemma and Visintin-types theorem in Bochner integration we refer to [1, 3, 5, 6, 7, 8, 9, 10, 28, 30, 31, 32, 33, 36].

2. Notations and terminology. We will use the following notions and notations and summarize some useful facts.

- E is a separable Banach space, $\overline{B}_E$ is the closed unit ball and $B_E(x, r)$ the open ball of center x with radius r.
- E' is the topological dual of E and $\overline{B}_{E'}$ is the closed unit ball of E'.
- $(\Omega, \mathcal{F}, \mu)$ is a probability space.
- $L^1_E(\mu) := L^1_E(\Omega, \mathcal{F}, \mu)$ is the Banach space of (equivalent classes of) Bochner integrable E-valued functions.

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- $P_E^1(\mu) := P_E^1(\Omega, \mathcal{F}, \mu)$ is the normed space of (equivalent classes of) Pettis integrable E-valued functions $f : \Omega \to E$, endowed with the Pettis norm (see, for instance, [23, 25, 29]) $||f||_{Pe} := \sup_{||x'||\leq 1} \int_\Omega |\langle x', f(\omega)\rangle| \, \mu(d\omega)$. Let us recall that the Pettis norm $||.||_{Pe}$ is equivalent to the norm $f \mapsto \sup_{A\in\mathcal{F}} || \int_A f(\omega)\, \mu(d\omega)||$. Indeed let $x' \in E'$ with $||x'|| \leq 1$. We have

$$\begin{aligned}
\sup_{A\in\mathcal{F}} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega) &\leq \int_\Omega |\langle x', f(\omega)\rangle| \, \mu(d\omega) \\
&= \int_{\langle x', f\rangle \geq 0} \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&\quad - \int_{\langle x', f\rangle < 0} \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&\leq 2 \sup_{A\in\mathcal{F}} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{A\in\mathcal{F}} || \int_A f(\omega)\mu(d\omega)|| &= \sup_{A\in\mathcal{F}} \sup_{||x'||\leq 1} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&= \sup_{||x'||\leq 1} \sup_{A\in\mathcal{F}} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&\leq \sup_{||x'||\leq 1} \int_\Omega |\langle x', f(\omega)\rangle \, | \mu(d\omega) \\
&= ||f||_{Pe} \\
&\leq 2 \sup_{||x'||\leq 1} \sup_{A\in\mathcal{F}} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&= 2 \sup_{A\in\mathcal{F}} \sup_{||x'||\leq 1} \int_A \langle x', f(\omega)\rangle \, \mu(d\omega) \\
&= 2 \sup_{A\in\mathcal{F}} || \int_A f(\omega)\mu(d\omega)||.
\end{aligned}$$

- By $ck(E)$ (resp. $cwk(E)$) (resp. $ccb(E)$) (resp. $\mathcal{L}wc(E)$) (resp. $cc(E)$) (resp. $c(E)$) we denote the collection of all nonempty convex compact (resp. convex weakly compact) (resp. closed convex bounded) (resp. line free closed convex locally weakly compact [18]) (resp. closed convex) (resp. closed) subsets of E.

- If K is a subset of E, we denote by $cl(K)$ (resp. $\delta^*(x', K)$) the closure (resp. the support function) of K.

- An element e of a convex subset K in a Hausdorff locally convex space F is an *extreme point* of K if , for any $x, y \in K$, $e = \frac{1}{2}(x+y) \Longrightarrow x = y = e$.

- An element e of a convex subset K in a Hausdorff locally convex space F is a *weak denting point* of K if, for any weak neighbourhood V of e, one has $e \notin \overline{co}(K \setminus V)$. If e is a weak denting point of K, e is an extreme point of K.

Indeed, if e is not extremal, there are x and y in K with $x \neq y$ such that $e = \frac{1}{2}(x+y)$. By Hahn-Banach theorem, there is x' in the dual F' of F such that $\alpha := \langle x', x-e \rangle > 0$. Let V be the weak neighbourhood of e defined by $V := \{z \in F : |\langle x', z-e \rangle| < \frac{\alpha}{2}\}$. Then $e \in \overline{co}(K \setminus V)$ because both x and y belong to $K \setminus V$.

- An element e of a convex subset K in E is a *strong extreme point* of K if for any sequences (x_n) and (y_n) in K, $\lim_{n\to\infty} ||\frac{1}{2}(x_n + y_n) - e|| = 0$ implies that $\lim_{n\to\infty} ||y_n - e|| = 0$. It is easy to check that e is a strong extreme point of K iff the following holds. $\forall \varepsilon > 0$, there exists $\eta > 0$, such that : $x, y \in K$ and $||e - \frac{x+y}{2}|| < \eta \Longrightarrow ||x-e|| < \varepsilon$ and $||y-e|| < \varepsilon$. Indeed we can suppose $e = 0$. Let $\varepsilon > 0$. Put

$$\eta := \inf\{||\frac{x+y}{2}|| : x, y \in K, \ ||x|| \geq \varepsilon \text{ or } ||y|| \geq \varepsilon\}.$$

Since 0 is extremal, $||\frac{x+y}{2}|| > 0$ whenever $||x|| \geq \varepsilon$ or $||y|| \geq \varepsilon$. We claim that $\eta > 0$. If $\eta = 0$ there are sequence (x_n) and (y_n) in K with $||x_n|| \geq \varepsilon$ or $||y_n|| \geq \varepsilon$ for every n such that $||\frac{x_n+y_n}{2}|| \to 0$. Since 0 is a strong extreme point, $||x_n|| \to 0$ and $||y_n|| \to 0$. A contradiction.

- A point e of a convex subset K in E is a point of continuity (shortly pc) of K if the identity mapping $(K, weak) \to (K, ||.||)$ is continuous at the point e.

- A point e of a convex subset K in E is a *denting point* (resp. *weak denting point*) of K if, for any $\varepsilon > 0$, $e \notin \overline{co}(K \setminus B_E(e, \varepsilon))$ (resp. for any weak neighbourhood V of e, $e \notin \overline{co}(K \setminus V)$).

- If K is a convex subset of E, we will denote by $\partial_e(K)$ (resp. $\partial_{se}(K)$) (resp. $\partial_{pc}(K)$) (resp. $\partial_{wd}(K)$) (resp. $\partial_d(K)$) the set of extreme points (resp. strong extreme points) (resp. pc points) (resp. weak denting points) (resp. denting points) of K. The following inclusions hold:

(a) $\partial_{se}(K) \subset \partial_e(K)$,

(b) $\partial_{se}(K) \subset \partial_{se}(cl(K))$,

(c) $\partial_d(K) \subset \partial_{se}(K) \cap \partial_{pc}(K)$, *if K is closed,*

(d) $\partial_d(K) \subset \partial_{wd}(K)$.

(a) is obvious. Let us prove (b). Suppose that $e \in \partial_{se}(K)$. Let (x_n) and (y_n) in $cl(K)$ such that $\lim_{n\to\infty} ||\frac{1}{2}(x_n + y_n) - e|| \to 0$. Let us consider the sequences (x'_n) and (y'_n) in K with $||x_n - x'_n|| < \frac{1}{n}$ and $||y_n - y'_n|| < \frac{1}{n}$ for every n. Since

$$\frac{1}{2}(x'_n + y'_n) - e = \frac{1}{2}(x_n + y_n) - e + \frac{1}{2}(x'_n - x_n) + \frac{1}{2}(y'_n - y_n)$$

we have that $\lim_{n\to\infty} ||\frac{1}{2}(x'_n + y'_n) - e|| \to 0$. It follows that $||x'_n - e|| \to 0$ and similarly $||y'_n - e|| \to 0$. Since $||x_n - e|| \leq ||x'_n - e|| + ||x'_n - x_n||$, we have that

$||x_n - e|| \to 0$ and $||y_n - e|| \to 0$. Whence $e \in \partial_{se}(cl(K))$. (d) is obvious, while (c) follows from [11] and [28].

- A multifunction $\Gamma : \Omega \to E$ is measurable if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E)$ where $\mathcal{B}(E)$ denotes the Borel tribe of E. A $ccb(E)$-valued measurable multifunction is scalarly measurable (resp. integrable) if, for every x' in the topological dual E' of E, the real valued function $\omega \mapsto \delta^*(x', \Gamma(\omega))$ is measurable (resp. integrable). For more on measurable multifunction, see [18]. Given a measurable multifunction $\Gamma : \Omega \to E$, we denote by $\mathcal{S}_\Gamma^{Pe}$ the set of Pettis-integrable selections of Γ. If $\mathcal{S}_\Gamma^{Pe}$ is not empty, the Pettis multivalued integral of Γ is defined by

$$\int_\Omega \Gamma(\omega)\,\mu(d\omega) := \{\int_\Omega f(\omega)\,\mu(d\omega) : f \in \mathcal{S}_\Gamma^{Pe}\}.$$

A subset $\mathcal{H}$ of $P_E^1(\mu)$ is *scalarly Pettis uniformly integrable* if the set $\{\langle x', u\rangle : ||x'|| \le 1,\ u \in \mathcal{H}\}$ is uniformly integrable in $L_{\mathbb{R}}^1(\mu)$, $\mathcal{H}$ is *Pettis uniformly integrable* (PUI) if, for every $\varepsilon > 0$, there is $\delta > 0$, such that

$$\mu(A) < \delta \Longrightarrow \sup_{u \in \mathcal{H}} ||1_A u||_{Pe} \le \varepsilon.$$

By [4] we have the implication: "scalarly Pettis uniformly integrable" $\Longrightarrow$ "Pettis uniformly integrable".

- A scalarly integrable $ccb(E)$-valued multifunction Γ is *Pettis integrable*, if the set $\{\delta^*(x', \Gamma(.)) : ||x'|| \le 1\}$ is uniformly integrable in $L_{\mathbb{R}}^1(\Omega, \mathcal{F}, \mu)$. By definition $\mathcal{S}_\Gamma^{Pe}$ is scalarly Pettis uniformly integrable. The multivalued integral of a $cwk(E)$-valued Pettis integrable multifunction is convex *weakly compact* in E because the set $\mathcal{S}_\Gamma^{Pe}$ of all Pettis integrable selections of Γ is nonempty convex and *sequentially* $\sigma(P_E^1(\mu), L^\infty \otimes E')$ *compact* by [2, Prop. 3.4]. If $f : \Omega \to E$ is a scalarly integrable function, then $f \in P_E^1(\mu) \Longleftrightarrow \{\langle x', f\rangle : ||x'|| \le 1\}$ is weakly compact in $L_{\mathbb{R}}^1(\Omega, \mathcal{F}, \mu)$. Indeed $\Longrightarrow$ follows from the sequential weak compactness of the unit ball $\overline{B}_{E'}$ and Eberlein-Smulian theorem while $\Longleftarrow$ follows from Banach-Dieudonné theorem; so a Pettis integrable function f is scalarly Pettis uniformly integrable. More generally, let $(f_1, ...f_n)$ be a finite sequence in $P_E^1(\Omega, \mathcal{F}, \mu)$ and let $K \in \mathcal{L}wc(E)$, then the multifunction $\Gamma : \omega \mapsto co(\{f_1(\omega), ..f_n(\omega)\}) + K$ is a $\mathcal{L}wc(E)$-valued measurable multifunction with $\mathcal{S}_\Gamma^{Pe} \neq \emptyset$; if K is convex weakly compact, Γ is a $cwk(E)$-valued Pettis integrable multifunction according to the preceding definition and $\mathcal{S}_\Gamma^{Pe}$ is scalarly Pettis uniformly integrable.

- A sequence (u_n) in $P_E^1(\mu)$ is $ck(E)$-tight if, for every $\varepsilon > 0$, there exists a $ck(E)$-valued Pettis-integrable multifunction Γ_ε satisfying:

$$\sup_n \mu(\{\omega \in \Omega : u_n(\omega) \notin \Gamma_\varepsilon(\omega)\}) \le \varepsilon.$$

- For all undefined statements and notations in Convex analysis and Measurable multifunctions we refer to [18].

2. Olech-types lemma. In this section we provide several variants of Olech's lemma in Pettis integration.

We will need first two easy lemmas.

LEMMA 2.1. *Let $\mathcal{M}$ be a decomposable set of vector E-valued measures, that is, for every pair m, n in $\mathcal{M}$, and for every $\mathcal{F}$-measurable set A, the vector measure $1_A m + 1_{\Omega \setminus A} n$ belongs to $\mathcal{M}$, and let K be a convex subset of E. Suppose that $m(\Omega) \in K$ for every $m \in \mathcal{M}$ and e is an extreme point of K, then there exists at most one $m \in \mathcal{M}$ such that $m(\Omega) = e$.*

Proof. Let $m_1, m_2 \in \mathcal{M}$ with $m_1(\Omega) = m_2(\Omega) = e$. For any fixed $\mathcal{F}$-measurable set A and for any $\mathcal{F}$-measurable set B, let us denote

$$m_{1,2}(B) := m_1(B \cap A^c) + m_2(B \cap A)$$

and

$$m_{2,1}(B) := m_2(B \cap A^c) + m_1(B \cap A).$$

As $\mathcal{M}$ is a decomposable, $m_{1,2}$ and $m_{2,1}$ belong to $\mathcal{M}$. Then $m_{1,2}(\Omega) \in K$ and $m_{2,1}(\Omega) \in K$ and we have that $e = \frac{1}{2}(m_{1,2}(\Omega) + m_{2,1}(\Omega))$. Since e is an extreme point of K, we deduce that $e = m_{1,2}(\Omega)$. Whence $m_{1,2}(\Omega) = m_1(\Omega)$ and $m_{2,1}(\Omega) = m_2(\Omega)$ which imply $m_1(A) = m_2(A)$.

REMARK. If $\Gamma : \Omega \to E$ is a convex valued measurable multifunction such that the set $\mathcal{S}^{Pe}_\Gamma$ of Pettis integrable selections is not empty and e is an extreme point of $\int_\Omega \Gamma(\omega)\,\mu(d\omega)$, then there is a unique $f \in \mathcal{S}^{Pe}_\Gamma$ such that $\int_\Omega f(\omega)\,\mu(d\omega) = e$. It is enough to apply lemma 2.1 to $\mathcal{M} = \{f\mu : f \in \mathcal{S}^{Pe}_\Gamma\}$ and $K = \int_\Omega \Gamma(\omega)\,\mu(d\omega)$.

LEMMA 2.2. *Let $\mathcal{M}$ be a decomposable set of E-valued vector measures and K a convex subset of E. Suppose that $m(\Omega) \in K$ for every $m \in \mathcal{M}$ and e is a strong extreme point of K, then, for every $\varepsilon > 0$, there is $\eta > 0$ such that*

$$\sup_{A \in \mathcal{F}} ||m_1(A) - m_2(A)|| \leq \varepsilon$$

whenever $m_1, m_2 \in \mathcal{M}$ and $||m_i(\Omega) - e|| < \eta$ for $i = 1, 2$.

Proof. Let $\varepsilon > 0$. As $e \in \partial_{se}(K)$, there is $0 < \eta < \frac{\varepsilon}{2}$ such that $x, y \in K$ and $||e - \frac{x+y}{2}|| < \eta \implies ||x - e|| < \frac{\varepsilon}{2}$ and $||y - e|| < \frac{\varepsilon}{2}$. Now let $m_1, m_2 \in \mathcal{M}$ and $||m_i(\Omega) - e|| < \eta$ for $i = 1, 2$ and let A be a fixed $\mathcal{F}$-measurable set. For any $\mathcal{F}$-measurable set B, let us denote

$$m_{1,2}(B) := m_1(B \cap A^c) + m_2(B \cap A)$$

and

$$m_{2,1}(B) := m_2(B \cap A^c) + m_1(B \cap A).$$

As $\mathcal{M}$ is decomposable, $m_{1,2}$ and $m_{2,1}$ belong to $\mathcal{M}$. Furthermore we have

$$\begin{aligned} ||e - \frac{m_{1,2}(\Omega) + m_{2,1}(\Omega)}{2}|| &= ||e - \frac{m_1(\Omega) + m_2(\Omega)}{2}|| \\ &\leq \frac{1}{2}(||e - m_1(\Omega)|| + ||e - m_2(\Omega)|| < \eta. \end{aligned}$$

Therefore we get $||m_{1,2}(\Omega) - e|| < \frac{\varepsilon}{2}$. Whence

$$\begin{aligned} ||m_2(A) - m_1(A)|| &= ||m_{1,2}(\Omega) - m_1(\Omega)|| \\ &\leq \frac{\varepsilon}{2} + \eta < \varepsilon. \end{aligned}$$

As the preceding inequality holds for all $A \in \mathcal{F}$, we have

$$\sup_{A \in \mathcal{F}} ||m_1(A) - m_2(A)|| \leq \varepsilon.$$

Now we proceed to state some Olech-types lemma.

THEOREM 2.3. *Suppose that $\Gamma : \Omega \to E$ is a convex valued measurable multifunction and (f_n) is a sequence in $\mathcal{S}_\Gamma^{Pe}$ satisfying:*

(i) $\lim_{n\to\infty} || \int_\Omega f_n(\omega)\, \mu(d\omega) - e|| = 0$,

(ii) $e \in \partial_{se}(\int_\Omega \Gamma(\omega)\, \mu(d\omega))$,

then (f_n) converges in the normed space $(P_E^1(\mu), ||.||_{Pe})$ to the unique selection $f \in \mathcal{S}_\Gamma^{Pe}$ with $\int_\Omega f(\omega)\, \mu(d\omega) = e$.

Proof. Let $\varepsilon > 0$. Applying lemma 2.2 to

$$K := \int_\Omega \Gamma(\omega)\, \mu(d\omega) \text{ and } \mathcal{M} := \{m = g\,\mu : g \in \mathcal{S}_\Gamma^{Pe}\}$$

provides $\eta > 0$ such that

$$\sup_{A \in \mathcal{F}} ||m_1(A) - m_2(A)|| \leq \varepsilon$$

whenever $m_1, m_2 \in \mathcal{M}$ and $||m_i(\Omega) - e|| < \eta$ for $i = 1, 2$. By (i) there is $N \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow || \int_\Omega f_n(\omega)\, \mu(d\omega) - e|| < \eta.$$

Now set $m_n := f_n\, \mu$ for $n \geq N$. It is clear that

$$||m_n(\Omega) - e|| = || \int_\Omega f_n(\omega)\, \mu(d\omega) - e|| < \eta.$$

By lemma 2.1 there is a unique $f \in \mathcal{S}_\Gamma^{Pe}$ with $\int_\Omega f(\omega)\, \mu(d\omega) = e$. Set $m := f\,\mu$ so that $||m(\Omega) - e|| = 0 < \eta$. Therefore we get

$$\sup_{A \in \mathcal{F}} ||m_n(A) - m(A)|| \leq \varepsilon,$$

that is

$$\sup_{A \in \mathcal{F}} || \int_A (f_n(\omega) - f(\omega))\, \mu(d\omega)|| \leq \varepsilon.$$

whence $\lim_{n\to\infty} ||f_n - f||_{Pe} \to 0$.

Combining the preceding theorem with the results obtained in [4] we obtain the following.

PROPOSITION 2.4. *Let $\Gamma : \Omega \to E$ be a $cwk(E)$-valued Pettis-integrable multifunction. Then the following hold:*

(a) The set $\mathcal{S}_\Gamma^{Pe}$ of Pettis integrable selections of Γ in nonempty, convex sequentially compact for the topology of pointwise convergence on $L^\infty \otimes E'$.

(b) If (f_n) is a sequence in $\mathcal{S}_\Gamma^{Pe}$ such that $\lim_{n\to\infty} \int_\Omega f_n(\omega)\,\mu(d\omega) = e$ strongly and that $e \in \partial_{se}(\int_\Omega \Gamma(\omega)\,\mu(d\omega))$, then (f_n) converges in the normed space $P_E^1(\mu)$ to the unique extreme selection $f \in \mathcal{S}_\Gamma^{Pe}$ (that is $f(\omega) \in \partial_e(\Gamma(\omega))$ a.e) with $\int_\Omega f(\omega)\,\mu(d\omega) = e$.

(c) If Γ is $ck(E)$-valued, then

$$\partial_{se}\Big(\int_\Omega \Gamma(\omega)\,\mu(d\omega)\Big) = \partial_e\Big(\int_\Omega \Gamma(\omega)\,\mu(d\omega)\Big) = \partial_d\Big(\int_\Omega \Gamma(\omega)\,\mu(d\omega)\Big).$$

Proof. (a) is in [4, Theorem 1.1]. (b) follows from Theorem 2.3 and the arguments given in [4, Theorem 2.5] by noting that

$$\partial_{se}\Big(\int_\Omega \Gamma(\omega)\,\mu(d\omega)\Big) \subset \partial_e\Big(\int_\Omega \Gamma(\omega)\,\mu(d\omega)\Big).$$

(c) follows from the norm compactness of $\int_\Omega \Gamma(\omega)\,\mu(d\omega)$. See again [4, page 11].

The following is a characterization of strong extreme points of a closed convex subset in E. See also [4, 28, 33] for related results.

THEOREM 2.5. *Let K be a closed convex subset of E and e an element of K. Let us consider the following conditions.*

(i) $e \in \partial_{se}(K)$.

(ii) If (f_n) is a sequence in $\mathcal{S}_K^{Pe}$ such that $\lim_{n\to\infty} || \int_\Omega f_n(\omega)\,\mu(d\omega) - e|| = 0$, then $\lim_{n\to\infty} ||f_n - e||_{Pe} = 0$.

(iii) If (f_n) is a sequence in $\mathcal{S}_K^{Pe}$ such that $\lim_{n\to\infty} || \int_\Omega f_n(\omega)\,\mu(d\omega) - e|| = 0$, then $\lim_{n\to\infty} || \int_A (f_n(\omega) - e)\,\mu(d\omega)|| = 0$ for every $A \in \mathcal{F}$.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). Suppose there exist $C \in \mathcal{F}$ with $\mu(C) = \frac{1}{2}$, then (iii) $\Longrightarrow$ (i).

Proof. Note that $\int_A f\,d\mu \in \mu(A)K$ for every $f \in \mathcal{S}_K^{Pe}$ and for every $A \in \mathcal{F}$. Now $(i) \Longrightarrow (ii)$ follows from Theorem 2.3 by taking $\Gamma(.) = K$, while $(ii) \Longrightarrow (iii)$ is obvious. Now suppose there exists $C \in \mathcal{F}$ with $\mu(C) = \frac{1}{2}$. Let us prove that $(iii) \Longrightarrow (i)$. Let (x_n) and (y_n) be sequences in K such that $\lim_{n\to\infty} ||\frac{1}{2}(x_n + y_n) - e|| = 0$. Set $f_n = x_n 1_C + y_n 1_{\Omega\setminus C}$. Then we have that $f_n \in \mathcal{S}_K^1$ and

$$\lim_{n\to\infty} || \int_\Omega f_n(\omega)\,\mu(d\omega) - e|| = 0.$$

By (iii) we have

$$\lim_{n\to\infty} \|\int_C f_n(\omega)\,\mu(d\omega) - e\| = \lim_{n\to\infty} \frac{1}{2}\|x_n - e\| = 0.$$

Therefore $e \in \partial_{se}(K)$.

We finish this section with an Olech-type lemma in $L^1_E(\mu)$ via the *pc-point condition.*

PROPOSITION 2.6. *Suppose that K be a closed convex subset of E and $e \in \partial_{pc}(K)$, (u_n) is a sequence in $L^1_E(\mu)$ with $u_n(\omega) \in K$ for all n and a.e $\omega \in \Omega$ such that, $\forall x' \in E'$, $\lim_{n\to\infty} \int_\Omega |\langle x', u_n(\omega) - e\rangle|\,\mu(d\omega) = 0$, then*

$$\lim_{n\to\infty} \int_\Omega \|u_n(\omega) - e\|\,\mu(d\omega) = 0.$$

Proof. We can suppose that $e = 0$. It is enough to show that every subsequence (v_n) of (u_n) admits a subsequence (w_n) such that $\lim_{n\to\infty} \|w_n - e\|_1 = 0$. Since $0 \in \partial_{pc}(K)$, for every $m \in \mathbb{N}^*$, there exist $e_1^m, \ldots, e_{\nu_m}^m$ in E' and $\alpha_m > 0$ such that

$$W_m := \{x \in K : \sup_{1\le i\le \nu_m} |\langle e_i^m, x\rangle| \le \alpha_m\} \subset B_E(0, \frac{1}{m}).$$

As $\lim_{n\to\infty} \int_\Omega |\langle e_i^1, u_n(\omega)|\,\mu(d\omega) = 0$, for $1 \le i \le \nu_1$, there is a subsequence (v_n^1) of (v_n) such that $\langle e_i^1, v_n^1\rangle \to 0$ a.e, for $1 \le i \le \nu_1$. Hence we have that $\lim_{n\to\infty} \sup_{1\le i\le\nu_1} |\langle e_i^1, v_n^1\rangle| = 0$ a.e. Repeating this argument, we see that for every $m \in \mathbb{N}^*$, there is a subsequence (v_n^{m+1}) of (v_n^m) such that

$$\lim_{n\to\infty} \sup_{1\le i\le \nu_{m+1}} |\langle e_i^{m+1}, v_n^{m+1}\rangle| = 0$$

a.e. Set $w_n = v_n^n$. Then for every $m \in \mathbb{N}^*$, we have

$$\lim_{n\to\infty} \sup_{1\le i\le \nu_m} |\langle e_i^m, w_n\rangle| = 0$$

a.e, thus for almost all ω and for every $m \in \mathbb{N}^*$, there is $k_m \in \mathbb{N}^*$ such that

$$k \ge k_m \Longrightarrow \sup_{1\le i\le\nu_m} |\langle e_i^m, w_k(\omega)\rangle| \le \alpha_m \Longrightarrow \|w_k(\omega)\| \le \frac{1}{m}.$$

Therefore $\|w_n\| \to 0$ a.e. Let $m \in \mathbb{N}^*$ be fixed. For every $n \in \mathbb{N}$, set

$$A_n^m := \{\omega \in \Omega : \sup_{1\le i\le\nu_m} |\langle e_i^m, w_n(\omega)\rangle| \le \alpha_m\}.$$

As $\|w_n(\omega))\| \to 0$ a.e, $\|1_{A_n^m} w_n\|_1 \to 0$ because $\|1_{A_n^m} w_n(\omega)\| \le \frac{1}{m}$ a.e. But $0 \in K$ and K is convex, for every $x \in K$, we get

$$\frac{\alpha_m}{\sup_{1\le i\le\nu_m} |\langle e_i^m, x\rangle|}\, x \in W_m$$

so that

$$\frac{\alpha_m}{\sup_{1\le i\le \nu_m} |\langle e_i^m, x\rangle|}\,||x|| \le \frac{1}{m}.$$

Therefore we have

$$\begin{aligned}||w_n||_1 &= ||1_{A_n^m} w_n||_1 + ||1_{\Omega\setminus A_n^m} w_n||_1\\ &\le ||1_{A_n^m} w_n||_1 + \frac{1}{m\alpha_m}\int_{\Omega\setminus A_n^m} \sup_{1\le i\le \nu_m} |\langle e_i^m, w_n(\omega)\rangle|\,\mu(d\omega)\\ &\le ||1_{A_n^m} w_n||_1 + \frac{1}{m\alpha_m}\sup_{1\le i\le \nu_m}\int_{\Omega} |\langle e_i^m, w_n(\omega)\rangle|\,\mu(d\omega).\end{aligned}$$

It follows that $||w_n||_1 \to 0$.

3. Visintin-types theorem in Pettis integration. We begin with a lemma.

LEMMA 3.1. *Let $\Gamma : \Omega \to E$ be a $cwk(E)$-valued measurable multifunction and (f_n) a sequence of Pettis integrable functions such that $f_n(\omega) \in \Gamma(\omega)$ for all n and a.e $\omega \in \Omega$. Suppose that (f_n) converges in the Pettis norm to $f \in P^1_E(\mu)$. Then there is a subsequence (g_m) of (f_n) such that (g_m) converges weakly almost everywhere to f. Consequently $f(\omega) \in \Gamma(\omega)$ a.e.*

Proof. Let (e'_k) be a dense sequence in $\overline{B}_{E'}$ for the Mackey topology. By the definition of Pettis norm, it is obvious that, for each k, the sequence $(\langle e'_k, f_n\rangle)_n$ converges to $\langle e'_k, f\rangle$ for the norm in $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. By an appropriate diagonal procedure we find a subsequence (g_m) of (f_n) such that

$$\forall k,\ \lim_{m\to\infty} \langle e'_k, g_m\rangle = \langle e'_k, f\rangle \ a.e.$$

It follows that

$$\forall k,\ \langle e'_k, f(\omega)\rangle \le \delta^*(e'_k, \Gamma(\omega)) \ a.e.$$

By [18, Lemma III.34] we have $f(\omega) \in \Gamma(\omega)$ a.e. Using a routine density argument we see that (g_m) converges weakly almost everywhere to f.

REMARK. The proof of the preceding lemma shows that the set $\mathcal{S}^{Pe}_\Gamma$ of all Pettis integrable selections of a $\mathcal{L}wc(E)$-valued measurable multifunction is a convex closed subset of the normed space $(P^1_E(\mu), ||.||_{Pe})$.

Using the preceding lemma we are able to establish the relationship between $\mathcal{S}^{Pe}_{\partial_d\Gamma}$ and $\partial_d(\mathcal{S}^{Pe}_\Gamma)$ where Γ is a $cwk(E)$–valued Pettis integrable multifunction and $\partial_d\Gamma : \omega \mapsto \partial_d(\Gamma(\omega))$.

PROPOSITION 3.2. *Suppose that $\Gamma : \Omega \to E$ is a $cwk(E)$-valued Pettis integrable multifunction and u is a Pettis integrable function such that $u(\omega) \in \partial_d(\Gamma(\omega))$ a.e, then $u \in \partial_d(\mathcal{S}^{Pe}_\Gamma)$.*

Proof. Without lost of generality we may suppose $u \equiv 0$. Assume by contradiction that $0 \notin \partial_d(\mathcal{S}^{Pe}_\Gamma)$. Then there exist, $\varepsilon > 0, (\lambda^n_i)_{1\le i\le \nu_n}$ with $0 \le \lambda^n_i \le 1$ and $\Sigma^{\nu_n}_{i=1}\lambda^n_i = 1$ and $(u^n_i)_{1\le i\le \nu_n}$ in $\mathcal{S}^{Pe}_\Gamma$ such that

$$||u^n_i||_{Pe} \ge \varepsilon,\ \forall n \in \mathbb{N} \text{ and } \forall i \in \{1, ...\nu_n\}$$

and

$$\lim_{n\to\infty} ||\Sigma_{i=1}^{\nu_n}\lambda_i^n u_i^n||_{Pe} = 0.$$

Since 0 is also a *pc* point of $\Gamma(\omega)$ by hypothesis, using lemma 3.1 and the weak compactness of $\Gamma(\omega)$, we may suppose that

$$\lim_{n\to\infty} ||\Sigma_{i=1}^{\nu_n}\lambda_i^n u_i^n(\omega)|| = 0 \ a.e.$$

As $0 \in \partial_d(\Gamma(\omega))$ a.e, it follows from [28] that

$$\lim_{n\to\infty} \Sigma_{i=1}^{\nu_n}\lambda_i^n ||u_i^n(\omega)|| = 0 \ a.e.$$

As Γ is Pettis integrable, $\mathcal{S}_\Gamma^{Pe}$ is Pettis uniformly integrable. Hence there exist $\eta > 0$, such that

$$\mu(A) < \eta \Longrightarrow \sup_{v\in\mathcal{S}_\Gamma^{Pe}} ||1_A v||_{Pe} \leq \frac{\varepsilon}{2}.$$

By Egorov's theorem, there is a subsequence still denoted $(\Sigma_{i=1}^{\nu_n}\lambda_i^n||u_i^n(.)||)_n$ which converges almost uniformly. So there is a $\mathcal{F}$-mesurable set B with $\mu(\Omega\setminus B) < \eta$ such that $\Sigma_{i=1}^{\nu_n}\lambda_i^n||u_i^n(\omega)|| \to 0$ uniformly on B. Hence there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, \Sigma_{i=1}^{\nu_n}\lambda_i^n||u_i^n(\omega)|| \leq \frac{\varepsilon}{2}$ uniformly on B. An easy computation gives

$$\Sigma_{i=1}^{\nu_n}\lambda_i^n||1_B u_i^n||_{Pe} \leq \int_\Omega \sup_{\omega\in B} \Sigma_{i=1}^{\nu_n}\lambda_i^n||u_i^n(\omega)||\, d\mu \leq \frac{\varepsilon}{2}.$$

As $\mu(\Omega\setminus B) < \eta$ we have $||1_{\Omega\setminus B}u_i^n||_{Pe} \leq \frac{\varepsilon}{2}$ for all n and for all $i \in \{1,...\nu_n\}$. That implies

$$\forall n, \Sigma_{i=1}^{\nu_n}\lambda_i^n||1_{\Omega\setminus B}u_i^n||_{Pe} \leq \frac{\varepsilon}{2}.$$

Whence we have

$$\begin{aligned}\Sigma_{i=1}^{\nu_n}\lambda_i^n||u_i^n||_{Pe} &\leq \Sigma_{i=1}^{\nu_n}\lambda_i^n(||1_B u_i^n||_{Pe} \\ &\quad + ||1_{\Omega\setminus B}u_i^n||_{Pe}) \\ &\leq \Sigma_{i=1}^{\nu_n}\lambda_i^n||1_B u_i^n||_{Pe} \\ &\quad + \Sigma_{i=1}^{\nu_n}\lambda_i^n||1_{\Omega\setminus B}u_i^n||_{Pe} \\ &\leq \varepsilon\end{aligned}$$

that contradicts the inequalities

$$||u_i^n||_{Pe} \geq \varepsilon, \ \forall n \in \mathbb{N} \text{ and } \forall i \in \{1,...\nu_n\}. \quad \square$$

The following is a version of Komlós-Visintin type theorem in Pettis integration. See also [7] for other related results in $L_E^1(\mu)$.

THEOREM 3.3. *Suppose that $\Gamma : \Omega \to E$ is a $cwk(E)$-valued measurable multifunction, (u_n) is a sequence in $\mathcal{S}_\Gamma^{Pe}$ satisfying :*

(i) the set $\{\langle x', u_n\rangle : ||x'|| \leq 1,\ n \in \mathbb{N}\}$ is uniformly integrable,

(ii) (u_n) *pointwise converges on* $L^\infty \otimes E'$ *to* $u \in P_E^1(\Omega, \mathcal{F}, \mu)$ *with* $u(\omega) \in \partial_{pc}(\Gamma(\omega))$ a.e,

then there is a subsequence (v_m) *of* (u_n) *such that*

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma_{i=1}^n w_i - u||_{Pe} = 0$$

for every subsequence $(w_l) = (v_{m_l})$ *of* (v_m).

Proof. Let us recall that the *scalarly Pettis uniformly integrable condition* (i) implies that (u_n) is Pettis uniformly integrable [4]. Let (e'_k) be a dense sequence in $\overline{B}_{E'}$ for the Mackey topology of E'. By (i) and (ii) the sequence $(\langle e'_k, u_n\rangle)_n$ is uniformly integrable for each k and converges $\sigma(L^1, L^\infty)$ to $\langle e'_k, u\rangle$, using Komlós theorem [26] and an appropriate diagonal procedure, we find a subsequence (v_m) of (u_n) such that

$$\forall k, \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^n \langle e'_k, w_i\rangle = \langle e'_k, u\rangle \text{ a.e.}$$

for every subsequence $(w_l) = (v_{m_l})$ of (v_m). By density argument, we see that $(s_n)_n := (\frac{1}{n}\Sigma_{i=1}^n w_i)_n$ weakly converges to u a.e. As $u(\omega)$ is a pc point of $\Gamma(\omega)$ for a.e $\omega \in \Omega$ by our assumption, it follows that

$$\lim_{n\to\infty} ||s_n - u|| = 0$$

a.e. Since (u_n) is scalarly Pettis uniformly integrable so is (s_n). Hence (s_n) is Pettis uniformly integrable. By Vitali theorem for Pettis uniformly integrable functions [4, Proposition 2.1] we have that

$$\lim_{n\to\infty} ||s_n - u||_{Pe} = 0,$$

thus completing the proof.

By combining Proposition 3.2 and Theorem 3.3 we obtain the following version of Visintin's theorem in Pettis integration.

THEOREM 3.4. *Suppose that* $\Gamma : \Omega \to E$ *is a* $cwk(E)$*-valued Pettis integrable multifunction and* (u_n) *is a sequence in* $\mathcal{S}_\Gamma^{Pe}$ *which converges* $\sigma(P_E^1(\mu), L^\infty \otimes E')$ *to* $u \in \mathcal{S}_\Gamma^{Pe}$ *with* $u(\omega) \in \partial_d(\Gamma(\omega))$ a.e, *then* $\lim_{n\to\infty} ||u_n - u||_{Pe} = 0$.

Proof. Without lost of generality we may suppose $u = 0$. Assume by contradiction that $\limsup_{n\to\infty} ||u_n||_{Pe} := \varepsilon > 0$. By extracting a subsequence we may suppose that $||u_n||_{Pe} > \frac{\varepsilon}{2}$ for all n. By Theorem 3.3 there is a subsequence (v_m) of (u_n) such that

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma_{i=1}^n v_i||_{Pe} = 0.$$

As $||v_m||_{Pe} > \frac{\varepsilon}{2}$ for all m, we deduce that

$$(*) \qquad 0 \in \overline{co}[\mathcal{S}_\Gamma^{Pe} \setminus B_{P_E^1(\mu)}(0, \frac{\varepsilon}{2})]$$

where $B_{P^1_E(\mu)}(0, \frac{\varepsilon}{2})$ denotes the open ball of center 0 and radius $\frac{\varepsilon}{2}$ of the normed space $P^1_E(\mu)$. Since 0 is a denting point of $\Gamma(\omega)$ for a.e $\omega \in \Omega$, by Proposition 3.2 the null function 0 is a denting point of the closed convex subset $\mathcal{S}^{Pe}_\Gamma$ of the normed space $P^1_E(\mu)$ that contradicts $(*)$.

REMARK. Using the arguments of the proof of Theorem 3.3 leads to a second version of Komlós-Visintin type theorem. We omit the proof.

THEOREM 3.5. *Suppose that the strong dual E' of E is separable, $\Gamma : \Omega \to E$ is a $ccb(E)$-valued measurable multifunction, (u_n) is a sequence in $\mathcal{S}^{Pe}_\Gamma$ satisfying :*

(i) the set $\{\langle x', u_n\rangle : ||x'|| \leq 1,\ n \in \mathbb{N}\}$ is uniformly integrable,

(ii) (u_n) pointwise converges on $L^\infty \otimes E'$ to $u \in \mathcal{S}^{Pe}_\Gamma$ with $u(\omega) \in \partial_{pc}(\Gamma(\omega))$ a.e,

then there is a subsequence (v_m) of (u_n) such that

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma^n_{i=1} w_i - u||_{Pe} = 0$$

for every subsequence $(w_l) = (v_{m_l})$ of (v_m).

Now we want to show that in some special situations Theorem 2.3 follows from Visintin-types theorem. We will need the following property of the denting points of the integral of a $\mathcal{L}wc(E)$-valued measurable multifunction.

PROPOSITION 3.6. *Suppose that $\Gamma : \Omega \to E$ is a $\mathcal{L}wc(E)$-valued measurable multifunction and $u \in \mathcal{S}^{Pe}_\Gamma$ such that $\int_\Omega u\, d\mu \in \partial_d(\overline{\int_\Omega \Gamma(\omega)\mu(d\omega)})$, then u is a denting point of the closed convex set $\mathcal{S}^{Pe}_\Gamma$ of the normed space $(P^1_E(\mu), ||.||_{Pe})$.*

Proof. Let us observe that $\mathcal{S}^{Pe}_\Gamma$ is a closed convex set of the normed space $P^1_E(\mu)$ (cf. the remark of lemma 3.1). Suppose that $u \notin \partial_d(\mathcal{S}^{Pe}_\Gamma)$. There is $\varepsilon > 0$ such that $u \in \overline{co}(\mathcal{S}^{Pe}_\Gamma \setminus B_{P^1_E(\mu)}(u, \varepsilon))$ where $B_{P^1_E(\mu)}(u, \varepsilon)$ denotes the open ball of center u and radius ε in the normed space $P^1_E(\mu)$. Since the application $f \in P^1_E(\mu) \mapsto \int f\, d\mu \in E$ is linear and continuous for the norm topologies of $P^1_E(\mu)$ and E, we have that

$$(3.6.1) \qquad \int_\Omega u(\omega)\,\mu(d\omega) \in \overline{co}\{\int_\Omega f(\omega)\,\mu(d\omega) : f \in \mathcal{S}^{Pe}_\Gamma \setminus B_{P^1_E(\mu)}(u, \varepsilon)\}.$$

Now we claim that

$$(3.6.2) \qquad \{\int_\Omega f(\omega)\,\mu(d\omega) : f \in \mathcal{S}^{Pe}_\Gamma \setminus B_{P^1_E(\mu)}(u, \varepsilon)\} \cap B_E(I(u), \alpha) \neq \emptyset$$

for every $\alpha > 0$, where $I(u) := \int_\Omega u\, d\mu$. For simplicity set

$$K := \overline{\int_\Omega \Gamma(\omega)\mu(d\omega)}$$

and

$$H := \{\int_\Omega f(\omega)\,\mu(d\omega) : f \in \mathcal{S}^{Pe}_\Gamma \setminus B_{P^1_E(\mu)}(u, \varepsilon)\}.$$

It is obvious that (3.6.2) is equivalent to

$$H \not\subset K \setminus B_E(I(u), \alpha). \tag{3.6.3}$$

Assume by contradiction that

$$H \subset K \setminus B_E(I(u), \alpha).$$

Then

$$\overline{co}(H) \subset \overline{co}[K \setminus B_E(I(u), \alpha)].$$

But

$$I(u) \notin \overline{co}[K \setminus B_E(I(u), \alpha)]$$

because $I(u)$ is a denting point of K. Therefore $I(u) \notin \overline{co}(H)$ that contradicts (3.6.1). By (3.6.2) there exist a sequence (u_n) in the norm closed set $\mathcal{S}_\Gamma^{Pe} \setminus B_{P_E^1(\mu)}(u, \varepsilon)$ in $P_E^1(\mu)$ such that

$$\lim_{n\to\infty} \left|\left| \int_\Omega u_n(\omega)\, \mu(d\omega) - \int_\Omega u(\omega)\, \mu(d\omega) \right|\right| = 0.$$

Since $I(u) \in \partial_{se}(\int_\Omega \Gamma(\omega)\mu(d\omega))$, by Lemma 2.1 and 2.2 we deduce that

$$\lim_{n\to\infty} ||u_n - u||_{Pe} = 0.$$

Whence we have $u \in \mathcal{S}_\Gamma^{Pe} \setminus B_{P_E^1(\mu)}(u, \varepsilon)$. That is impossible.

In the same vein as Theorem 3.3 and 3.5 we present a convergence result in Pettis norm via a vector-valued version of Komlós theorem (see, for instance, [14, 22, 24]) ensuring Komlós convergence for the Pettis norm of L^1-bounded PUI sequences in $L_E^1(\mu)$ where E is a B convex reflexive separable Banach space.

THEOREM 3.7. *Suppose that E is a B convex reflexive separable Banach space (u_n) is a L^1-bounded and PUI sequence in $L_E^1(\Omega, \mathcal{F}, \mu)$, then there exist $u \in L_E^1(\Omega, \mathcal{F}, \mu)$ and a subsequence (v_m) of (u_n) such that*

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma_{i=1}^n w_i - u||_{Pe} = 0$$

for every subsequence $(w_l) = (v_{m_l})$ of (v_m).

Proof. Since (u_n) is bounded in $L_E^1(\Omega, \mathcal{F}, \mu)$ and E is B convex reflexive Banach space, by [14] there is $u \in L_E^1(\Omega, \mathcal{F}, \mu)$ such that (u_n) Komlós converges to u a.e, that is, there exists a subsequence (v_m) of (u_n) such that

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma_{i=1}^n w_i - u|| = 0$$

almost everywhere, for every subsequence $(w_l) = (v_{m_l})$ of (v_m). As (u_n) is Pettis uniformly integrable, so is the sequence $(s_n) = (\frac{1}{n}\Sigma_{i=1}^n w_i)$. By Vitali theorem for

Pettis integrable functions [4, Proposition 2.1], we have that $\lim_{n\to\infty} ||s_n - u||_{Pe} = 0$. The proof is complete.

COROLLARY 3.8. *Suppose that E is a B convex reflexive separable Banach space, $\Gamma : \Omega \to E$ is a convex valued measurable multifunction and (u_n) is a L^1- bounded and PUI sequence in $\mathcal{S}^1_\Gamma$ which converges $\sigma(P^1_E(\mu), L^\infty \otimes E')$ to $u \in P^1_E(\Omega, \mathcal{F}, \mu)$ with $u \in \partial_d(\mathcal{S}^{Pe}_\Gamma)$, then $u \in L^1_E(\Omega, \mathcal{F}, \mu)$ and $\lim_{n\to\infty} ||u_n - u||_{Pe} = 0$.*

Proof. Assume by contradiction that $\limsup_{n\to\infty} ||u_n - u||_{Pe} := \varepsilon > 0$. By extracting a subsequence we may suppose that $||u_n - u||_{Pe} > \frac{\varepsilon}{2}$ for all n. As (u_n) is PUI, applying Theorem 3.7 provides a subsequence (v_m) of (u_n) and $v \in L^1_E(\mu)$ such that

$$\lim_{n\to\infty} ||\frac{1}{n}\Sigma_{i=1}^n v_i - v||_{Pe} = 0.$$

Since (u_n) converges $\sigma(P^1_E(\mu), L^\infty \otimes E')$ to u, we have $u = v$ a.e. But $||v_m - u||_{Pe} > \frac{\varepsilon}{2}$ for all m, so we deduce that

$$u \in \overline{co}[\mathcal{S}^{Pe}_\Gamma \setminus B_{P^1_E(\mu)}(u, \frac{\varepsilon}{2})]$$

where $B_{P^1_E(\mu)}(u, \frac{\varepsilon}{2})$ denotes the open ball of center u and radius $\frac{\varepsilon}{2}$ of the normed space $P^1_E(\mu)$. That is impossible.

Combining Proposition 3.6 and the arguments of Corollary 3.8 we get easily a version of Olech's lemma in $L^1_E(\mu)$.

PROPOSITION 3.9. *Suppose that E is a B convex reflexive separable Banach space, $\Gamma : \Omega \to E$ is a $\mathcal{L}wc(E)$-valued measurable multifunction and (u_n) is a L^1-bounded and PUI sequence in $\mathcal{S}^1_\Gamma$ which converges $\sigma(P^1_E, L^\infty \otimes E')$ to $u \in \mathcal{S}^{Pe}_\Gamma$ with $\int u\, d\mu \in \partial_d(\overline{\int \Gamma\, d\mu})$, then $u \in L^1_E(\mu)$ and $\lim_{n\to\infty} ||u_n - u||_{Pe} = 0$.*

There is another denting property of the Pettis integral of a $ccb(E)$-valued Pettis integrable multifunction.

PROPOSITION 3.10. *Suppose that $\Gamma : \Omega \to E$ is a Pettis integrable $ccb(E)$-valued multifunction and $u \in \mathcal{S}^{Pe}_\Gamma$ such that $\int_\Omega u\, d\mu \in \partial_d(\int_\Omega \Gamma(\omega)\mu(d\omega))$, then $u(\omega) \in \partial_{wd}(\Gamma(\omega))$ a.e.*

Proof. We may suppose that $u \equiv 0$. Let V be a weak neighbourhood of 0 in E. There exist $e'_1, e'_2, ..e'_k$ in $\overline{B}_{E'}$ and $\eta > 0$ such that

$$W := \{x \in E : \sup_{1\leq i\leq k} \langle e'_i, x\rangle < \eta\} \subset V.$$

Then we have

$$\overline{co}(\Gamma(\omega) \setminus V) \subset \overline{co}(\Gamma(\omega) \setminus W)$$

so that we need only to prove that

$$0 \notin \overline{co}(\Gamma(\omega) \setminus W) \ a.e.$$

As 0 is a denting point of $\int_\Omega \Gamma(\omega)\mu(d\omega)$, we have $0 \notin \overline{co}(\int \Gamma d\mu \setminus \eta B_E)$. Hence there exist $x' \in \overline{B}_{E'}$ such that

$$\alpha := \delta^*(x', \int_\Omega \Gamma d\mu \setminus \eta B_E)) < 0.$$

Set $A := \{\omega \in \Omega : \delta^*(x', \Gamma(\omega) \setminus W) \geq \frac{\alpha}{2}\}$. Since W is open in $(E, \sigma(E, E'))$, it is not difficult to check that $A \in \mathcal{F}$. We claim that $\mu(A) = 0$. Assume by contradiction that $\mu(A) > 0$. Let us consider the multifunction Δ defined on A by

$$\Delta(\omega) := \{x \in \Gamma(\omega) \setminus W : \langle x', x \rangle \geq \frac{\alpha}{2}\}.$$

Then Δ is a $c(E)$-valued measurable multifunction. By [18] Δ admits a measurable selection $v : A \to E$. By the Pettis integrability assumption on Γ, v is Pettis integrable. Set $A_i = \{\omega \in A : \langle e_i', v(\omega) \rangle \geq \eta\}$ for $i = 1, 2, ...k$. Set $B_1 = A_1$ and for $2 \leq i \leq k, B_i = A_i \setminus \cup_{j=1}^{i-1} A_j$. Then $(B_i)_{1 \leq i \leq k}$ is a mesurable partition of A. There is a mesurable set B_i with $\mu(B_i) > 0$. By integrating on B_i we get

$$|| \int_{B_i} v \, d\mu || \geq \int_{B_i} \langle e_i', v \rangle \, d\mu \geq \eta \mu(B_i)$$

and

$$\langle x', \int_{B_i} v \, d\mu \rangle = \int_{B_i} \langle x', v \rangle \, d\mu \geq \frac{\alpha}{2} \mu(B_i).$$

As $||x'|| \leq 1$ and $\alpha < 0$, we get

$$|| \int_{B_i} v \, d\mu || \leq -\frac{\alpha}{2} \mu(B_i).$$

It follows that $\eta < -\frac{\alpha}{2}$ which contradicts the definition of α. Hence we conclude that $0 \notin \overline{co}(\Gamma(\omega) \setminus W)$ a.e.

The variations of our techniques allow to obtain other variants in $L_E^1(\mu)$. Let us mention only the following. Let us recall that a subset $\mathcal{H}$ in $L_E^1(\mu)$ has the weak Talagrand property (shortly WTP) [11] if, given any L^1-bounded sequence (u_n) in $\mathcal{H}$ there is a sequence $(\widetilde{u}_n)$ with $\widetilde{u}_n \in co\{u_m : m \geq n\}$ such that $(\widetilde{u}_n)$ converges weakly a.e in E. Any bounded and weak tight sequence in $L_E^1(\mu)$ has the WTP [11, Theorem 2.8]. $L_E^1(\mu)$ has the WTP iff E is reflexive. For more on WTP sets in $L_E^1(\mu)$, we refer to [11, 24]. Now we proceed to a variant of Theorem 3.7.

PROPOSITION 3.11. *Suppose that E is separable Banach space (u_n) is an L^1-bounded, WTP and PUI sequence in $L_E^1(\Omega, \mathcal{F}, \mu)$, then there exist $u \in P_E^1(\Omega, \mathcal{F}, \mu)$ and a sequence (v_n) with $v_n \in co\{u_m : m \geq n\}$ such that*

$$\lim_{n \to \infty} ||v_n - u||_{Pe} = 0.$$

Proof. Since (u_n) is bounded WTP in $L^1_E(\Omega, \mathcal{F}, \mu)$, by [11, Theorem 2.9] there is a sequence (v_n) in $L^1_E(\Omega, \mathcal{F}, \mu)$ with $v_n \in co\{u_m : m \geq n\}$ such that (v_n) converges *strongly* a.e to a function $u \in L^1_E(\Omega, \mathcal{F}, \mu)$. As (u_n) is Pettis uniformly integrable, so is the sequence (v_n). By Vitali theorem for Pettis integrable functions [4, Proposition 2.1], we have that $\lim_{n\to\infty} ||v_n - u||_{Pe} = 0$. The proof therefore is complete.

Using Proposition 3.11 and Proposition 3.6 we get easily another version of Olech's lemma in $L^1_E(\mu)$. We omit the proof since it follows from the arguments given in Corollary 3.8.

PROPOSITION 3.12. *Suppose that E is a separable Banach space, $\Gamma : \Omega \to E$ is a $\mathcal{L}wc(E)$-valued measurable multifunction and (u_n) is a L^1-bounded and PUI sequence in S^1_Γ which converges $\sigma(P^1_E, L^\infty \otimes E')$ to $u \in S^{Pe}_\Gamma$ with $\int_\Omega u\, d\mu \in \partial_d(\overline{\int_\Omega \Gamma\, d\mu})$, then $u \in L^1_E(\mu)$ and $\lim_{n\to\infty} ||u_n - u||_{Pe} = 0$.*

It is enough to observe that (u_n) has the WTP by [11, Theorem 2.9] so that Proposition 3.12 is a direct consequence of Proposition 3.6 and Proposition 3.11.

Now let us mention some remarks concerning Olech-types lemma under extreme point condition.

REMARKS. 1.- If E is $\mathbb{R}^d$ the techniques given in the proof of Theorem 2.3 allow to recover easily Olech's lemma [30]. Firstly, if μ is atomless, the multivalued integral of closed valued multifunction is always convex. Secondly, an extreme point of a closed convex subset of $\mathbb{R}^d$ is a denting point [33, Lemma 1, page 5.4] so that if one suppose in Theorem 2.3, μ is atomless, E is $\mathbb{R}^d$ and $e \in \partial_e\, cl(\int_\Omega \Gamma(\omega)\, \mu(d\omega))$ where $\Gamma : \Omega \to \mathbb{R}^d$ is a closed valued measurable multifunction, then by [33, Lemma 2, page 5.5] asserting that

$$\forall v \in L^1_{\mathbb{R}^d}(\Omega, \mathcal{F}, \mu),\ ||v||_{L^1} \leq 2d \sup_{A \in \mathcal{F}} ||\int_A v\, d\mu||_{\mathbb{R}^d},$$

we get an alternative proof of Olech's lemma. We refer to [5, 8] for other related results.

2.- The next version of Olech's lemma in $P^1_E(\mu)$ is based essentially on the following Visintin-type theorem.

PROPOSITION 3.13. *Suppose that E is a separable Banach space, $\Phi : \Omega \to E$ is a $\mathcal{L}wc(E)$-valued measurable multifunction, $(u_n)_{n\in\mathbb{N}}$ is a Pettis uniformly integrable and $ck(E)$-tight sequence in S^{Pe}_Φ which converges $\sigma(P^1_E(\mu), L^\infty \otimes E')$ to $u \in S^{Pe}_\Phi$ with $u(\omega) \in \partial_e(\Phi(\omega))$ a.e, then $||u_n - u||_{Pe} \to 0$.*

Proof. See [4, Theorem 2.4].

Using the preceding result we get

PROPOSITION 3.14. *Suppose that E is a separable Banach space, $\Phi : \Omega \to E$ is a $\mathcal{L}wc(E)$-valued measurable multifunction, $(u_n)_{n\in\mathbb{N}}$ is a Pettis uniformly integrable and $ck(E)$-tight sequence in S^{Pe}_Φ which converges $\sigma(P^1_E(\mu), L^\infty \otimes E')$ to $u \in P^1_E(\mu)$ with $\int_\Omega u(\omega)\, \mu(d\omega) \in \partial_e(\int_\Omega \Phi(\omega)\mu(d\omega))$, then $||u_n - u||_{Pe} \to 0$.*

Proof. Note that $u \in \mathcal{S}_{\phi}^{Pe}$ because $u_n \to u$ for the $\sigma(P_E^1(\mu), L^\infty \otimes E')$ topology and Φ is $\mathcal{L}wc(E)$-valued (see e.g. [4]). In view of Proposition 3.13, it is enough to check that $u(\omega) \in \partial_e(\Phi(\omega))$ a.e. But this fact follows easily from the arguments given in [4, Theorem 2.4]. Suppose not. Then there exist a $\mathcal{F}$-measurable set A such that

$$A \subset \{\omega \in \Omega : u(\omega) \notin \partial_e(\Phi(\omega)\}$$

with $\mu(A) > 0$. As in [4, Theorem 1.3], it is easy to find two Pettis integrable selections g and h of Φ such that

$$g \neq h \text{ and } u = \frac{1}{2}(g+h).$$

Let us consider a $\mathcal{F}$-measurable set $B \subset A$ of positive measure such that

$$\int_B g\, d\mu \neq \int_B h\, d\mu$$

and set

$$g_1 = 1_B g + 1_{\Omega \setminus B} u \text{ and } g_2 = 1_B h + 1_{\Omega \setminus B} u.$$

Then we have

$$e = \int_\Omega u\, d\mu = \frac{1}{2}(\int_\Omega g_1\, d\mu + \int_\Omega g_2\, d\mu)$$

with $\int_\Omega g_1\, d\mu \neq \int_\Omega g_2\, d\mu$, thus contradicting the extreme nature of e.

4. Visintin-type theorem in $L^1_{E'}[E]$. Let E be a Banach space. For the sake of completeness we will recall the following notations and notions and summarize some useful results [12] in the space $L^1_{E'}[E]$ before we state the main result in this section. We denote by $\mathcal{L}^1_{E'}[E]$ the vector space of scalarly measurable functions $f : \Omega \to E'$ such that there exists a positive integrable function h (depending on f) such that $\forall \omega \in \Omega, ||f(\omega)|| \leq h(\omega)$. A semi-norm on $\mathcal{L}^1_{E'}[E]$ is defined by

$$N_1(f) = \int_\Omega^* ||f(\omega)||\, \mu(d\omega) = \inf\{\int_\Omega h\, d\mu : h \text{ integrable; } h \geq ||f||\}.$$

Two functions $f, g \in \mathcal{L}^1_{E'}[E]$ are *equivalent* (shortly $f \equiv g \ (w^*)$) if, $\langle f(.), x\rangle = \langle g(.), x\rangle$ a.e. for every $x \in E$. The equivalence class of f is denoted by $\overline{f}$. The quotient space $L^1_{E'}[E]$ is equipped with the norm $\overline{N}_1$ given by

$$\overline{N}_1(\overline{f}) = \inf\{\overline{N}_1(g) : g \in \overline{f}\}.$$

Let $\dot{\rho}$ [12] be the lifting in $\mathcal{L}^\infty_{E'}[E]$ associated to a lifting ρ in $\mathcal{L}^\infty_{\mathbb{R}}(\mu)$. We denote by $\mathcal{L}^{1;\rho}_{E'}[E]$ the vector space of all mappings $f \in \mathcal{L}^1_{E'}[E]$ such that there exists a sequence $(A_n)_{n\geq 1}$ in $\mathcal{F}$ satisfying :

$$\bigcup_{n\geq 1} A_n = \Omega \text{ and } \forall n \geq 1,\ 1_{A_n} f \in \mathcal{L}^\infty_{E'}[E] \text{ and } \dot{\rho}(1_{A_n} f) = 1_{\rho(A_n)} f.$$

If $f \in \mathcal{L}^{1,\rho}_{E'}[E]$, $||f(.)||$ is measurable [12, Prop. 2.4 (6)] and the quotient space $L^{1,\rho}_{E'}[E]$ is equipped with the norm

$$N_{1,\rho}(\overline{f}) = N_1(||f||) = \int_\Omega ||f||\, d\mu.$$

By [12, Theorem 2.5] there is a linear isometric isomorphism $\widetilde{\rho} : (L^1_{E'}[E], \overline{N}_1) \to (L^{1,\rho}_{E'}[E], N_{1,\rho})$ so that $L^1_{E'}[E]$ and $L^{1,\rho}_{E'}[E]$ can be indentified. In this identification $\overline{f} \in L^1_{E'}[E]$ is identified with $\widetilde{\rho}(\overline{f})$ and for notational convenience, $\overline{f}$ is identified with a function $f \in \mathcal{L}^{1,\rho}_{E'}[E]$. Let $ck(E')$ (resp. $cwk(E')$) be the set of all nonempty convex norm compact (resp. $\sigma(E', E'')$ compact) subsets of the Banach space E'. A $cwk(E')$-valued multifunction $\Gamma : \Omega \to E'$ is scalarly measurable (resp. integrable) if, for every $x \in E$, the function $\delta^*(x, \Gamma(.))$ is measurable (resp. integrable), where $\delta^*(x, K)$ denotes the support function of $K \in cwk(E')$.

PROPOSITION A [12, Proposition 4.1, page 30]. *Suppose that $\Gamma : \Omega \to E'$ is a $cwk(E')$-valued multifunction and (f_n) is a uniformly integrable sequence in $L^1_{E'}[E]$ such that $f_n(\omega) \in \Gamma(\omega)$ for a.e $\omega \in \Omega$ and for all n, then (f_n) is relatively $\sigma(L^1_{E'}[E](\mu), (L^1_{E'}[E](\mu))')$ (weakly) compact in $L^1_{E'}[E]$.*

We only sketch the proof. By Theorem 3.9 in [12] there are a sequence (g_n) with $g_n \in co\{f_m : m \geq n\}$ and two measurable sets A and B in Ω with $\mu(A \cup B) = 1$ such that

(a) $\forall \omega \in A, (g_n(\omega))$ is $\sigma(E', E'')$ Cauchy in E'.

(b) $\forall \omega \in B$, there exists $k \in \mathbb{N}$ such that the sequence $(g_n(\omega))_{n \geq k}$ is equivalent to the vector unit basis of l^1.

As $\Gamma(\omega)$ is $\sigma(E', E'')$-compact for all $\omega \in \Omega$, using (b) one has $\mu(B) = 0$. Hence there is a sequence (g_n) in $L^1_{E'}[E]$ with $g_n \in co\{f_m : m \geq n\}$ such that $(g_n(\omega))$ is $\sigma(E', E'')$-convergent a.e. By Theorem 3.5 in [12] (g_n) is $\sigma(L^1_{E'}[E](\mu), (L^1_{E'}[E](\mu))')$ convergent in $L^1_{E'}[E]$. Hence (f_n) is relatively weakly compact in $L^1_{E'}[E]$ by a general criterion for weak compactness in Banach spaces.

In the remainder of this section we shall suppose that E is a *separable* Banach space. Using Proposition A and the separability of E we have

COROLLARY B. *Suppose that $\Gamma : \Omega \to E'$ is a scalarly measurable $cwk(E')$-valued multifunction and there is $g \in L^1_{\mathbb{R}+}$ such $\Gamma(\omega) \subset g(\omega)\overline{B}_{E'}$ for all ω in Ω, then the set $\mathcal{S}_\Gamma$ of all scalarly integrable selections of Γ is convex weakly compact in $L^1_{E'}[E]$.*

PROOF. See [12, Corollary 4.2].

For notational convenience such a multifunction Γ is said to be *integrably bounded*. Unlike the space $L^1_E(\mu)$, the preceding results are not standard and rely on a deep result involving the Talagrand decomposition in $L^{1,\rho}_{E'}[E](\mu)$ [12, Theorem 3.9].

A uniformly integrable sequence (u_n) in $L^1_{E'}[E](\mu)$ is *norm-tight* if for every $\varepsilon > 0$ there is a scalarly $ck(E')$-valued measurable and integrably bounded multifunction $\Phi_\varepsilon : \Omega \to E'$ with $0 \in \Phi_\varepsilon(\omega)$ for all $\omega \in \Omega$ such that

$$\sup_n \mu(\{\omega \in \Omega : u_n(\omega) \in \Phi_\varepsilon(\omega)\}) \leq \varepsilon.$$

It is easily seen that u_n can be written as $u_n = 1_{A_n} u_n + 1_{\Omega\setminus A_n} u_n$ where $A_n \in \mathcal{F}$ and $1_{A_n} u_n \in S_{\Phi_\varepsilon}$ and $||1_{\Omega\setminus A_n} u_n||_{L^1_{E'}[E]} \leq \varepsilon$, so that a uniformly integrable norm-tight sequence (u_n) in $L^1_{E'}[E](\mu)$ is relatively weakly compact in view of Proposition A and Grothendiek lemma, see e.g [3, page 183] for details.

Now we are able to present a version of Visintin theorem in $L^1_{E'}[E](\mu)$ in same style as in [3, Lemme 10 and Théorème 11] and [31]. Since the proof follows the same lines, we don't want to give details so much. Yet this needs a careful look. In the following by *weakly converges* we mean $\sigma(L^1_{E'}[E](\mu), (L^1_{E'}[E](\mu))')$ *converges.*

THEOREM 4.1. *Suppose that (u_n) is uniformly integrable norm-tight sequence in $L^1_{E'}[E](\mu)$ weakly converging to $u \in L^1_{E'}[E](\mu)$ such that*

$$u(\omega) \in \partial_e(\cap_{n\in\mathbb{N}} \overline{co}\,[\{u_k(\omega) : k \geq n\}]) \ a.e,$$

then $\int_\Omega ||u_n(\omega) - u(\omega)||\, \mu(d\omega) \to 0$.

Proof. We will divide the proof in two steps.

Step 1. We will prove the theorem in the particular case when (u_n) is a uniformly integrable sequence in $L^1_{E'}[E](\mu)$ weakly converging to $u \in L^1_{E'}[E](\mu)$ and satisfying:

(a) there is a convex norm compact multifunction $\Gamma : \Omega \to E'$ such that $u_n(\omega) \in \Gamma(\omega)$ for all n and all $\omega \in \Omega$,

(b) $u(\omega) \in \partial_e(\cap_{n\in\mathbb{N}} \overline{co}[\{u_k(\omega) : k \geq n\}])$.

For every $r > 0$, let us denote by $B_{E'}(0, r)$ the open ball

$$B_{E'}(0, r) := \{x' \in E' : ||x'|| < r\}$$

in the Banach space E'. Since $(||u_n||)_n$ is uniformly integrable it suffices to prove that $||u_n - u|| \to 0$ in measure. We may suppose $u \equiv 0$. Suppose not. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that

$$\mu(\{\omega \in \Omega : u_n(\omega) \notin B_{E'}(0, \varepsilon)\}) \geq \eta. \tag{4.1.1}$$

(*) for infinitely many n; namely there exists an infinite subset $S_1 \subset \mathbb{N}$ such that the preceding inequality holds for all $n \in S_1$. For every $\omega \in \Omega$, let

$$\Sigma_n(\omega) := \overline{co}\,[u_k(\omega) : k \geq n\}] \ \text{ and } \ \Sigma(\omega) := \cap_{n\in\mathbb{N}} \Sigma_n(\omega).$$

Since the function $x' \mapsto ||x'||$ is lower semicontinuous on $E'_{w^*}, B_{E'}(0, r)$ is a Borel subset of E'_{w^*}. As E'_{w^*} is a Lusin space and any scalarly integrable multifunction

(*) The measurability of $\{\omega \in \Omega : u_n(\omega) \notin B_{E'}(0, \varepsilon)\}$ will be demonstrated later.

from Ω to the set $ck(E'_{w^*})$ of nonempty convex compact subsets of E'_{w^*} has its graph in $\mathcal{F} \otimes \mathcal{B}(E'_{w^*})$ where $\mathcal{B}(E'_{w^*})$ is the Borel tribe of E'_{w^*}. Now set

$$A := \{\omega \in \Omega : \Sigma(\omega) \setminus B_{E'}(0, \varepsilon) \neq \emptyset\}$$

$$B := \{\omega \in \Omega : \Sigma(\omega) \subset B_{E'}(0, \varepsilon)\}$$

$$B_n := \{\omega \in \Omega : \Sigma_n(\omega) \subset B_{E'}(0, \varepsilon)\}.$$

As the graph of the multifunctions Σ and Σ_n belong to $\mathcal{F} \otimes \mathcal{B}(E'_{w^*})$ and $B_{E'}(0, r)$ is a Borel subset of E'_{w^*}, by a classical mesurable projection theorem [18, Theorem III.23] we see that $\{w \in \Omega : u_n(\omega) \notin B_{E'}(0, \varepsilon)\}$, A, B_n, B are $\mathcal{F}$-measurable. Furthermore we have $B_n \uparrow B$ because if $\omega \in B$ we have that $\cap_n \Sigma_n(\omega) \setminus B_{E'}(0, \varepsilon) = \emptyset$ so that by finite intersection property of compact spaces there is an integer m such that $\cap_{n \geq m} \Sigma_n(\omega) \setminus B_{E'}(0, \varepsilon) = \emptyset$. Pick N_1 such that $n \geq N_1$ implies $\mu(B \setminus B_n) < \frac{\eta}{2}$. Since

$$\{\omega \in B : u_n(\omega) \notin B_{E'}(0, \varepsilon)\} \subset B \setminus B_n$$

we get

$$n \geq N_1 \implies \mu(\{\omega \in B : u_n(\omega) \notin B_{E'}(0, \varepsilon)\} < \frac{\eta}{4}. \tag{4.1.2}$$

Let us write $u_n = v_n + w_n$ where

$$v_n := 1_{\{\omega \in \Omega : u_n(\omega) \notin B_{E'}(0, \varepsilon)\}} u_n$$

and

$$w_n := 1_{\{\omega \in \Omega : u_n(\omega) \in B_{E'}(0, \varepsilon)\}} u_n.$$

Then the sequence (v_n) is relatively sequentially $\sigma(L^1_{E'}[E], (L^1_{E'}[E])')$ compact in view of Proposition A and Eberlein-Smulian theorem. There is a subsequence $(v_n)_{n \in S_2}$ where S_2 is an infinite subset of S_1 such that $(v_n)_{n \in S_2}$ converges to $v \in L^1_{E'}[E](A \cap \mathcal{F}, \mu)$ for this topology. It follows that $(w_n)_{n \in S_2}$ weakly converges to $u - v$. Using a version of Mazur's theorem in $L^1_{E'}[E]$ [12, Lemma 3.12] and [3, Lemma 4] we get

$$v(\omega) \in \cap_{n \in S_2} \overline{co}\,[\{v_k(\omega) : k \geq n, k \in S_2\}] \subset \Sigma(\omega) \ a.e.$$

Similarly we have $w(\omega) \in \Sigma(\omega)$ *a.e.* Since $0 \in \partial_{ext}(\Sigma(\omega))$ a.e and u_n weakly converges to 0, we get $v = w = 0$. As $\Sigma(\omega)$ is norm compact and convex in E' we have that $\partial_e(\Sigma(\omega)) = \partial_d(\Sigma(\omega))$ in view of [3, Lemme 1]. So we have

$$0 \notin \overline{co}\,[\Sigma(\omega) \setminus B_{E'}(0, \varepsilon)] \ a.e.$$

It is obvious that $\overline{co}\,[\Sigma(\omega) \setminus B_{E'}(0, \varepsilon)]$ is nonempty convex norm compact (a fortiori $\sigma(E', E)$ compact in E') whenever $\omega \in A$. Hence the multifunction Ψ defined from

A with nonempty values in the closed unit ball $\overline{B}_E$ of E (thanks to the Hahn-Banach Theorem) :

$$\Psi(\omega) := \{x \in \overline{B}_E : \delta^*(x, \overline{co}\,[\Sigma(\omega) \setminus B_{E'}(0,\varepsilon)] < 0\}$$

has its graph in $A \cap \otimes \mathcal{B}(\overline{B}_E)$ [18, Lemma III.14]. By [18, Theorem II. 22], Ψ admits a $\mathcal{F}$-measurable selection $\sigma : A \to \overline{B}_E$. Since $\Sigma_n(\omega) \setminus B_{E'}(0,\varepsilon) \downarrow \Sigma_(\omega) \setminus B_{E'}(0,\varepsilon)$ we get

$$\delta^*(\sigma(\omega), \overline{co}\,[\Sigma_n(\omega) \setminus B_{E'}(0,\varepsilon)]) \to \delta^*(\sigma(\omega), \overline{co}\,[\Sigma_(\omega) \setminus B_{E'}(0,\varepsilon)]). \tag{4.1.3}$$

By [3, Lemma 3], there are $a < 0$ and N_2 such that

$$n \geq N_2 \implies \mu(\{\omega \in A : \delta^*(\sigma(\omega), \overline{co}\,[\Sigma_(\omega) \setminus B_{E'}(0,\varepsilon)]) > a\}) < \frac{\eta}{4}. \tag{4.1.4}$$

As v_n is either $= 0$ or belongs to $[\Sigma_n(\omega) \setminus B_{E'}(0,\varepsilon)]$, we have $\limsup_n \langle \sigma(\omega), v_n(\omega)\rangle \leq 0$. Now since $v_n \to 0$ for $\sigma(L^1_{E'}[E], (L^1_{E'}[E])')$ and $L^\infty_E(\mu) \subset (L^1_{E'}[E])'$ in view of [12, page 18], for every $h \in L^\infty_{\mathbb{R}}(A \cap \mathcal{F}, \mu)$ we get

$$\int_A h(\omega)\langle \sigma(\omega), v_n(\omega)\rangle\, \mu(d\omega) \to 0. \tag{4.1.5}$$

Therefore by [3, Cor. D], $\langle \sigma(.), v_n(.)\rangle \to 0$ in measure. Consequently there exist N_3 such that

$$n \geq N_3, (n \in S_2) \implies \mu(\{\omega \in A : \langle \sigma(\omega), v_n(\omega)\rangle \leq a\}) < \frac{\eta}{4}. \tag{4.1.6}$$

Now observe that

$$\begin{aligned} &\{\omega \in A : v_n(\omega) \neq 0\} \\ &\subset \{\omega \in A : v_n(\omega) \neq 0 \text{ and} \langle \sigma(\omega), v_n(\omega)\rangle > a\} \cup \{\omega \in A : \langle \sigma(\omega), v_n(\omega)\rangle \leq a\} \\ &\subset \{\omega \in A : \delta^*(\sigma(\omega), \overline{co}\,[\Sigma_n(\omega) \setminus B_{E'}(0,\varepsilon)] > a\} \cup \{\omega \in A : \langle \sigma(\omega), v_n(\omega)\rangle > a\}. \end{aligned}$$

By (4.1.5) and (4.1.6) we get

$$n \geq \max(N_2, N_3) \implies \mu(\{\omega \in A : v_n(\omega) \neq 0\}) < \frac{\eta}{2}. \tag{4.1.7}$$

But

$$\begin{aligned} &\{\omega \in \Omega : u_n(\omega) \notin B_{E'}(0,\varepsilon)\} \\ &= \{\omega \in A : u_n(\omega) \notin B_{E'}(0,\varepsilon)\} \cup \{\omega \in B : u_n(\omega) \notin B_{E'}(0,\varepsilon)\} \\ &= \{\omega \in A : v_n(\omega) \neq 0\} \cup \{\omega \in B : u_n(\omega) \notin B_{E'}(0,\varepsilon)\}. \end{aligned}$$

So by (4.1.4) and (4.1.7) and for $n \geq \max(N_1, N_2, N_3)$, we get

$$n \geq \max(N_1, N_2, N_3) \implies \mu(\{\omega \in A : u_n(\omega) \notin B_{E'}(0,\varepsilon)\}) < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

That contradicts (4.1.1).

Step 2. Now we pass to the general case. We suppose that (u_n) is *norm-tight*. Let $\varepsilon > 0$. There is a scalarly $ck(E')$-valued measurable and integrably bounded multifunction $\Phi_\varepsilon : \Omega \to E'$ with $0 \in \Phi_\varepsilon(\omega)$ for all $\omega \in \Omega$ such that u_n can be written as

$$u_n = 1_{A_n} u_n + 1_{\Omega \setminus A_n} u_n$$

where $A_n \in \mathcal{F}$ and $1_{A_n} u_n \in \mathcal{S}_{\Phi_\varepsilon}$ and $||1_{\Omega\setminus A_n} u_n||_{L^1_{E'}[E]} \leq \varepsilon$. By Corollary B , we may suppose that $(v_n) = (1_{A_n} u_n)$ weakly converges to $v \in \mathcal{S}^{Pe}_\Phi$, by extracting a subsequence if necessary. Hence we have

$$0 = \text{weak-} \lim_{n\to\infty} u_n = \text{weak-} \lim_{n\to\infty} [v_n + w_n] = v + w$$

with $w_n = 1_{\Omega\setminus A_n} u_n$ and $w \in \mathcal{S}^{Pe}_\Phi$ similarly. By Mazur's theorem in $L^1_{E'}[E]$ [12, Lemma 3.12] and [3, Lemma 4] we have

$$v(\omega) \in \cap_{n\in\mathbb{N}} \overline{co}[\{v_k(\omega) : k \geq n\}] \subset \cap_{n\in\mathbb{N}} \overline{co}\,[\{u_k(\omega) : k \geq n\}] \ \ a.e. \tag{4.1.8}$$

Similarlly

$$w(\omega) \in \cap_{n\in\mathbb{N}} \overline{co}\,[\{u_k(\omega) : k \geq n\}] \ \ a.e. \tag{4.1.9}$$

As $0 \in \partial_e(\cap_{n\in\mathbb{N}} \overline{co}\,[\{u_k(\omega) : k \geq n\}])$ by hypothesis, applying the arguments of Step 1 to v_n and w_n gives $v = w = 0$. Again by [3, Lemma 4] we get

$$0 = v(\omega) \in \partial_e(\cap_{n\in\mathbb{N}} \overline{co}\,[\{v_k(\omega) : k \geq n\}]) \ \ a.e. \tag{4.1.10}$$

By (4.1.10) we can apply the results stated in Step 1 to the sequence $(v_n)_n$ showing that $||v_n||_{L^1_{E'}[E]} \to 0$. Since

$$||u_n||_{L^1_{E'}[E]} \leq ||v_n||_{L^1_{E'}[E]} + ||w_n||_{L^1_{E'}[E]} \leq ||v_n||_{L^1_{E'}[E]} + \varepsilon$$

for all $n \in \mathbb{N}$ and ε is arbitrary > 0, $||u_n||_{L^1_{E'}[E]} \to 0$.

To finish the paper let us mention an easy variant of Theorem 4.1.

PROPOSITION 4.2. *Suppose that E is a Banach space with strong separable dual, $\Gamma : \Omega \to E$ is a closed convex measurable multifunction and (u_n) is a sequence in $\mathcal{S}^1_\Gamma$ which converges $\sigma(L^1, L^\infty)$ to $u \in \mathcal{S}^1_\Gamma$ with $u(\omega) \in \partial_{pc}(\Gamma(\omega))$ a.e, then $||u_n - u||_{L^1} \to 0$.*

Proof. Using the separability of the dual of E', it is easy to check that (u_n) has the weak Komlós property. Namely there there exist $u \in L^1_E(\Omega, \mathcal{F}, \mu)$ and a subsequence (v_m) of (u_n) such that for all $e^* \in E'$ and almost all $\omega \in \Omega$

$$\lim_{n\to\infty} \langle e^*, \frac{1}{n}\Sigma^n_{j=1} w_j(\omega)\rangle = \langle e^*, u(\omega)\rangle$$

a.e for each subsequence $(w_l) = (v_{m_l})$. Since $u(\omega) \in \partial_{pc}(\Gamma(\omega))$ a.e we have

$$\lim_{n\to\infty} \frac{1}{n}\Sigma^n_{j=1} w_j(\omega) = u(\omega)$$

a.e and strongly in $L^1_E(\mu)$.

COMMENTS. (1) Visintin-types theorem were extensively studied by classical methods [36, 33, 5], Young measures [32, 6], Truncation methods [3, 4], K-convergence (alias Komlós convergence) [6, 7]. Several refinements of this problem were stated in [6, 7] as well as the connection between Olech-types lemma and Visintin-types theorem [8]. A good synthesis of this problem is in Valadier's exposé [33].

Visintin-types theorem and Olech-types lemma in Pettis integration have been studied recently in [4].

(2) The present paper is a continuation of the preceding studies, mainly we give several variants of the problem of " norm convergence is implied by the weak'" under denting point conditions in both Bochner and Pettis integration as well we provide a characterization of strong extreme points in the same vein as in [28]. So it is worth to mention the characterization of the denting points of the convex set of Bochner integrable selections [10, 19, 27]. Unfortunately we are unable to give a complete characterization for the analoguous ones in Pettis integration [see Proposition 3.2]. That is an open problem. Theorem 4.1 is related to the problem of " norm convergence is implied by the weak" under extreme point condition in $L^1_{E'}[E]$ extending Lemma 3 and Theorem 11 in [3] and Theorem 4.3 in [12]. Yet the proof is based upon the truncation techniques developped in [3] and several delicate results in [12]. Actually we are unable to derive Theorem 4.1 from the theory of Young measures [6, 9, 32, 33] or Komlós convergence [6, 7, 9]. The problem of "norm convergence is implied by the convergence in $\mathcal{D}'$ " under strict convexity was stated by Brezis [15] using a quite different technique. An other approach of the problem of "norm convergence is implied by the weak" under strict convexity is in [13].

(4) It is worth to pose the following question. Suppose that $(u_n)_n$ is a sequence in $L^1_E(\mu)$ which converges $\sigma(L^1_E(\mu), L^\infty_{E'}(\mu))$ to $u \in L^1_E(\mu)$ with $u(\omega) \in \partial_d(\cap_{n\geq 0}\overline{co}\{u_k(\omega) : k \geq n\})$ a.e. Does $||u_n - u||_{L^1} \to 0$. The answer is negative by considering the following example (see also an analogous one in [3, page 180]) which has been kindly communicated to us by M. Valadier. Let $(e_n)_n$ denotes the orthonormal basis of a separable Hilbert space H. Set

$$\overline{co}\{e_k : k \geq n\} = \{\Sigma_{k=n}^{\infty}\lambda_k e_k : \lambda_k \geq 0 \text{ and } \Sigma_{k=n}^{\infty}\lambda_k \leq 1\}.$$

We have

$$\cap_{n\geq 0}\ \overline{co}\{e_k : k \geq n\} = \{0\}.$$

Set $u_n = e_n$ and $u \equiv 0$. Then $u_n \to u$ for $\sigma(L^1_H(\mu), L^\infty_H(\mu))$ using the dominated convergence theorem while $||u_n||_{L^1}$ does not tend to 0. But now suppose that E is a separable reflexive Banach space and $\Gamma : \Omega \to E$ is a closed convex valued measurable multifunction, (u_n) is a sequence in $L^1_E(\mu)$ which converges $\sigma(L^1_E(\mu), L^\infty_{E'}(\mu))$ to $u \in L^1_E(\mu)$ with $u_n(\omega) \in \Gamma(\omega)$ for all n and all $\omega \in \Omega$ and $u(\omega) \in \partial_d(\Gamma(\omega))$ a.e, then $\lim_{n\to\infty} ||u_n - u||_{L^1} = 0$. Namely that version of Visintin's theorem is valid in separable reflexive Banach spaces. See e.g [33, Theorem 7] using Hahn-Banach theorem and measurable selection theorem as in [3, Lemma

10] and [4, Lemma 2.2]. An extension of this result to separable Banach space has been obtained in [7] under weak tightness assumption using Komlós convergence. It turns out that the above mentioned example does not satisfy the assumption of the preceding version of Visintin's theorem. See also [7, Corollary 3.6] and Proposition 4.2. Indeed $u(\omega)$ is not a denting point of $\Gamma(\omega)$ when $\Gamma(\omega)$ is a closed convex multifunction satisfying $\forall n, u_n(\omega) \in \Gamma(\omega)$ because, for every $\varepsilon \in]0,1[, 0 \in \overline{co}\{e_n : n \geq 0\} \subset \overline{co}[\Gamma(\omega) \setminus B_E(0,\varepsilon)]$.

Apart from the use of truncation method and Komlós arguments, we stress the fact that most techniques employed here are elementary.

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References

[1] A. Amrani, *Entaillabilité et convergence dans l'espace L^1*, Bull. Polish Acad. Sc. **42** (1994), 229-236.

[2] A. Amrani and C. Castaing, *Weak compactness in Pettis integration*, Bull. Polish Acad. Sc. **45** (1997), 139-150.

[3] A. Amrani, C. Castaing and M. Valadier, *Méthodes de troncatures appliquées à des problèmes de convergence faible où forte dans L^1*, Arch. Rational Mech. Anal. **117** (1992), 167-191.

[4] A. Amrani, C. Castaing and M. Valadier, *Convergence in Pettis norm under extreme point condition*, to appear in Vietnam Journal of Mathematics **4** (1998).

[5] Z. Artstein and T. Rzeżuchowski, *A note on Olech's lemma*, Studia Math. **98** (1991), 91-94.

[6] E. J. Balder, *On equivalence of strong and weak convergence in L^1 -spaces under extreme point condition*, Israel J. Math. **75** (1991), 21-47.

[7] E. J. Balder, *From weak to strong convergence in L^1-spaces via K-convergence*, Annali di Matematica pura ed applicata **165** (1993), 337-349.

[8] E. J. Balder, *Connections between recent Olech-types lemmas and Visintin's theorem*, Banach Center Publications **32** (1995), 47-52.

[9] E. J. Balder, *Lectures on Young measures, Preprint no 9517*, CEREMADE, Université Paris-Dauphine, France (1995).

[10] H. Benabdellah, *Extrémalité, stricte convexité et convergence in L^1_E*, Séminaire Anal. Convexe Montpellier **21** (1991), 4.1-4.18.

[11] H. Benabdellah and C. Castaing, *Weak compactness criteria and convergences in $L^1_E(\mu)$*, Collectanea Mathematica **XLVIII** (1997), 423-448..

[12] H. Benabdellah and C. Castaing, *Weak compactness and convergences in convergences in $L^1_{E'}[E]$*, Preprint (1998), 1-40.

[13] J. M. Borwein and A. S Lewis, *Strong rotundity and optimization*, Siam J. Optimization No 1 **4** (1994), 146-158.

[14] J. Bourgain, *The Komlós theorem for vector valued functions*, Wrije Universiteit Brussel (1979), 9 pages.

[15] H. Brezis, *Convergence in $\mathcal{D}'$ and in L^1 under strict convexity jour To appear in the volume in honor of E. Magenes.*

[16] C. Castaing, *Convergences faible et sections extrémales*, Séminaire Analyse Convexe Montpellier (1988), 2.1-2.18.

[17] C. Castaing, *Weak compactness and convergences in Bochner and Pettis integration*, Vietnam Journal of Math. No 3 **24** (1996), 241-286.

[18] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lectures Notes in Math., **580** Springer-Verlag, Berlin (1977).

[19] C. Castaing and R. Pluciennik, *Denting points in Köthe-Bochner Spaces*, Set-Valued Analysis **2(3)** (1994), 439-458.

[20] C. Dellacherie et P. Meyer, *Probabilités et Potentiel*, Hermann, Paris (1975).

[21] J. Diestel and J.J. Uhl, *Vector measures*, Amer. Math. Soc. Providence (1977).

[22] D. J. H. Garling, *Subsequence principles for vector-valued random variables*, Math. Proc. Camb. Philos. Soc. **86** (1979), 301-311.

[23] R. Geitz, *Pettis integration*, Proc. Amer. Math. Soc., **82** (1981), 81-86.

[24] M. Guessous, *An elementary proof of Komlós-Révész theorem in Hilbert spaces*, Journal of Convex Analysis **4** (1997), 321-332.

[25] R. Huff, *Remarks on Pettis integration*, Proc. Amer. Math. Soc. **96** (1986), 402-404.

[26] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. **18** (1967), 217-229.

[27] B. L. Lin, P. K. Lin and S. L. Troyanski, *Denting points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **97(4)** (1986), 629-633.

[28] B. L. Lin, P. K. Lin and S. L. Troyanski, *Characterizations of denting points*, Proc. Amer. Math. Soc. **102** (1988), 526-528.

[29] K. Musial, *Topics in the theory of Pettis integration*, Rendiconti dell'istituto di matematica dell'Università di Trieste, School on Measure Theory and Real Analysis Grado (Italy), 14-25 ottobre 1991, 176-262.

[30] C. Olech, *Existence theory in optimal control*, in: Control theory and topics in functional analysis, IAEA, Vienna **1** (1976), 191-228.

[31] T. Rzeżuchowski, *Impact of dentability on weak convergence in L^1*, Bolletino U.M.I **7** (1992), 71-80.

[32] L. Tartar, *Compensated compactnesss and applications to partial differential equations*, Nonlinear Analysis and Mechanics; Heriot-Watt Symposium, (ed. R.J. Knops), Research Notes in Math. 39, London, **IV** (1979), 136-212.

[33] M. Valadier, *Différents cas où grâce à une propriété d'extrémalité une suite de fonctions intégrables faiblement convergente, converge fortement*, Séminaire Analyse Convexe Montpellier (1989), (20 pages).

[34] M. Valadier, *Young measures, Methods of Nonconvex Analysis (A. Cellina, ed.)*, Lectures in Math. Springer-Verlag, Berlin **1448** (1990), 152-188.

[35] M. Valadier, *A course of Young measures*, Workshop in Teoria della Misura e Analsisi Reale, Grado, September 19-October 2, Rend. Inst. Mat. Univ. Trieste suppl. **26**, 349-394.

[36] M. Visintin, *Strong convergence results related to strict convexity*, Comment. Partial Diff. Equation **9** (1984), 439-466.

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