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A NONLINEAR ERGODIC THEOREM AND ERGODIC SEQUENCES FOR AMENABLE SEMIGROUPS OF NON-EXPANSIVE MAPPINGS

by

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1. Introduction

In this note we discuss a recent nonlinear ergodic theorem for amenable semigroups [6] and introduce the notion of an ergodic sequence for semigroups of non-expansive mappings on a non-empty closed convex subset of a Hilbert space. It contains part of my talk given during the third symposium on Nonlinear Analysis and Applications held in July, 1998, at Josai University, Japan. I would like to thank the organizers of this symposium for their kind invitation to speak and their warm hospitality during the conference.

2. An Ergodic Theorem for Amenable Semigroups

A semigroup S is called *amenable* if there is a linear functional m on $\ell^\infty(S)$, the Banach space of all bounded real-valued functions on S with supremum norm such that

(i) $m(f) \geq 0$ for all $f \in \ell^\infty(S)$, $f \geq 0$.

(ii) $m(1) = 1$

(iii) $m(\ell_a f) = m(r_a f) = m(f)$ for all $f \in \ell^\infty(S)$ where $(\ell_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$ for all $t \in S$.

A linear functional m on $\ell\infty(S)$ satisfying (i) and (ii) is called a *mean*; m satisfying (i), (ii), (iii) is called an *invariant mean*. As well known, any commutative semigroup is

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amenable. A finite semigroup S is amenable if and only if S has a (unique) minimal ideal. However, the free group (or semigroup) on two generators is not amenable (see [3] and [4]).

A well-known result of Day [2] (see also [4]) asserts that if S is an amenable semigroup, then whenever $\mathcal{S} = \{T_s;\, s \in S\}$ is a bounded representation of S as bounded linear operators on a Banach space E, there exists a net of finite averages A_α of $\mathcal{S}$ (i.e. for each $x \in E$, $A_\alpha(x)$ is in the convex hull of $\{T_s(x);\, s \in S\}$) such that $\lim_\alpha \|A_\alpha(T_s - I)(x)\| = 0$ and $\lim_\alpha \|(T_s - I)A_\alpha(x)\| = 0$, for each $x \in E$.

In this case, if $E(\mathcal{S}) = F(\mathcal{S}) + D(\mathcal{S})$, where $F(\mathcal{S})$ is the fixed point set of $\mathcal{S}$, and $D(\mathcal{S})$ is the closed linear span of $\{T_s x - x;\, s \in S$ and $x \in E\}$, then there is a projection P onto $F(\mathcal{S})$ along $D(\mathcal{S})$ and $PT_s = T_s P = P$ for all $s \in S$. Furthermore, if $x \in E(\mathcal{S})$, then $P(x)$ is the unique common fixed point in $\overline{\text{co}}\{T_s x;\, s \in S\}$, where $\overline{\text{co}}\, A$ is the closed convex hull of A.

The first nonlinear ergodic theorem for nonexpansive maps was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and T a nonexpansive mapping of C into itself. It the fixed point set $F(T)$ of T is non-empty, then for each $x \in C$, the Cesàrio means

$$S_n(x) = \frac{1}{n} \sum_{k=1}^{n-1} T^k x$$

converges weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{co}}\{T^n x;\, n = 1, 2, \dots\}$ for each $x \in C$. In [8], Takahashi proved:

Theorem 1([8]). *Let S be an amenable semigroup, C be a non-empty closed convex subset of a Hilbert space H, and $\mathcal{S} = \{T_s;\, s \in S\}$ be a representation of S as non-expansive mappings from C into C. Assume that $F(\mathcal{S})$ = fixed point set of $\mathcal{S}$ is non-empty. Then there is a non-expansive retraction P of C onto $F(\mathcal{S})$ such that $T_s P = PT_s = P$ for every $s \in S$, and $Px \in \overline{\text{co}}\{T_s x;\, s \in S\}$ for every $x \in C$, where*

$\overline{co}\,A$ *is the closed convex full of* A.

Takahashi's result was extended to uniformly convex Banach space with a Fréchet differentiable norm when S is commutative by Hirano, Kido and Takahashi [5]. However, it has been an open problem for some time (see [9]), whether Takashi's result can be fully extended to such Banach spaces for amenable semigroups. Recently this problem was answered by Lau, Shioji and Takahashi in [6]:

Theorem 2 ([6]). *Let* C *be a closed convex subset of a uniformly convex Banach space* E, *let* S *be an amenable semigroup, let* $\mathcal{S} = \{T_t;\, t \in S\}$ *be a nonexpansive semigroup on* C *such that* $F(\mathcal{S}) \neq \emptyset$. *Then there exists a net* $\{A_\alpha\}$ *of finite averages of* $\mathcal{S}$ *such that for each* $t \in S$ *and for each bounded subset* B *of* C, $\lim_\alpha \|A_\alpha T_t x - A_\alpha x\| = 0$ *and* $\lim_\alpha \|T_t A_\alpha x - A_\alpha x\| = 0$ *uniformly for each* $x \in B$.

3. Ergodic Sequences

Let S be a semigroup and $\ell^1(S)$ denote the Banach space of all $f : S \to \mathbb{R}$ such that $\|f\|_1 = \sum |f(x)| < \infty$. Let $(\ell^1(S))_1^+ =$ all $\theta \in \ell^1(S)$ such that $\theta \geq 0$ and $\|\theta\|_1 = 1$ (countable means). There is a natural convolution on $\ell^1(S)$:

$$(\theta_1 * \theta_2)(s) = \sum \{\theta(s_1)\theta(s_2);\; s_1 s_2 = s\}.$$

Then $(\ell^1(S), *)$ is a Banach algebra, i.e.

$$\|\theta_1 * \theta_2\| \leq \|\theta_1\|\, \|\theta_2\|.$$

Let H be a Hilbert space, C be a closed convex subset of H, and $\mathcal{S} = \{T_s;\, s \in S\}$ be a representation of S as non-expansive mappings from C into C such $F(\mathcal{S}) \neq \emptyset$.

Let $x \in C$. For each $y \in H$, consider the bounded real-valued function on S

$s \mapsto \langle T_s x, y\rangle$. Let θ be a *mean* on $\ell^\infty(S)$, define

$$\begin{aligned}\langle T_\theta(x), y\rangle &= \theta_s\big(\langle T_s x, y\rangle\big)\\ &= \sum\{\langle T_s x, y\rangle\theta(s);\ s\in S\} \quad \text{if} \quad \theta\in\big(\ell^1(S)\big)_1^+.\end{aligned}$$

Then T_θ is a non-expansive mapping from $C \to C$ (see [7]).

Call a *sequence* (*net*) $\{\theta_n\}$ of means on S an *ergodic sequence* (*net*) *for non-expansive mappings* if for any representation $\mathcal{S} = \{T_s;\ s\in S\}$ of S as non-expansive mappings on a closed convex subset C of a Hilbert space into C such that $F(\mathcal{S}) \neq \emptyset$, then for each $x\in C$, the sequence (net) $T_{\theta_n}(x)$ converges weakly to a fixed point of $\mathcal{S}$.

A net of means $\{\mu_\alpha\}$ on $\ell^\infty(S)$ is called "asymptotically invariant" if

$$\lim_\alpha \big(\mu_\alpha(\ell_s f) - \mu_\alpha(f)\big) = 0 \quad \text{and}$$

$$\lim_\alpha \big(\mu_\alpha(r_s f) - \mu_\alpha(f)\big) = 0 \quad \text{for all} \quad s\in S.$$

Theorem ([7]). *Let S be an amenable semigroup. Then any "asymptotically invariant net" of means is an ergodic net for non-expansive mappings.*

Note:

(1) Every invariant mean on $\ell^\infty(S)$ is asymptotically invariant.

(2) If m is an invariant mean on $\ell^\infty(S)$, then there is a net $\theta_\alpha \in \big(\ell^1(S)\big)_1^+$ such that θ_α has *finite* support i.e. $\theta_\alpha = \sum_{i=1}^{n}\lambda_i\delta_{s_i}$ (convex combination) such that $\theta_\alpha \xrightarrow{w^*} m$. In particular the net $\{\theta_\alpha\}$ is asymptotically invariant. Hence $\{\theta_\alpha\}$ is an ergodic *net* of finite means on S for non-expansive mappings.

Example ([7]): $S = (\{0,1,2,\dots\},+)$

$$\theta_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_k,$$

then $\{\theta_n\}$ is an asymptotically invariant sequence of finite means on S. Consequently, $\{\theta_n\}$ is an ergodic *sequence* of finite means on S for non-expansive mappings.

Problem 1: Given an amenable semigroup S, when does there exist a *ergodic sequence* of countable (or finite) means on S for non-expansive mappings?

Problem 2: When can the net $\{A_\alpha\}$ of finite average of S in Theorem 2 be chosen to be a sequence dependent on the semigroup S?

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