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THE HOMOTOPIC INVARIANCE FOR FIXED POINTS OF SET-VALUED NONEXPANSIVE MAPPINGS*

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ABSTRACT

The homotopic invariance of fixed points of set-valued contractions and nonexpansive mappings is studied. As application, nonlinear alternative principles are given. A Leray-Schauder alternative and an antipodal theorem for set-valued nonexpansive mappings are also included.

1. Introduction

Let U be a nonempty bounded open subset of a Banach space X containing the origin and $f : \overline{U} \to X$ a contraction. It is known that either f has a fixed point in $\overline{U}$ or there exists $\lambda \in (0,1)$ such that λf has a fixed point on the boundary ∂U of U. This kind of property is known as an alternative principle. To the best of our knowledge, Leray and Schauder [16] in 1934 were the first to establish such an alternative principle for nonlinear operators in Banach spaces. Since then, the Leray-Schauder type of alternative principle, its various extensions and variants have played a basic role in numerous applications of nonlinear analysis. In this

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direction, Granas and his school have made extensive studies, see Granas [11–14], Frigon [6], Frigon and Granas [7], Frigon et al. [8] and references therein; see also T.C. Lim [17] for a stability result.

It is the objective of the present paper to study the homotopic invariance of the fixed point theory of set-valued mappings. This will be done first for contractions and then for nonexpansive mappings in Banach spaces having Opial's property. Alternative principles for multivalued nonexpansive mappings are established and a Leray-Schauder alternative and an antipodal theorem for set-valued nonexpansive mappings are given. We would like to point out here that there is a gap in the proof of Theorem 4.6 [6] (That $y \in \overline{U}$ must be verified to assert that y belongs to the asymptotic center $A(\overline{U}, \{x_n\})$.) It therefore remains an open question whether the conclusion of Theorem 4.6 [6] is true.

2. Preliminaries

Let (X, d) be a metric space with distance d. Let U be a subset of X. We shall denote by ∂U the boundary of U in X and by $B(x, r)$ ($\overline{B}(x, r)$) the open (closed) ball in X with center x and radius r. The symbol $C(X)$ will stand for the collection of nonempty closed subsets of X and $K(X)$ for the collection of nonempty compact subsets of X. Let D be the (generalized) Hausdorff metric on $C(X)$ induced by d; namely, for $A,\ B \in C(X)$, we have

$$D(A, B) := \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},$$

where $d(x, C) = \inf_{c \in C} d(x, c)$ is the distance from a point x to a subset C of X.

Given a number $\lambda \in [0, 1)$. Recall that a set-valued mapping $T : U \to C(X)$ is called a λ-*contraction* if

$$D(Tx, Ty) \leq \lambda d(x, y) \quad \forall x, y \in U. \tag{2.1}$$

If (2.1) holds with $\lambda = 1$, then T is said to be *nonexpansive.*

Recall that a Banach space X is said to satisfy *Opial's property* [21] if given any sequence $\{x_n\}$ in X with $x_n \rightharpoonup x_0$, it follows that

$$\limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - x\| \quad \forall x \in X \setminus \{x_0\}.$$

Here we use $\rightharpoonup$ to denote weak convergence and will use $\to$ to denote strong convergence.

We recall the demiclosedness principle for single-valued nonexpansive mappings. Assume C is a closed convex subset of a Banach space X. A mapping $f : C \to X$ is said to be *demiclosed (at y)* if for any sequence $\{x_n\}$ in C, we have

$$x_n \rightharpoonup x \text{ and } f(x_n) \to y \ \Rightarrow\ f(x) = y.$$

It is known that for every (single-valued) nonexpansive mapping $T : C \to X$, $I - T$ is demiclosed if the underlying Banach space X is either uniformly convex [1] or has Opial's property [3]. Demiclosedness plays an important role in the fixed point theory of (single-valued) nonexpansive mappings. The counterpart for set-valued mappings is defined as follows.

Definition 2.1. Let C be a nonempty closed convex subset of a Banach space X and $f : C \to C(X)$ a set-valued mapping. Then f is said to satisfy the *demiclosedness principle* on C (or f is said to be *demiclosed* on C) if the graph of f,

$$G(f) := \{(x, y) \in X \times X : x \in C,\ y \in f(x)\}$$

is closed in the product topology of $\sigma(X, X^*) \times (X, \|\cdot\|)$, where $\sigma(X, X^*)$ and $\|\cdot\|$ denote the weak and strong topology, respectively.

An equivalent description is that a set-valued mapping $T : C \to C(X)$ is demiclosed if and only if for any sequence $\{x_n\}$ in C weakly convergent to x_0 and $\{y_n\}$ strongly convergent to y_0 with $y_n \in f(x_n)$ for each $n \in \mathbb{N}$, it follows that $y_0 \in f(x_0)$.

The demiclosedness principle for multivalued mappings is more delicate than that for single-valued mappings. In fact, the only known class of Banach spaces in which the demiclosedness principle holds for multivalued nonexpansive mappings is the class of Banach spaces which satisfy Opial's property. This is a result of Lami-Dozo [15]. Here we include it as a lemma and its proof for completeness. However, in contrast to the single-valued case, it is still unknown whether the demiclosedness principle is valid for multivalued nonexpansive mappings in another important class of uniformly convex Banach spaces.

Lemma 2.1. Assume that C is a closed convex subset of a Banach space X having Opial's property and $T : C \to K(X)$ is nonexpansive. Then $I - T$ is demiclosed.

Proof. Assume that $\{x_n\} \subset C$ converges weakly to $x \in C$ and that $\{y_n\}$ strongly converges to y, with $y_n \in (I - T)x_n$ for all n. Let $z_n \in Tx_n$ be such that $y_n = x_n - z_n$. Take $v_n \in Tx$ for which $\|z_n - v_n\| = d(z_n, Tx)$. Since Tx is compact, we may assume that $v_n \to v \in Tx$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty} \|x_n - (y + v)\| &= \limsup_{n\to\infty} \|z_n - v_n\| \\
&= \limsup_{n\to\infty} d(z_n, Tx) \\
&\le \limsup_{n\to\infty} D(Tx_n, Tx) \\
&\le \limsup_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

Opial's property then implies that $y + v = x$ and hence $y = x - v \in (I - T)x$. $\square$

Asymptotic centers play an important role in studies of the fixed point theory for both single-valued and multivalued nonexpansive mappings. Let C be a nonempty subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. The number

$$r(C, \{x_n\}) := \inf\left\{\limsup_{n\to\infty} \|x_n - x\| : x \in C\right\}$$

and the set (possibly empty)

$$A(C,\{x_n\}) := \left\{x \in C : \limsup_{n\to\infty} \|x_n - x\| = r(C,\{x_n\})\right\}$$

are called the *asymptotic radius* and *asymptotic center* of $\{x_n\}$ in C, respectively. A sequence $\{x_n\}$ is said to be *regular* in C if $r(C,\{x_n\}) = r(C,\{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. The following lemma was proved independently by K. Goebel [9] and T.C. Lim [18].

Lemma 2.2. Let C be a subset of a Banach space X. Then every bounded sequence admits a subsequence which is regular in C.

We shall need the following localization of the multivalued contraction principle [19] which may be found in quite many textbooks on nonlinear functional analysis; see, for example, Deimling [2, p. 317].

Lemma 2.3. Let (X,d) be a complete metric space, $x \in X$ and r a given positive number. Suppose $T : \overline{B}(x,r) \to C(X)$ is a λ-contraction satisfying $d(x,T(x)) < (1-\lambda)r$. Then T has fixed point $\hat{x}$ in the ball $\overline{B}(x,r)$, i.e., $\hat{x} \in T(\hat{x})$ and $d(x,\hat{x}) \le r$.

3. Homotopic Invariance for Multivalued Contractions

In this section, we shall establish the homotopic invariance for fixed points of set-valued contractions. We begin with the definition below.

Definition 3.1. Let (X,d) be a complete metric space and U a nonempty open subset of X. Let $T,G : \overline{U} \to C(X)$ be two multivalued contractions. Then T,G are said to be *homotopic* if there exists a function (which is called a homotopy) $H : [0,1] \times \overline{U} \to C(X)$ such that

(1) $H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$;

(2) $x \notin H(t,x)$ for all $x \in \partial U$ and $t \in [0,1]$, where ∂U is the boundary of U;

(3) there exists a $\lambda \in [0,1)$ such that $H(t,\cdot)$ is a $\lambda-$contraction for all $t \in [0,1]$; i.e.,

$$D(H(t,x),H(t,y) \le \lambda d(x,y) \quad \forall x,y \in \overline{U},$$

(4) $H(t,x)$ is equi-continuous in $t \in [0,1]$ over $x \in \overline{U}$. This means that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that whenever $t,s \in [0,1]$ with $|t-s| < \delta$, we have $D(H(t,x),H(s,x)) < \varepsilon$ for all $x \in \overline{U}$.

Remark 3.1. Frigon [6, Definition 3.2] uses a stronger condition than (4) above. She requires that there is a continuous increasing function $\varphi : [0,1] \to \mathbb{R}$ such that $D(H(t,x),H(s,x)) \le |\varphi(t) - \varphi(s)|$ for all $t,s \in [0,1]$ and $x \in \overline{U}$. Indeed, our condition (4) above is equivalent to the condition that

$$\varphi(t,s) := \sup_{x\in\overline{U}} D(H(t,x),H(s,x))$$

is a continuous function in $t,s \in [0,1]$; while Frigon's condition is equivalent to the requirement that $\varphi(t,s) \le |\varphi(t) - \varphi(s)|,\ t,s \in [0,1]$ for some increasing continuous function $\varphi : [0,1] \to \mathbb{R}$.

Lemma 3.1. Let U be a nonempty open subset of a complete metric space (X, d). Let $\{T_t\}_{0\leq t\leq 1}$ be a family of $\lambda-$contractions from $\overline{U} \to C(X)$, where $\lambda \in [0,1)$. Suppose that T_t is equi-continuous in $t \in [0,1]$ over $x \in \overline{U}$ and for some $t \in [0,1]$, T_t has a fixed point. Assume

$$\inf_{\substack{x\in\partial U\\ t\in[0,1]}} d(x, T_t x) > 0. \tag{3.1}$$

Then for each $t \in [0,1]$, T_t has a fixed point and

$$\lim_{s\to t} D(F(T_s), F(T_t)) = 0, \tag{3.2}$$

where $F(T_t) := \{x \in \overline{U} : x \in T_t x\}$ is the fixed point set of T_t.

Proof. Let

$$V = \{t \in [0,1] : T_t \text{ has a fixed point in } \overline{U}\}.$$

Then V is nonempty by assumption. We shall show that V is actually the entire interval $[0,1]$ by verifying that V is both open and closed in $[0,1]$. To see that V is open in $[0,1]$, we take any $t_0 \in V$ and $x_0 \in F(T_{t_0})$. As T_{t_0} is fixed point free on ∂U, we have $x_0 \in U$ and hence $\overline{B}(x_0, r) \subset U$ for some $r > 0$. Select $\eta > 0$ small enough so that $D(T_{t_0}x_0, T_t x_0) < (1-\lambda)r$ for all $t \in [0,1]$ such that $|t - t_0| < \eta$. It then follows that for such t,

$$d(x_0, T_t x_0) \leq D(T_{t_0}x_0, T_t x_0) < (1-\lambda)r.$$

Therefore, Lemma 2.3 yields a fixed point for T_t in the ball $\overline{B}(x_0, r)$, which verifies that V is open.

To prove that V is closed, we assume $\{t_n\} \subset V$ such that $t_n \to t$. We are going to prove that $F(T_t) \neq \emptyset$. Take any $x_n \in F(T_{t_n})$ and let $\eta_n = \varphi(t_n, t) \to 0$ as $n \to \infty$, where $\varphi(t,s) = \sup_{x\in\overline{U}} D(T_t x, T_s, x)$. We claim that

$$\inf_{n\geq 1} d(x_n, \partial U) > 0.$$

For otherwise we have a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a sequence $\{z_i\}$ in ∂U such that

$$d(x_{n_i}, z_i) < \frac{1}{i} \quad \forall i \geq 1.$$

We then arrive at the following contradiction:

$$\begin{aligned}
0 < \inf_{\substack{x\in\partial U\\ t\in[0,1]}} d(x, T_t x) &\leq \liminf_{i\to\infty} d(z_i, T_{t_{n_i}} z_i) \\
&\leq \liminf_{i\to\infty} [d(z_i, x_{n_i}) + d(x_{n_i}, T_{t_{n_i}} z_i)] \\
&\leq \liminf_{i\to\infty} D(T_{t_{n_i}} x_{n_i}, T_{t_{n_i}} z_i) \\
&\leq \liminf_{i\to\infty} d(x_{n_i}, z_i) = 0.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded away from ∂U and hence we have some $\delta > 0$ for which

$$\overline{B}(x_n, \delta) \subset U \quad \forall n \geq 1.$$

Now for any fixed $0 < \delta' < \delta$, we have an integer N big enough so that

$$d(x_n, T_t x_n) \leq \eta_n < (1 - \lambda)\delta' \quad \forall n \geq N.$$

Lemma 2.3 again ensures that T_t has a fixed point $x_{t,n} \in \overline{B}(x_n, \delta')$ for all $n \geq N$. Thus t belongs to V and V is closed.

Finally to prove (3.2), we observe that the above argument gives us that

$$d(x_n, F(T_t)) \leq (1 - \lambda)\delta' \quad \forall n \geq N.$$

Since N is independent of the choice $x_n \in F(T_{t_n})$, it follows that

$$\sup_{x_n \in F(T_{t_n})} d(x_n, F(T_t)) \leq (1 - \lambda)\delta' \quad \forall n \geq N.$$

Interchange t_n and t to get

$$\sup_{x \in F(T_t)} d(x, F(T_{t_n})) \leq (1 - \lambda)\delta' \quad \forall n \geq N.$$

We therefore obtain that $\lim_{t_n \to t} D(F(T_{t_n}), F(T_t)) = 0$. $\Box$

Remark 3.2. It is not clear whether in Lemma 3.1 it is possible to weaken the assumption (3.1) to the assumption that every T_t ($t \in [0, 1]$) is fixed point free on ∂U. A partial answer is given below in a Banach space with Opial's property.

Theorem 3.1. Let U be a nonempty convex open subset of a Banach space X having Opial's property. Assume U is bounded weakly relatively compact (i.e., each bounded sequence in U admits a weakly convergent subsequence). Let $\{T_t\}_{0 \leq t \leq 1}$ be a family of $\lambda-$contractions from $\overline{U}$ to $K(X)$ which is equi-continuous in $t \in [0, 1]$ over $x \in \overline{U}$. Assume that some T_t has a fixed point in U and every T_t is fixed point free on ∂U. Then every T_t has a fixed point in U.

Proof. Define the same set V as above. The openness of V is proved by exactly the same argument as above. To show the closedness of V, we assume $\{t_n\}_{n=1}^{\infty} \subset V$ and $t_n \to t_0$ as $n \to \infty$. Take any $x_n \in F(T_{t_n})$. (Note that $\{x_n\} \subset U$ as every T_{t_n} is fixed point free on ∂U.) For any $x \in U$, by compactness, we have some $y_n \in T_{t_0} x$ satisfying $\|x_n - y_n\| = d(x_n, T_{t_0} x)$. Let $b = \sup\{\|y\| : y \in T_{t_0} x\}$. It then follows that

$$\begin{aligned}
\|x_n\| &\leq \|x_n - y_n\| + \|y_n\| \\
&\leq d(x_n, T_{t_0} x) + b \\
&\leq D(T_{t_n} x_n, T_{t_0} x) + b \\
&\leq D(T_{t_n} x_n, T_{t_0} x_n) + D(T_{t_0} x_n, T_{t_0} x) + b \\
&\leq \varphi(t_n, t_0) + \lambda \|x_n - x\| + b.
\end{aligned}$$

Hence $\|x_n\| \le \frac{1}{1-\lambda}(\varphi(t_n, t_0) + \lambda\|x\| + b)$ and $\{x_n\}$ is bounded. Since U is bounded weakly relatively compact, we may assume that $x_n \rightharpoonup z \in \overline{U}$. Take $z_n \in T_{t_0}x_n$ such that $\|x_n - z_n\| = d(x_n, T_{t_0}x_n)$. Noting $x_n - z_n \in (I - T_{t_0})x_n$ and $\|x_n - z_n\| \le \varphi(t_n, t_0) \to 0$, we get by the demiclosedness principle (Lemma 2.1) that z is a fixed point of T_{t_0}. □

4. Homotopic Invariance for Set-valued Nonexpansive Mappings

Frigon [6] was the first to investigate the homotopic invariance for set-valued nonexpansive mappings. However, the proof of the main result of Section 4, Theorem 4.6, of [6] contains a gap (on page 29, to ensure that the y belongs to the asymptotic center $A(\overline{U}, \{x_n\})$, one must show that y lies in $\overline{U}$). Thus nothing has been established so far regarding the homotopic invariance of fixed points for set-valued nonexpansive mappings. As a matter of fact, even for single-valued nonexpansive mappings, fixed points are not homotopically invariant if we only require that the homotopy H be nonexpansive, as the following simple example shows.

Example [6]. Let $X = l^2$ and $U = B(0,1)$. Define a homotopy $H : [0,1] \times \overline{U} \to X$ by

$$H(t, x) = (ta, x_1, x_2, \cdots),$$

where $a \in \mathbb{R} \setminus \{0\}$ is a fixed constant. It is easily seen that for each $t \in [0,1]$, $H(t, \cdot)$ is nonexpansive (actually, isometric) and $\|H(t,x) - H(s,x)\| \le |a||t - s|$ for all $x \in \overline{U}$ and $t, s \in [0,1]$. It is also easily seen that $H(t, \cdot)$ has a fixed point if and only if $t = 0$ with $x = 0$ the only fixed point.

Our results below show that we can obtain homotopic invariance by putting some additional conditions on the homotopy H.

Theorem 4.1. Let C be a nonempty weakly compact convex subset of a Banach space X which has Opial's property. Suppose $T, G : C \to K(X)$ are two set-valued nonexpansive mappings and there exists a set-valued homotopy $H : [0,1] \times C \to K(E)$ such that

(1) $H(0, \cdot) = T(\cdot)$ and $H(1, \cdot) = G(\cdot)$;
(2) for each $t \in [0,1]$, $H(t, \cdot)$ is a set-valued nonexpansive mapping;
(3) $H(t,x)$ is equi-continuous in $t \in [0,1]$ over $x \in \overline{U}$; and
(4) for each sequence $\{t_n\}$ in $[0,1]$ with $\inf_{x\in C} d(x, H(x, t_n)) > 0$ and $\lim_{n\to\infty} t_n = t_0$, it follows that $\inf_{x\in C} d(x, H(x, t_0)) > 0$.

Then T has a fixed point in C if and only if G has a fixed point in C.

Proof. Without loss of generality, we may assume that T has a fixed point in C. We denote by V the set $\{t \in [0,1] :$ there exists $x \in C$ such that $x \in H(t,x)\}$. In order to prove G has a fixed point, it suffices to show that $V = [0,1]$. First as T has a fixed point, V is nonempty. Also, it is not hard to see that V is closed. Indeed, assume that $\{t_n\}_{n=1}^\infty \subset V$ converges to t_0. For each $n \in \mathbb{N}$, we can find some $x_n \in C$ such that $x_n \in H(t_n, x_n)$. As $\{x_n\}_{n=1}^\infty$ is bounded, we may assume $x_n \rightharpoonup x_0 \in C$. With φ defined as before, we see that

$$d(x_n, H(t_0, x_n)) \le D(H(t_n, x_n), H(t_0, x_n)) \le \varphi(t_n, t_0).$$

It then follows that $\lim_{n\to\infty} d(x_n, H(t_0, x_n)) = 0$. As X satisfies Opial's property, $I - H(t_0, \cdot)$ satisfies the demiclosedness principle. We thus conclude by Lemma 2.1 that $x_0 \in H(t_0, x_0)$ which means $t_0 \in V$. Therefore V is closed.

Next, we shall prove that V is open. Suppose not; then there exist $t_0 \in V$ and a sequence $\{t_n\}$ in $[0,1] \setminus V$ such that $\lim_{n\to\infty} t_n = t_0$. Since for each n, $t_n \notin V$, we see that $d(x, H(t_n, x)) > 0$ for all $x \in C$. Now we claim that

$$\inf_{z \in C} d(z, H(t_n, z)) > 0.$$

Otherwise, there exists a sequence $\{x_m\} \subset C$ such that $\lim_{m\to\infty} d(x_m, H(t_n, x_m)) = 0$. As $\{x_m\}_{n=1}^{\infty}$ is bounded, we may assume that $x_m \rightharpoonup x_0 \in C$. From the demiclosedness of $I - H(t_n, \cdot)$, it follows that $x_0 \in H(t_n, x_0)$. This is a contradiction to the assumption that $t_n \notin V$. Therefore for each n, we have

$$\inf_{x \in C} d(x, H(t_n, x)) =: \delta_n > 0.$$

Condition (4) then implies $\inf_{z\in C} d(z, H(t_0, z)) > 0$, which in turn implies $t_0 \notin V$, contradicting the fact that $t_0 \in V$. Therefore, V must be open and hence $V = [0,1]$. $\square$

Remark 4.1. In the above proof we see that the only reason for requiring X to have the Opial property is to ensure that for each $t \in [0,1]$, the mapping $I - H(t, \cdot) : C \to K(X)$ is demiclosed. As the demiclosedness principle is valid for single-valued nonexpansive mappings in uniformly convex Banach spaces, the result below follows.

Theorem 4.2. Let C be a nonempty bounded closed and convex subset of a uniformly convex Banach space X and let $T, G : C \to X$ be two single-valued nonexpansive mappings. Assume there exists a single-valued mapping $H : [0,1] \times C \to X$ with

(1) $H(0, \cdot) = T(\cdot)$ and $H(1, \cdot) = G(\cdot)$;
(2) for each $t \in [0,1]$, $H(t, \cdot)$ is a single-valued nonexpansive mapping;
(3) $H(t, x)$ is equi-continuous in $t \in [0,1]$ over $x \in \overline{U}$; and
(4) for each sequence $\{t_n\}$ in $[0,1]$ with $\inf_{x\in C} d(x, H(x, t_n)) > 0$ and $\lim_{n\to\infty} t_n = t_0$, it follows that $\inf_{x\in C} d(x, H(x, t_0)) > 0$.

Then T has a fixed point in C if and only if G has a fixed point in C.

If we further strengthen the homotopy H, we can drop the assumption (4) in Theorem 4.1.

Theorem 4.3. Let X be a Banach space satisfying Opial's property and U a nonempty convex open subset of X containing the origin. Let $F : \overline{U} \to K(X)$ be a nonexpansive mapping. Assume U is relatively weakly compact and assume there exists a homotopy $H : [0,1] \times \overline{U} \to C(X)$ satisfying the conditions:

(a) $H(1, \cdot) = F$;
(b) $H(0, \cdot)$ has a fixed point in U;

(c) for each $t \in [0, 1)$, we have a $\lambda_t \in [0, 1)$ such that $H(s, \cdot)$ is a λ_t-contraction for all $s \in [0, t]$;
(d) $H(t, \cdot)$ is equi-continuous in $t \in [0, 1]$ over $x \in \overline{U}$;
(e) for each $t \in [0, 1)$, $H(t, \cdot)$ is fixed point free on ∂U.

Then F has a fixed point.

Proof. Apply Theorem 3.1 to the homotopy H restricted to $[0, t] \times \overline{U}$ to get a fixed point of $H(s, \cdot)$ for each $s \in [0, t]$. Taking $t_n \uparrow 1$ yields a sequence $\{x_n\}$ in $\overline{U}$ such that $x_n \in H(t_n, x_n)$ for all n. It follows that

$$d(x_n, Fx_n) \leq D(H(t_n, x_n), H(1, x_n)) \leq \varphi(t_n, 1) \to 0.$$

Since $\{x_n\}$ is bounded and $\overline{U}$ is weakly compact, the demiclosedness of $I - F$ implies that each weak cluster point of $\{x_n\}$ is a fixed point of F. $\square$

Corollary 4.1 (Nonlinear Alternative). Let X, U and F be as in Theorem 4.3. Then either F has a fixed point, or there are some $x \in \partial U$ and some $t \in (0, 1)$ such that $x \in tFx$.

Proof. The homotopy $H : [0, 1] \times \overline{U} \to K(X)$ given by $H(t, x) = tFx$ satisfies the conditions (a)-(d) above. Thus either F has a fixed point, or the condition (e) above is violated. $\square$

Remark 4.2. For other alternative principles, homotopic invariance of fixed points of set-valued contractions and single-valued nonexpansive mappings in metric and Banach spaces and their various applications, the interested readers are referred to Dugundji and Granas [3], Frigon [6], Frigon and Granas [7], Frigon et al. [8], Granas [11–14], Nussbaum [20] and references therein.

5. A Leray-Schauder Alternative and an Antipodal Theorem

In this section, as applications of our alternative principles we establish a Leray-Schauder alternative and an antipodal theorem for set-valued nonexpansive mappings in Banach spaces. We first have the following result.

Theorem 5.1 (Leray-Schauder Alternative). Let X be a reflexive Banach space which satisfies Opial's property or is uniformly convex. Suppose $T : X \to K(X)$ is a set-valued nonexpansive mapping and let $E_T = \{x \in X : x \in \lambda T(x)$ for some $\lambda \in (0, 1)\}$. Then either E_T is unbounded or T has a fixed point.

Proof. Assume E_T is bounded. Let $r > 0$ be large enough so that the open ball $B(0, r)$ contains E_T. Then $x \notin \lambda T(x)$ for any $x \in \partial B(0, r)$ and $\lambda \in (0, 1)$.

If X satisfies Opial's property, then T has a fixed point by Corollary 4.1 and the proof is finished. So assume that X is uniformly convex. For each integer $n \geq 1$, the contraction $(1 - \frac{1}{n})T(\cdot) : X \to K(X)$ has a fixed point $x_n \in X$; thus $x_n = (1 - \frac{1}{n})y_n$ for some $y_n \in Tx_n$. Since $x_n \in E_T$ for all n, $\{x_n\}$ (and $\{y_n\}$ as well) is bounded. Hence $\|x_n - y_n\| = \frac{1}{n}\|y_n\| \to 0$ as $n \to \infty$. It follows from Lemma 2.2 that $\{x_n\}$

contains a regular subsequence which we still denote by $\{x_n\}$. Let z be the unique point in the asymptotic center of $\{x_n\}$ in X. By compactness we have, for each n, a $z_n \in Tz$ for which $\|y_n - z_n\| = d(y_n, Tz) \leq D(Tx_n, Tz) \leq \|x_n - z\|$. Since Tz is compact, we may assume that $z_n \to v \in Tz$. It then follows that

$$\limsup_{n\to\infty} \|x_n - v\| = \limsup_{n\to\infty} \|y_n - z_n\| \leq \limsup_{n\to\infty} \|x_n - z\|.$$

The uniqueness of the asymptotic center of $\{x_n\}$ in X guarantees that $v = z$ and hence $z \in Tz$. □

As an immediate consequence of Theorem 5.1, we have the following:

Corollary 5.1. Let X be a reflexive Banach space which satisfies Opial's property or is uniformly convex. Suppose $T : X \to X$ is a single-valued nonexpansive mapping and let $E_T = \{x \in X : x = \lambda T(x) \text{ for some } \lambda \in (0,1)\}$. Then either E_T is unbounded or T has a fixed point.

Finally we have the following result.

Corollary 5.2. Let $B(0,r)$ be an open ball of a reflexive Banach space X which satisfies Opial's property. Suppose $T : \overline{B}(0,r) \to X$ is a single-valued mapping such that $T(x) = -T(-x)$ for all $x \in \partial B(0,r)$. Then T has a fixed point in $\overline{B}(0,r)$.

Proof. Since $D(T(x), T(-x)) = \|T(x) - T(-x)\| \leq \|x - (-x)\|$ for each $x \in \partial B(0,r)$, it follows that $D(0, T(x)) = \|0 - T(x)\| \leq \|x\|$ for each $x \in \partial B(0,r)$. This easily implies that there are no $x \in \partial B(0,r)$ and $\lambda \in (0,1)$ satisfying $x \in \lambda Tx$. Thus by Corollary 4.1, T has a fixed point. □

The following example given by Rouhani shows that the set-version version of Corollary 5.2, in general, is not true.

Example 5.1. Let $X = \mathbb{R}$ and consider a set-valued mapping $T : [-1,1] \to K(\mathbb{R})$ defined by $T(x) := [x - a, x + a]$ for each $x \in X$. Then obviously T is an odd set-valued and nonexpansive mapping, which is fixed point free (indeed we also have that $D(0, T(x)) > \|x\|$ for each $x \in X$).

Remark 5.1. We note that Theorems 4.1 and 4.3 extend the corresponding results given by Dugundji and Granas [3] and Granas [14].

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