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Existence Theorems in Metric Spaces and Characterizations of Metric Completeness

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ABSTRACT. In this article, we first study existence theorems in complete metric spaces which generalize the Banach contraction principle. Next, we introduce the concept of w-distances and prove some results in complete metric spaces. Finally we study the relationship between contractive mappings and Kannan mappings and then discuss characterizations of metric completeness.

1 Introduction

Let X be metric space with metric d. A mapping T from X into itself is called *contractive* if there exists a real number $r \in [0,1)$ such that $d(Tx, Ty) \leq rd(x,y)$ for every $x, y \in X$. It is well known that if X is a complete metric space, then every contractive mapping from X into itself has a unique fixed point in X. This theorem is called the Banach contraction principle. However, we exhibit a metric space X such that X is not complete and every contractive mapping from X into itself has a fixed point in X; see Section 4. A mapping T from X into itself is also called *Kannan* if there exists a real number $r \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq r\{d(Tx, x) + d(Ty, y)\}$ for every $x, y \in X$. We know that a metric space X is complete if and only if every Kannan mapping from X into itself has a fixed point in X.

On the other hand, in 1976, Caristi [3] proved a fixed point theorem in a complete metric space which generalizes the Banach contraction principle. Ekeland [5] also obtained a nonconvex minimization theorem for a proper lower semicontinuous function, bounded from below, in a complete metric space. The theorem is also called the ε-variational principle. The two theorems are very useful and have many applications. Later Takahashi [22] proved the following nonconvex minimization theorem: Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$.

Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$. This theorem was used to obtain Caristi's fixed point theorem [3], Ekeland's ε-variational principle [5] and Nadler's fixed point theorem [11]. In [22], Takahashi also studied characterizations of metric completeness by using the nonconvex minimization theorem. Recently, Kada, Suzuki and Takahashi [8] introduced the concept of w-distances on a metric space and improved Caristi's fixed point theorem, Ekeland's ε-variational principle and the nonconvex minimization theorem according to Takahashi.

In this article, we first study existence theorems in complete metric spaces which generalize the Banach contraction principle. Next, we introduce the concept of w-distances and prove some results in complete metric spaces. Finally we study the relationship between contractive mappings and Kannan mappings and then discuss characterizations of metric completeness.

2 Existence Theorems

The Banach contraction principle in a complete metric space is very important. The following theorem obtained by Takahashi [22] is also used to prove Ekeland's ε-variational principle (Theorem 2.2) and Caristi's fixed point theorem (Theorem 2.4) which generalize the principle. Throughout this article, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers.

Theorem 2.1. *Let X be a complete metric space and let $F : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $\inf_{x \in X} F(x) < F(u)$, there exists a $v \in X$ such that $v \neq u$ and $F(v) + d(u,v) \leq F(u)$. Then, there exists an $x_0 \in X$ such that $F(x_0) = \inf_{x \in X} F(x)$.*

Proof. Suppose $\inf_{x \in X} F(x) < F(y)$ for every $y \in X$ and let $u \in X$ with $F(u) < \infty$. Then, we define inductively a sequence $\{u_n\}$ in X, starting $u_0 = u$. Suppose $u_{n-1} \in X$ is known. Then, choose $u_n \in S_n$ such that

$$S_n = \{w \in X : F(w) \leq F(u_{n-1}) - d(u_{n-1}, w)\}$$

and

$$F(u_n) \leq \inf_{w \in S_n} F(w) + \frac{1}{2}\{F(u_{n-1}) - \inf_{w \in S_n} F(w)\}. \qquad \cdots (*)$$

We claim that this is a Cauchy sequence. Indeed, if $m > n$, then

$$\begin{aligned} d(u_n, u_m) &\leq \sum_{i=n}^{m-1} d(u_i, u_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \{F(u_i) - F(u_{i+1})\} \\ &= F(u_n) - F(u_m). \qquad \cdots (**) \end{aligned}$$

This implies that $\{u_n\}$ is a Cauchy sequence in X. Let $u_n \to v$. Then, if $m \to \infty$ in $(**)$, we have

$$d(u_n, v) \leq F(u_n) - \lim_{m\to\infty} F(u_m) \leq F(u_n) - F(v).$$

On the other hand, by hypothesis, there exists a $z \in X$ such that $z \neq v$ and $F(z) \leq F(v) - d(v, z)$. Hence we have

$$\begin{aligned} F(z) &\leq F(v) - d(v, z) \\ &\leq F(v) - d(v, z) + F(u_n) - F(v) - d(u_n, v) \\ &= F(u_n) - \{d(u_n, v) + d(v, z)\} \\ &\leq F(u_n) - d(u_n, z). \end{aligned}$$

This implies $z \in S_{n+1}$. Using $(*)$, we have

$$2F(u_n) - F(u_{n-1}) \leq \inf_{x\in S_n} F(x) \leq F(z).$$

Hence we have

$$F(v) \leq \lim_{n\to\infty} F(u_n) \leq F(z) \leq F(v) - d(v, z) < F(v).$$

This is a contradiction. Therefore, there exists an $x_0 \in X$ such that $F(x_0) = \inf_{x\in X} F(x)$. □

Using Theorem 2.1, we prove Ekeland's ε-variational principle [5] in a complete metric space.

Theorem 2.2 (Ekeland's ε-variational principle). *Let X be a complete metric space and let $F : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Then, for any $\varepsilon > 0$ and $u \in X$ with*

$$F(u) \leq \inf_{x\in X} F(x) + \varepsilon,$$

there exists $v \in X$ satisfying the following conditions:

(1) $F(v) \leq F(u)$;

(2) $d(u, v) \leq 1$;

(3) $F(w) > F(v) - \varepsilon d(v, w)$ *for all $w \in X$ with $w \neq v$.*

Proof. Let

$$X_0 = \{x \in X : F(x) \leq F(u) - \varepsilon d(x, u)\}.$$

Then, it is obvious that X_0 is nonempty and closed. Further, for each $x \in X_0$,

$$\varepsilon d(u, x) \leq F(u) - F(x) \leq F(u) - \inf_{x\in X} F(x) \leq \varepsilon$$

and hence $d(u,x) \leq 1$. We also have $F(x) \leq F(u)$. Suppose that for every $x \in X_0$, there exists $w \in X$ such that $w \neq x$ and $F(w) \leq F(x) - \varepsilon d(x,w)$. Then

$$\begin{aligned}\varepsilon d(w,u) &\leq \varepsilon d(w,x) + \varepsilon d(x,u)\\ &\leq F(x) - F(w) + F(u) - F(x)\\ &= F(u) - F(w)\end{aligned}$$

and hence $w \in X_0$. By Theorem 2.1, there exists $x_0 \in X_0$ such that $F(x_0) = \inf_{x\in X_0} F(x)$. On the other hand, there exists $w_0 \in X_0$ with $F(w_0) < F(x_0)$. This is a contradiction. □

Corollary 2.3. *Let X be a complete metric space and let $F : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Then, for any $\varepsilon > 0$, there exists $v \in X$ satisfying the following two conditions:*

(1) $F(v) \leq \inf_{x\in X} F(x) + \varepsilon$;

(2) $F(w) \geq F(v) - \varepsilon d(v,w)$ *for all* $w \in X$.

Proof. For any $\varepsilon > 0$, there exists $u \in X$ such that

$$F(u) \leq \inf_{x\in X} F(x) + \varepsilon.$$

By Theorem 2.2, there exists $v \in X$ satisfying

$$F(v) \leq F(u) \leq \inf_{x\in X} F(x) + \varepsilon$$

and

$$F(w) \geq F(v) - \varepsilon d(v,w) \quad \text{for all } w \in X. \quad \square$$

Using Theorem 2.1, we prove Caristi's fixed point theorem [3] in a complete metric space.

Theorem 2.4 (Caristi's fixed point theorem). *Let X be a complete metric space and let f be a mapping of X into itself such that*

$$d(x, f(x)) + F(f(x)) \leq F(x) \quad \text{for all } x \in X,$$

where $F : X \to (-\infty, \infty]$ is a proper lower semicontinuous function, bounded from below. Then there exists $z \in X$ such that $f(z) = z$ and $F(z) < \infty$.

Proof. Since F is proper, there exists $u \in X$ with $F(u) < \infty$. So, let

$$X' = \{x \in X : F(x) \leq F(u) - d(u,x)\}.$$

Then, X' is nonempty and closed. We can also see that X' is invariant under the mapping f. In fact, for each $x \in X'$, we have

$$F(f(x)) + d(x, f(x)) \leq F(x) \leq F(u) - d(u, x)$$

and hence

$$\begin{aligned} F(f(x)) &\leq F(u) - \{d(u,x) + d(x, f(x))\} \\ &\leq F(u) - d(u, f(x)). \end{aligned}$$

This implies $f(x) \in X'$. Suppose that $f(x) \neq x$ for every $x \in X'$. Then, for every $x \in X'$, there exists $w \in X'$ such that $x \neq w$ and $F(w) + d(x,w) \leq F(x)$. So, by Theorem 2.1, we obtain an $x_0 \in X'$ with $F(x_0) = \inf_{x \in X'} F(x)$. For such an $x_0 \in X'$, we have

$$0 < d(x_0, f(x_0)) \leq F(x_0) - F(f(x_0)) \leq F(f(x_0)) - F(f(x_0)) = 0.$$

This is a contradiction. □

Finally, using Theorem 2.1, we obtain Nadler's fixed point theorem [11]. Before obtaining it, we give some definitions and notations. Let X be a metric space. Then, for $x \in X$ and $A \subset X$, define

$$d(x, A) = \inf\{d(x,y) : y \in A\}.$$

We also denote by $CB(X)$ the class of all nonempty bounded closed subsets of X. For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\}$$

and for $A, B \in CB(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

Then, H is a metric on $CB(X)$. A metric H on $CB(X)$ is said to be the *Hausdorff metric*. We know that for any $x \in X$ and $B, C \in CB(X)$,

$$|d(x, B) - d(x, C)| \leq H(B, C).$$

Let T be a mapping of a metric space X into $CB(X)$. Then T is called *nonexpansive* if

$$H(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in X.$$

If there exists $k < 1$ such that

$$H(Tx, Ty) \leq k d(x, y) \quad \text{for all } x, y \in X,$$

T is called *contractive* or *k-contractive*. If T is nonexpansive or k-contractive, the real valued function g on X defined by

$$g(x) = d(x, Tx) \quad \text{for all } x \in X$$

is continuous.

Corollary 2.5 (Nadler's fixed point theorem). *Let X be a complete metric space and let T be a k-contractive mapping of X into $CB(X)$. Then T has a fixed point in X.*

Proof. Suppose that $d(x, Tx) > 0$ for every $x \in X$ and choose a positive number ε with $\varepsilon < 1/k - 1$. Then, for every $x \in X$, we can choose $y \in Tx$ satisfying $d(x, y) \leq (1+\varepsilon)d(x, Tx)$. Since

$$d(y, Ty) \leq H(Tx, Ty) \leq kd(x, y) \leq k(1+\varepsilon)d(x, Tx),$$

we have $\inf_{x \in X} d(x, Tx) = 0$. Further, we have

$$\begin{aligned} d(x, Tx) - d(y, Ty) &\geq d(x, Tx) - kd(x, y) \\ &\geq \frac{1}{1+\varepsilon}d(x, y) - kd(x, y) \\ &= \left(\frac{1}{1+\varepsilon} - k\right)d(x, y). \end{aligned}$$

Defining $F : X \to R$ by

$$F(x) = \left(\frac{1}{1+\varepsilon} - k\right)^{-1} d(x, Tx) \quad \text{for all } x \in X,$$

we have $d(x, y) \leq F(x) - F(y)$ from the above inequality. Now, using Theorem 2.1, we obtain $x_0 \in X$ such that $F(x_0) = 0$. This implies $d(x_0, Tx_0) = 0$. This is a contradiction. □

3 Distances on Metric Spaces

Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a *w-distance* on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

(2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The concept of w-distances was first introduced by Kada, Suzuki and Takahashi [8]. Let us give some examples of w-distances.

Example 3.1. *Let X be a metric space with metric d. Then $p = d$ is a w-distance on X.*

Example 3.2. *Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number.*

Example 3.3. *Let X be a normed linear space with norm $\|\cdot\|$. If $p : X \times X \to [0, \infty)$ is defined by*

$$p(x, y) = \|x\| + \|y\| \quad \textit{for every } x, y \in X,$$

then p is a w-distance on X.

Example 3.4. *Let X be a normed linear space with norm $\|\cdot\|$. If $p : X \times X \to [0, \infty)$ is defined by*

$$p(x, y) = \|y\| \quad \textit{for all } x, y \in X,$$

then p is a w-distance on X.

Example 3.5. *Let X be a metric space and let T be a continuous mapping from X into itself. Then a function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \max\{d(Tx, y), d(Tx, Ty)\} \quad \textit{for every } x, y \in X$$

is a w-distance on X.

Example 3.6. *Let F be a bounded and closed subset of a metric space X. Assume that F contains at least two points and c is a constant with $c \geq \delta(F)$, where $\delta(F)$ is the diameter of F. Then a function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \begin{cases} d(x, y) & \textit{if } x, y \in F, \\ c & \textit{if } x \notin F \textit{ or } y \notin F \end{cases}$$

is a w-distance on X.

Let $\varepsilon \in (0, \infty]$. A metric space X with metric d is called *ε-chainable* [4] if for every $x, y \in X$ there exists a finite sequence $\{u_0, u_1, \ldots, u_k\}$ in X such that $u_0 = x$, $u_k = y$ and $d(u_i, u_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, k-1$. Such a sequence is called an *ε-chain* in X linking x and y.

Example 3.7. *Let $\varepsilon \in (0, \infty]$ and let X be an ε-chainable metric space with metric d. Then the function $p : X \times X \to [0, \infty)$ defined by*

$$p(x, y) = \inf \left\{ \sum_{i=0}^{k-1} d(u_i, u_{i+1}) : \{u_0, u_1, \cdots, u_k\} \textit{ is an } \varepsilon\textit{-chain linking } x \textit{ and } y \right\}$$

is a w-distance on X.

Next, we discuss properties of w-distances. The following lemmas are crucial in the proofs of the theorems in this article.

Lemma 3.8. *Let X be a metric space with metric d and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In paticular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*

(ii) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*

(iii) *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(iv) *if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 3.9. *Let X be a metric space and let p_1 and p_2 be w-distances on X. If $p : X \times X \to [0,\infty)$ is defined by*

$$p(x, y) = \max\{p_1(x, y), p_2(x, y)\} \quad \text{for all } x, y \in X,$$

then p is a w-distance on X.

Lemma 3.10. *Let X, p_1 and p_2 be as in Lemma 3.9. If $p : X \times X \to [0,\infty)$ is defined by*

$$p(x, y) = \alpha p_1(x, y) + \beta p_2(x, y) \quad \text{for all } x, y \in X,$$

where α and β are nonnegative real numbers such that $\alpha \neq 0$ or $\beta \neq 0$, then p is a w-distance on X.

Lemma 3.11. *Let X be a metric space, let p be a w-distance on X and let f be a function of X into $[0,\infty)$. If $g : X \times X \to [0,\infty)$ is defined by*

$$g(x, y) = \max\{f(x), p(x, y)\} \quad \text{for all } x, y \in X,$$

then g is a w-distance on X.

We first prove a nonconvex minimization theorem [8] which improves the result in Section 2.

Theorem 3.12. *Let X be a complete metric space, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Assume that there exists a w-distance p on X such that for any $u \in X$ with $\inf_{x \in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and*

$$f(v) + p(u, v) \leq f(u).$$

Then there exists $x_0 \in X$ such that $\inf_{x \in X} f(x) = f(x_0)$.

Proof. Suppose $\inf_{x\in X} f(x) < f(u)$ for every $y \in X$ and choose $u \in X$ with $f(u) < \infty$. Then we define inductively a sequence $\{u_n\}$ in X, starting with $u_1 = u$. Suppose $u_n \in X$ is known. Then choose $u_{n+1} \in S(u_n)$ such that

$$\begin{aligned} S(u_n) &= \{x \in X : f(x) + p(u_n, x) \leq f(u_n)\}, \\ k(u_n) &= \inf_{x \in S(u_n)} f(x) \end{aligned}$$

and

$$f(u_{n+1}) \leq k(u_n) + \frac{1}{n}.$$

Since $f(u_{n+1}) + p(u_n, u_{n+1}) \leq f(u_n)$, $\{f(u_n)\}$ is nonincreasing. So, $\lim_{n\to\infty} f(u_n)$ exists. Put $k = \lim_{n\to\infty} f(u_n)$. We claim that $\{u_n\}$ is a Cauchy sequence. In fact, if $n < m$, then

$$\begin{aligned} p(u_n, u_m) &\leq \sum_{j=n}^{m-1} p(u_j, u_{j+1}) \\ &\leq \sum_{j=n}^{m-1} \{f(u_j) - f(u_{j+1})\} \\ &= f(u_n) - f(u_m) \leq f(u_n) - k. \qquad (*) \end{aligned}$$

From Lemma 3.8, $\{u_n\}$ is a Cauchy sequence. Let $u_n \to v_0$. Then, if $m \to \infty$ in $(*)$, we have

$$p(u_n, v_0) \leq f(u_n) - k \leq f(u_n) - f(v_0).$$

On the other hand, by hypothesis, there exists $v_1 \in X$ such that $v_1 \neq v_0$ and $f(v_1) + p(v_0, v_1) \leq f(v_0)$. Hence, we obtain

$$\begin{aligned} f(v_1) + p(u_n, v_1) &\leq f(v_1) + p(u_n, v_0) + p(v_0, v_1) \\ &\leq f(v_0) + p(u_n, v_0) \\ &\leq f(u_n) \qquad (**) \end{aligned}$$

and hence $v_1 \in S(u_n)$. Since

$$f(v_0) \leq f(u_{n+1}) \leq k(u_n) + \frac{1}{n} \leq f(v_1) + \frac{1}{n}$$

for every $n \in N$, we have $f(v_0) \leq f(v_1)$. Then, $f(v_0) = f(v_1)$. So, we have $p(v_0, v_1) = 0$. By hypothesis, there exists $v_2 \in X$ such that $v_2 \neq v_1$ and $f(v_2) + p(v_1, v_2) \leq f(v_1)$. As in $(**)$, we have $f(v_2) + p(u_n, v_2) \leq f(u_n)$ and hence $v_2 \in S(u_n)$. So, we have $f(v_1) = f(v_0) \leq f(v_2)$. This implies $p(v_1, v_2) = 0$. From $p(v_0, v_2) \leq p(v_0, v_1) + p(v_1, v_2) = 0$, we have $p(v_0, v_2) = 0$. Hence, from $p(v_0, v_1) = 0$, $p(v_0, v_2) = 0$ and Lemma 3.8, we have $v_1 = v_2$. This is a contradiction. □

The following theorem [8] is a generalization of Caristi's fixed point theorem.

Theorem 3.13. *Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Let T be a mapping from X into itself. Assume that there exists a w-distance p on X such that $f(Tx) + p(x, Tx) \leq f(x)$ for every $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.*

Proof. Since f is proper, there exists $u \in X$ such that $f(u) < \infty$. Put

$$Y = \{x \in X : f(x) \leq f(u)\}.$$

Then, since f is lower semicontinuous, Y is closed. So Y is complete. Let $x \in Y$. Then, since $f(Tx) + p(x, Tx) \leq f(x) \leq f(u)$, we have $Tx \in Y$. So, Y is invariant under T. Assume that $Tx \neq x$ for every $x \in Y$. Then by Theorem 3.12, there exists $v_0 \in Y$ such that $f(v_0) = \inf_{x \in Y} f(x)$. Since $f(Tv_0) + p(v_0, Tv_0) \leq f(v_0)$ and $f(v_0) = \inf_{x \in Y} f(x)$, we have $f(Tv_0) = f(v_0) = \inf_{x \in Y} f(x)$ and $p(v_0, Tv_0) = 0$. Similarly we obtain $f(T^2 v_0) = f(Tv_0) = \inf_{x \in Y} f(x)$ and $p(Tv_0, T^2 v_0) = 0$. Since $p(v_0, T^2 v_0) \leq p(v_0, Tv_0) + p(Tv_0, T^2 v_0) = 0$, we have $p(v_0, T^2 v_0) = 0$ and hence $Tv_0 = T^2 v_0$ by Lemma 3.8. This is a contradiction. Therefore T has a fixed point x_0 in Y. Since $f(x_0) < \infty$ and

$$f(x_0) + p(x_0, x_0) = f(Tx_0) + p(x_0, Tx_0) \leq f(x_0),$$

we have $p(x_0, x_0) = 0$. $\square$

Let X be a metric space with metric d. Then, a set-valued mapping T from X into itself is called *weakly contractive* or *p-contractive* if there exists a w-distance p on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1, y_2) \leq rp(x_1, x_2)$. In particular, a single-valued mapping T from X into itself is called *weakly contractive* or *p-contractive* if there exists a w-distance p on X and $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in X$. The following theorem was proved by Suzuki and Takahashi [16].

Theorem 3.14. *Let X be a complete metric space and let T be a set-valued p-contractive mapping from X into itself such that for any $x \in X$, Tx is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$ and $p(x_0, x_0) = 0$.*

Proof. Let p be a w-distance on X and let r be a real number with $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ with $p(y_1, y_2) \leq rp(x_1, x_2)$. Let $u_0 \in X$ and $u_1 \in Tu_0$ be fixed. Then there exists $u_2 \in Tu_1$ such that $p(u_1, u_2) \leq rp(u_0, u_1)$. Thus, we have a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and $p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n)$ for every $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \leq r^2 p(u_{n-2}, u_{n-1}) \leq \cdots \leq r^n p(u_0, u_1)$$

and hence, for any $n, m \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} p(u_n, u_m) &\leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\ &\leq r^n p(u_0, u_1) + r^{n+1} p(u_0, u_1) + \cdots + r^{m-1} p(u_0, u_1) \\ &\leq \frac{r^n}{1-r} p(u_0, u_1). \end{aligned}$$

By Lemma 3.8, $\{u_n\}$ is a Cauchy sequence. Then $\{u_n\}$ converges to a point $v_0 \in X$. Let $n \in \mathbb{N}$ be fixed. Then since $\{u_m\}$ converges to v_0 and $p(u_n, \cdot)$ is lower semicontinuous, we have

$$p(u_n, v_0) \leq \liminf_{m \to \infty} p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1). \tag{$*$}$$

By hypothesis, we also have $w_n \in Tv_0$ such that $p(u_n, w_n) \leq rp(u_{n-1}, v_0)$. So, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} p(u_n, w_n) &\leq rp(u_{n-1}, v_0) \\ &\leq \frac{r^n}{1-r} p(u_0, u_1). \end{aligned}$$

By Lemma 3.8, $\{w_n\}$ converges to v_0. Since Tv_0 is closed, we have $v_0 \in Tv_0$. For such v_0, there exists $v_1 \in Tv_0$ such that $p(v_0, v_1) \leq rp(v_0, v_0)$. Thus, we also have a sequence $\{v_n\}$ in X such that $v_{n+1} \in Tv_n$ and $p(v_0, v_{n+1}) \leq rp(v_0, v_n)$ for every $n \in \mathbb{N}$. So, we have

$$p(v_0, v_n) \leq rp(v_0, v_{n-1}) \leq \cdots \leq r^n p(v_0, v_0).$$

By Lemma 3.8, $\{v_n\}$ is a Cauchy sequence. Then $\{v_n\}$ converges to a point $x_0 \in X$. Since $p(v_0, \cdot)$ is lower semicontinuous, we have $p(v_0, x_0) \leq \liminf_n p(v_0, v_n) \leq 0$ and hence $p(v_0, x_0) = 0$. Then, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} p(u_n, x_0) &\leq p(u_n, v_0) + p(v_0, x_0) \\ &\leq \frac{r^n}{1-r} p(u_0, u_1). \end{aligned}$$

So, using $(*)$ and Lemma 3.8, we obtain $v_0 = x_0$ and hence $p(v_0, v_0) = 0$. □

Corollary 3.15. *Let X be a complete metric space. If a mapping T from X into itself is p-contractive, then T has a unique fixed point $x_0 \in X$. Further such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. Let p be a w-distance and let r be a real number with $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in X$. Then from Theorem 3.14, there exists $x_0 \in X$ with $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. If $y_0 = Ty_0$, then we have

$$p(x_0, y_0) = p(Tx_0, Ty_0) \leq rp(x_0, y_0).$$

Since $r \in [0, 1)$, we have $p(x_0, y_0) = 0$. So, by $p(x_0, x_0) = 0$ and (i) of Lemma 3.8, we have $x_0 = y_0$. □

4 Characterizations of Metric Completeness

In this section, we discuss characterizations of metric completeness. We first give the following example [16] : Difine

$$A_n = \left\{ \left(t, \frac{t}{n} \right) \in \mathbb{R}^2 : t \in (0,1] \right\} \text{ for every } n \in \mathbb{N}$$

and

$$S = \bigcup_{n \in \mathbb{N}} A_n \cup \{0\}.$$

Then S is not complete and every continuous mapping on S has a fixed point in S.

Motivated by this example, we shall discuss characterizations of metric completeness. Before discussing them, let us study the relationship between contractive mappings and Kannan mappings [15]. Let X be a metric space with metric d. Then we denote by $W(X)$ the set of w-distances on X. A w-distance p on X is called *symmetric* if $p(x,y) = p(y,x)$ for all $x, y \in X$. We denote by $W_0(X)$ the set of all symmetric w-distances on X. Note that the metric d is an element in $W_0(X)$. We denote by $WC_1(X)$ the set of all mappings T from X into itself such that there exist $p \in W(X)$ and $r \in [0,1)$ satisfying

$$p(Tx, Ty) \leq rp(x,y) \quad \text{for all } x, y \in X,$$

i.e., the set of all weakly contractive mappings from X into itself. We define the sets $WC_2(X), WC_0(X), WK_1(X), WK_2(X)$ and $WK_0(X)$ of mappings from X into itself as follows: $T \in WC_2(X)$ if and only if there exist $p \in W(X)$ and $r \in [0,1)$ such that

$$p(Tx, Ty) \leq rp(y,x) \quad \text{for all } x, y \in X;$$

$T \in WC_0(X)$ if and only if there exist $p \in W_0(X)$ and $r \in [0,1)$ such that

$$p(Tx, Ty) \leq rp(x,y) \quad \text{for all } x, y \in X;$$

$T \in WK_1(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\} \quad \text{for all } x, y \in X;$$

$T \in WK_2(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(y, Ty)\} \quad \text{for all } x, y \in X;$$

$T \in WK_0(X)$ if and only if there exist $p \in W_0(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\} \quad \text{for all } x, y \in X.$$

For proving the theorems in this section, we need some lemmas.

Lemma 4.1. *Let X be a metric space with metric d, let p be a w-distance on X and let f be a function from X into $[0, \infty)$. Then a function q from $X \times X$ into $[0, \infty)$ given by $q(x, y) = f(x) + p(x, y)$ for each $(x, y) \in X \times X$ is also a w-distance.*

Lemma 4.2. *Let X be a metric space with metric d, let p be a w-distance on X, let T be a mapping from X into itself and let u be a point of X such that*

$$\lim_{m,n\to\infty} p(T^m u, T^n u) = 0.$$

Then for every $x \in X$, $\lim_{k\to\infty} p(T^k u, x)$ and $\lim_{k\to\infty} p(x, T^k u)$ exist. Moreover, let β and γ be functions from X into $[0, \infty)$ defined by

$$\beta(x) = \lim_{k\to\infty} p(T^k u, x) \quad \text{and} \quad \gamma(x) = \lim_{k\to\infty} p(x, T^k u).$$

Then the following hold:

(i) *β is lower semicontinuous on X;*

(ii) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\beta(x) \leq \delta$ and $\beta(y) \leq \delta$ imply $d(x, y) \leq \varepsilon$. In particular, the set $\{x \in X : \beta(x) = 0\}$ consists of at most one point;*

(iii) *the functions q_1 and q_2 from $X \times X$ into $[0, \infty)$ defined by*

$$q_1(x, y) = \beta(x) + \beta(y) \quad \text{and} \quad q_2(x, y) = \gamma(x) + \beta(y)$$

are w-distances on X.

We study the relationship between the classes of mappings by using Lemmas 4.1 and 4.2.

Lemma 4.3. $WC_1(X) \subset WK_0(X)$.

Proof. Suppose $T \in WC_1(X)$, i.e., there exist a w-distance p and $r \in [0, 1)$ such that $p(Tx, Ty) \leq rp(x, y)$ for all $x, y \in X$. Fix $u \in X$. Then we have, for each $m, n \in \mathbb{N}$,

$$p(T^m u, T^n u) \leq \frac{r^{\min\{m,n\}}}{1-r} \max\{p(u, u), p(Tu, u), p(u, Tu)\}.$$

Since $0 \leq r < 1$, we have $\lim_{m,n\to\infty} p(T^m u, T^n u) = 0$. So, by Lemma 4.2, $\beta(x) = \lim_{k\to\infty} p(T^k u, x)$ is well-defined and $q_1(x, y) = \beta(x) + \beta(y)$ is a w-distance on X. From $\beta(Tx) \leq r\beta(x)$ for every $x \in X$, we have

$$q_1(Tx, Ty) \leq r(1+r)^{-1}\{q_1(Tx, x) + q_1(Ty, y)\} \quad \text{for all } x, y \in X.$$

This implies $T \in WK_0(X)$. □

Lemma 4.4. $WK_1(X) \subset WC_0(X)$.

Proof. Suppose $T \in WK_1(X)$, i.e., there exist a w-distance p and $\alpha \in [0, 1/2)$ such that $p(Tx, Ty) \le \alpha p(Tx, x) + \alpha p(Ty, y)$ for all $x, y \in X$. We put $r = \alpha(1-\alpha)^{-1}$. Note that $p(T^2x, Tx) \le rp(Tx, x)$ for every $x \in X$. Fix $u \in X$. For $m, n \in \mathbb{N}$, we have

$$p(T^m u, T^n u) \le \alpha p(T^m u, T^{m-1}u) + \alpha p(T^n u, T^{n-1}u) \le \alpha(r^{m-1} + r^{n-1})p(Tu, u)$$

and hence $\lim_{m,n\to\infty} p(T^m u, T^n u) = 0$. So, by Lemma 4.2, $\beta(x) = \lim_{k\to\infty} p(T^k u, x)$ is well-defined and $q_1(x, y) = \beta(x) + \beta(y)$ is a w-distance on X. We next prove that $\beta(Tx) \le r\beta(x)$ for every $x \in X$. In fact, from

$$p(Tx, x) \le p(Tx, T^k u) + p(T^k u, x) \le \alpha p(Tx, x) + \alpha p(T^k u, T^{k-1}u) + p(T^k u, x),$$

we have

$$p(T^k u, Tx) \le \alpha p(T^k u, T^{k-1}u) + \alpha p(Tx, x) \le rp(T^k u, T^{k-1}u) + rp(T^k u, x).$$

Hence $\beta(Tx) \le r\beta(x)$. So we have $q_1(Tx, Ty) \le rq_1(x, y)$ for all $x, y \in X$. This implies $T \in WC_0(X)$. $\Box$

Lemma 4.5. $WC_2(X) = WK_2(X)$.

Proof. We first show $WC_2(X) \subset WK_2(X)$. Suppose $T \in WC_2(X)$, i.e., there exist a w-distance p and $r \in [0, 1)$ such that $p(Tx, Ty) \le rp(y, x)$ for all $x, y \in X$. Fix $u \in X$ and $m, n \in \mathbb{N}$. If $m > n$, then

$$\begin{aligned} p(T^m u, T^n u) + p(T^n u, T^m u) &\le \sum_{i=n}^{m-1} \{p(T^{i+1}u, T^i u) + p(T^i u, T^{i+1}u)\} \\ &\le \frac{r^n}{1-r}\{p(Tu, u) + p(u, Tu)\}. \end{aligned}$$

If $m = n$, then $p(T^m u, T^n u) \le r^m p(u, u)$. So, we have

$$p(T^m u, T^n u) \le \frac{r^{\min\{m,n\}}}{1-r}\{p(u, u) + p(Tu, u) + p(u, Tu)\}$$

and hence $\lim_{m,n\to\infty} p(T^m u, T^n u) = 0$. So, by Lemma 4.2, $\beta(x) = \lim_{k\to\infty} p(T^k u, x)$ and $\gamma(x) = \lim_{k\to\infty} p(x, T^k u)$ are well-defined and $q_2(x, y) = \gamma(x) + \beta(y)$ is a w-distance on X. From $\beta(Tx) \le r\gamma(x)$ and $\gamma(Tx) \le r\beta(x)$ for every $x \in X$, we have

$$q_2(Tx, Ty) \le r(1 + r)^{-1}\{q_2(Tx, x) + q_2(y, Ty)\} \quad \text{for all } x, y \in X.$$

This implies $T \in WK_2(X)$.

We next show $WK_2(X) \subset WC_2(X)$. Suppose $T \in WK_2(X)$, i.e., there exist a w-distance p and $\alpha \in [0, 1/2)$ such that $p(Tx, Ty) \leq \alpha p(Tx, x) + \alpha p(y, Ty)$ for all $x, y \in X$. We put $r = \alpha(1-\alpha)^{-1}$. Note that $p(T^2x, Tx) \leq rp(x, Tx)$ and $p(Tx, T^2x) \leq rp(Tx, x)$ for every $x \in X$. Fix $u \in X$. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned} p(T^m u, T^n u) &\leq p(T^m u, T^{m-1}u) + p(T^{n-1}u, T^n u) \\ &\leq (r^{m-1} + r^{n-1})\{p(Tu, u) + p(u, Tu)\} \end{aligned}$$

and hence $\lim_{m,n\to\infty} p(T^m u, T^n u) = 0$. So, by Lemma 4.2, $\beta(x) = \lim_{k\to\infty} p(T^k u, x)$ and $\gamma(x) = \lim_{k\to\infty} p(x, T^k u)$ are well-defined and $q_2(x, y) = \gamma(x) + \beta(y)$ is a w-distance on X. We next prove that $\beta(Tx) \leq r\gamma(x)$ for every $x \in X$. In fact, from

$$p(x, Tx) \leq p(x, T^k u) + p(T^k u, Tx) \leq p(x, T^k u) + \alpha p(T^k u, T^{k-1}u) + \alpha p(x, Tx),$$

we have

$$p(T^k u, Tx) \leq \alpha p(T^k u, T^{k-1}u) + \alpha p(x, Tx) \leq rp(T^k u, T^{k-1}u) + rp(x, T^k u).$$

So $\beta(Tx) \leq r\gamma(x)$. Similarly, we have $\gamma(Tx) \leq r\beta(x)$. Hence we have $q_2(Tx, Ty) \leq rq_2(y, x)$ for all $x, y \in X$. This implies $T \in WC_2(X)$. □

Now we can state the first theorem [15] in this section.

Theorem 4.6. *Let X be a metric space. Then*

$$WC_1(X) = WC_0(X) = WK_1(X) = WK_0(X) \subset WC_2(X) = WK_2(X).$$

Proof. It is clear that $WC_0(X) \subset WC_1(X)$ and $WK_0(X) \subset WK_1(X)$. So, by Lemmas 4.3 and 4.4, we have

$$WC_0(X) = WC_1(X) = WK_0(X) = WK_1(X).$$

Hence by Lemma 4.5, we obtain the desired result. □

Next, we discuss characterizations of metric completeness. Before discussing them, let us give a definition. Let μ be a mean on $\mathbb{N}$, i.e., a continuous linear fucntional on l_∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \dots) \in l_\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is to be a *Banach limit* if it satisfies $\mu_n(a_{n+1}) = \mu_n(a_n)$. We know that if μ is a Banach limit and $a_n \to k$, then $\mu_n(a_n) = k$. The following theorem was proved in [22] and [15] by using the nonconvex minimization theorem.

Theorem 4.7. *Let X be a metric space with metric d. Then the following are equivalent:*

(i) *X is complete;*

(ii) *every Kannan mapping T from X into itself has a fixed point in X;*

(iii) *for every bounded sequence $\{x_n\}$ in X and every mean μ on N such that $\inf_{x \in X} \mu_n d(x_n, x) = 0$, there exists $x_0 \in X$ with $\mu_n d(x_n, x_0) = 0$.*

(iv) *if for every uniformly continuous function $\phi : X \to [0, \infty)$ and every $u \in X$ with $\inf_{x \in X} \phi(x) < \phi(u)$, there exists $v \in X$ such that $v \neq u$ and $\phi(v) + d(u, v) \leq \phi(u)$, then there exists $z \in X$ such that $\phi(z) = \inf_{x \in X} \phi(x)$.*

Proof. (i) $\Rightarrow$ (ii) is obvious from Corollary 3.15 and Theorem 4.6. (We also leave another proof to the reader.) We show (ii) $\Rightarrow$ (iii). Let $\{x_n\}$ be a bounded sequence in X and let μ be a mean on $\mathbb{N}$ such that $\inf_{x \in X} \mu_n d(x_n, x) = 0$. Let us define a mapping T from X into itself as follows. For each $x \in X$, we choose a point $Tx \in X$ with $\mu_n d(x_n, Tx) \leq 1/4 \mu_n d(x_n, x)$. We show that T is a Kannan mapping. Let x and y be arbitrary points in X. Then

$$\mu_n d(x_n, Tx) \leq \frac{1}{4} \mu_n d(x_n, x) \leq \frac{1}{4} \{\mu_n d(x_n, Tx) + \mu_n d(Tx, x)\}.$$

Hence $\mu_n d(x_n, Tx) \leq 1/3 d(Tx, x)$. Similarly, $\mu_n d(x_n, Ty) \leq 1/3 d(Ty, y)$. So we have

$$d(Tx, Ty) = \mu_n d(Tx, Ty) \leq \mu_n d(x_n, Tx) + \mu_n d(x_n, Ty) \leq \frac{1}{3} d(Tx, x) + \frac{1}{3} d(Ty, y).$$

Hence T is a Kannan mapping. From (ii), there exists a point $x_0 \in X$ such that $Tx_0 = x_0$. So we have

$$\mu_n d(x_n, x_0) = \mu_n d(x_n, Tx_0) \leq \frac{1}{4} \mu_n d(x_n, x_0).$$

Hence $\mu_n d(x_n, x_0) = 0$. This implies (iii). We next show that (iii) $\Rightarrow$ (i). Let $\{x_n\}$ be a Cauchy sequence in X and let μ be a Banach limit. Then it is easy to see that

$$\mu_n d(x_n, x) = \lim_{n \to \infty} d(x_n, x)$$

for every $x \in X$ and

$$\inf_{x \in X} \mu_n d(x_n, x) = 0.$$

So from (iii), there exists a point $x_0 \in X$ such that $\mu_n d(x_n, x_0) = 0$. Hence $\lim_n d(x_n, x_0) = 0$. Therefore X is complete. (i) $\Rightarrow$ (iv) is immediate from Theorem 2.1. Let us prove (iv) $\Rightarrow$ (i). Let $\{x_n\} \subset X$ be a Cauchy sequence and consider the function $\phi : X \to [0, \infty)$ given by

$$\phi(x) = \lim_{n \to \infty} d(x_n, x).$$

Then, ϕ is uniformly continuous and $\inf_{x \in X} \phi(x) = 0$. Let $0 < \phi(u)$. Then, there exists an $x_m \in X$ such that $x_m \neq u$, $\phi(x_m) < \frac{1}{3}\phi(u)$ and $d(x_m, u) - \phi(u) < \phi(u)$. Hence, we have

$$3\phi(x_m) + d(x_m, u) < \phi(u) + 2\phi(u) = 3\phi(u).$$

So, there exists an $x_0 \in X$ with $0 = \phi(x_0) = \lim_{n \to \infty} d(x_n, x_0)$. This completes the proof. □

Using the above theorems, we obtain the following:

Corollary 4.8. *Let X be a metric space. Then the following are equivalent:*

(i) *X is complete;*

(ii) *every weakly contractive mapping from X into itself has a fixed point in X.*

Proof. (i) $\Rightarrow$ (ii) is proved in Section 3. By Theorem 4.6, we have $WK_0(X) = WC_1(X)$. Since $WK_0(X)$ contains all Kannan mappings from X into itself, we can prove (ii) $\Rightarrow$ (i) from Theorem 4.7. □

We also know the following theorem [16]

Theorem 4.9. *Let X be a normed linear space and let D be a convex subset of X. Then D is complete if and only if every contractive mapping from D into itself has a fixed point in D.*

As a direct consequence of Thorem 4.9, we obtain the following.

Corollary 4.10. *Let X be a normed linear space. Then X is a Banach space if and only if every contractive mapping form X into itself has a fixed point in X.*

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