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STRONG CONVERGENCE OF ITERATES OF NONEXPANSIVE MAPPINGS AND APPLICATIONS

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ABSTRACT. In this article, we deal with generalizations of Wittmann's strong convergence theorem for nonexpansive mappings and then study strong convergence theorems for a finite family of nonexpansive mappings in a Banach space. Further, we deal with convergence theorems which are connected with the problem of image recovery. Finally, we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings in a Banach space.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Wittmann [28] considered the following iteration scheme in a Hilbert space H:

$$x_1 = x \in C,\ x_{n+1} = \beta_n x + (1 - \beta_n)Tx_n \quad \text{for every } n = 1, 2, \dots \tag{1.1}$$

and showed that $\{x_n\}$ converges strongly to the element of $F(T)$ which is nearest to x in $F(T)$ if $\{\beta_n\}$ satisfies $0 \le \beta_n \le 1, \lim_{n\to\infty} \beta_n = 0, \sum_{k=1}^{\infty} \beta_k = \infty$ and $\sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty$ (see also [9]). Shioji and Takahashi [19, 20] extended Wittmann's result to a Banach space. See [1, 16, 23, 24, 26] for some results concerning weak convergence of iteration schemes of nonexpansive mappings.

On the other hand, the problem of image recovery in a Hilbert space setting may be stated as follows: The original (unknown) image z is known a priori to belong the intersection C_0 of r well-defined sets $C_1, C_2, \dots, C_r$ in a Hilbert space H; given only the metric projections P_i of H onto C_i $(i = 1, 2, \dots, r)$, recover z by an iterative scheme. In 1991, Crombez [8] proved the following: Let $T = \alpha_0 I + \Sigma_{i=1}^{r} \alpha_i T_i$ with $T_i = I + \lambda_i(P_i - I)$ for all i, $0 < \lambda_i < 2$, $\alpha_i > 0$ for $i = 0, 1, 2, \dots, r$, $\Sigma_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \bigcap_{i=1}^{r} C_i$ is nonempty. Then starting from an arbitrary

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element x of H, $\{T^n x\}$ converges weakly to an element of C_0. Later, Kitahara and Takahashi [11] and Takahashi and Tamura [25] dealt with the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space.

In this article, we first state generalizations of Wittmann's result [28]. We consider an iteration scheme given by a finite family of nonexpansive mappings which is connected with (1.1) and then state a strong convergence theorem for a finite family of nonexpansive mappings in a Banach space which generalizes Shioji and Takahashi's result [19]. Further, we deal with convergence theorems which are connected with the problem of image recovery. Finally, we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings in a Banach space.

2. Preliminaries

Throughout this paper, E is a real Banach space and E^* is the dual space of E. We denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. We write $x_n \rightharpoonup x$ (or $w\text{-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x. Similarly, $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) will symbolize strong convergence. For a subset A of E, coA and $\overline{co}A$ mean the convex hull of A and the closure of the convex hull of A, respectively.

A Banach space E is said to be strictly convex if $\dfrac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \mid \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well-known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. The following result was proved by Kirk [10].

Theorem 2.1 ([10]). Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then, $F(T)$ is nonempty.

The multi-valued mapping J from E into E^* defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for every} \quad x \in E$$

is called the duality mapping of E. From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$. A Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be Fréchet differentiable if for each x in S_1, this limit is attained uniformly for y in S_1. And the norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E. Let C be a nonempty closed convex subset of E and let K be a nonempty subset of C. A mapping P of C onto K is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. P is a retraction if $Px = x$ for each $x \in K$. We know from [5, 15] that if E is smooth, then a retraction P of C onto K is sunny and nonexpansive if and only if

$$\langle x - Px, J(y - Px) \rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in K.$$

Hence, there is at most one sunny nonexpansive retraction of C onto K. If there is a sunny nonexpansive retraction of C onto K, K is said to be a sunny nonexpansive retract of C.

The following theorem related to the existence of nonexpansive retractions was proved by Bruck [6, 7].

Theorem 2.2 ([6, 7]). Let E be a reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. If T has a fixed point in every nonempty bounded closed convex subset of E such that T leaves invariant, then $F(T)$ is a nonexpansive retract of C.

The following theorem related to the existence of sunny nonexpansive retractions was proved by Reich [17] (see also [12, 27]).

Theorem 2.3 ([17]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, the set $F(T)$ is a sunny nonexpansive retract of C.

3. Strong convergence theorems for a nonexpansive mapping

Let C be a nonempty closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. Halpern [9] studied the convergence of the following iteration scheme: $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x + (1-\beta_n)Tx_n \quad \text{for every } n = 1, 2, \dots, \tag{3.1}$$

where $\{\beta_n\}$ is a sequence of real numbers such that $0 \leq \beta_n \leq 1$ and $\lim_{n\to\infty} \beta_n = 0$. Reich [17, 18] studied the convergence of the sequence defined by (3.1) and posed the following problem.

Problem 1 ([18]). Let E be a Banach space. Is there a sequence $\{\beta_n\}$ of real numbers such that whenever a weakly compact convex subset C of E possessed the fixed point property for nonexpansive mappings, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T for all $x \in C$ and all nonexpansive mapping T of C into itself ?

Though Reich [17, 18] obtained a partial result in a Banach space, the problem had been generally open. Later, Wittmann [28] solved the problem in the framework of a Hilbert space.

Theorem 3.1 ([28]). Let C be a nonempty closed convex subset of a Hilbert space H, let T be a nonexpansive mapping from C into itself with $F(T) \neq \emptyset$ and let P be the metric projection from C onto $F(T)$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x + (1-\beta_n)Tx_n \quad \text{for every } n = 1, 2, \dots .$$

If a sequence $\{\beta_n\}$ of real numbers satisfies $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \dots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, then $\{x_n\}$ converges strongly to $Px \in F(T)$.

In Theorem 3.1, if $\beta_n = \frac{1}{n}$ and T is affine, the following weak convergence theorem which was proved by Baillon [3] is deduced.

Theorem 3.2 ([3]). Let C be a nonempty closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to a fixed point of T.

Recently Shioji and Takahashi [19] extended Wittmann's result [28] to a Banach space.

Theorem 3.3 ([19]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a closed convex subset of E. Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \dots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$ and suppose that $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x + (1-\beta_n)Tx_n$$

for every $n = 1, 2, \dots$. Then, $\{x_n\}$ converges strongly to $Px \in F(T)$, where P is a unique sunny nonexpansive retraction from C onto $F(T)$.

4. Strong convergence theorems for a finite family of mappings

By using ideas of [17, 18, 19, 28], we consider an iteration scheme given by a finite family of nonexpansive mappings which is connected with (3.1) and give strong convergence theorems for a finite family of nonexpansive mappings.

Let C be a nonempty convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping W of C into itself as follows (see [2, 23, 24]).

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned} \tag{4.1}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr} (n = 1, 2, \dots)$ be real numbers such that $0 \leq \alpha_{ni} \leq 1$ for every $i = 1, 2, \dots, r$. Let $W_n (n = 1, 2, \dots)$ be the W-mappings generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr}$. Now consider the following iteration scheme:

$$x_1 = x \in C, \quad x_{n+1} = \beta_n x + (1-\beta_n) W_n x_n \quad \text{for every} \quad n = 1, 2, \dots,$$

where $\{\beta_n\}$ is a sequence of real numbers such that $0 \le \beta_n \le 1$ for every $n = 1, 2, \dots$ (see [2]). The following lemma is obvious from the definition of (4.1).

Lemma 4.1 ([2]). Let C be a non closed convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i = 1, 2, \dots, r$. Let $U_1, U_2, \dots, U_{r-1}$ and W be the mappings defined by (4.1). Then, $U_1, U_2, \dots, U_{r-1}$ and W are also nonexpansive.

To prove the main theorem (Theorem 4.4), we need the following lemma.

Lemma 4.2 ([2]). Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_r \le 1$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^r F(T_i)$.

The following theorem which is a generalization of Browder's result [4] plays an important role in the proofs of main theorem (Theorem 4.4) and Theorem 3.3. For the proof of the theorem, see [17, 27].

Theorem 4.3 ([17, 27]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then, there is a unique sunny nonexpansive retraction P from C onto $F(T)$. Further, let $x \in C$ and suppose that $\{u_k\} \subset C$ is given by

$$u_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right)Tu_k \quad \text{for every } k = 2, 3, \dots . \tag{4.2}$$

Then, $\{u_k\}$ converges strongly to $Px \in F(T)$.

Now we can state a strong convergence theorem for a finite family of nonexpansive mappings.

Theorem 4.4 ([2]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr}$ $(n = 1, 2, \dots)$ be real numbers such that $0 < \alpha_{ni} < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_{nr} \le 1$ and let $T_1, T_2, \dots, T_r$ be finite nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $W_n (n = 1, 2, \dots)$ be the W-mappings of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr}$. Let $\{\beta_n\}$ be a sequence of real numbers such that

$0 \leq \beta_n \leq 1$ for every $n = 1, 2, \dots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$ and suppose that $\sum_{n=1}^{\infty} |\alpha_{n+1i} - \alpha_{ni}| < \infty$ for every $i = 1, \dots, r$. If $\{x_n\}$ is given by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x + (1 - \beta_n) W_n x_n$$

for every $n = 1, 2, \dots$, then $\{x_n\}$ converges strongly to $Px \in \bigcap_{i=1}^{r} F(T_i)$, where P is a unique sunny nonexpansive retraction from C onto $\bigcap_{i=1}^{r} F(T_r)$.

As a direct consequence of Theorem 4.4, we have the following theorem.

Theorem 4.5 ([2]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_r \leq 1$ and let $T_1, T_2, \dots, T_r$ be finite nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \dots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $\{x_n\}$ is given by $y_1 = x \in C$ and

$$y_{n+1} = \beta_n x + (1 - \beta_n) W y_n$$

for every $n = 1, 2, \dots$. Then, $\{y_n\}$ converges strongly to $Px \in F(W) = \bigcap_{i=1}^{r} F(T_i)$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} F(T_i)$.

We show that Theorem 3.3 is a direct consequence of Theorem 4.4 (see also [2]). In Theorem 4.5, if $T_i = I$, $0 < \alpha_i < 1$ for every $i = 1, 2 \dots, r-1$, $\alpha_r = 1$ and T_r is a nonexpansive mapping T of C into itself, then $W = T$. So, from Theorem 4.5, $\{x_n\}$ converges strongly to $Px \in F(T)$, where P is a unique sunny nonexpansive retraction from C onto $F(T)$. Hence, Theorems 4.4 and 4.5 are generalizations of Theorem 3.3.

5. APPLICATIONS

In this section, we first deal with convergence theorems which are connected with the problem of image recovery in Banach spaces. Secondly, we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings in a Banach space.

The problem of image recovery in a Hilbert space setting may be stated as follows: The original (unknown) image z is known a priori to belong the intersection C_0 of r well-defined sets $C_1, C_2, \dots, C_r$ in a Hilbert space H; given only the metric projections P_i of

H onto C_i $(i = 1, 2, \dots, r)$, recover z by an iterative scheme. In 1991, Crombez [8] proved a weak convergence theorem which is connected with the problem of image recovery in a Hilbert space setting.

Theorem 5.1 ([8]). Let $C_1, C_2, \dots, C_r$ be closed convex subsets of a Hilbert space H such that $\bigcap_{i=1}^r C_i \neq \emptyset$. Let $T = \alpha_0 I + \Sigma_{i=1}^r \alpha_i T_i$ with $T_i = I + \lambda_i (P_i - I)$ for all i, $0 < \lambda_i < 2$, $\alpha_i > 0$ for $i = 0, 1, 2, \dots, r$, $\Sigma_{i=0}^r \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \bigcap_{i=1}^r C_i$ is nonempty. Then, starting from an arbitrary element x of H, $\{T^n x\}$ converges weakly to an element of C_0.

Now we consider convergence theorems which are connected with the problem in a Banach spaces setting. Before stating them, we give a definition. Let C be a closed convex subset of E. A mapping U of C into itself is said to be asymptotically regular if, for each $x \in C$, $U^n x - U^{n+1} x \to 0$.

Kitahara and Takahashi [11] dealt with the problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces. In [11], they proved that a mapping given by convex combinations of sunny nonexpansive retractions in a uniformly convex Banach space is asymptotically regular and the set of fixed points of the mapping is equal to the intersection of the ranges of sunny nonexpansive retractions in a strictly convex Banach space. Further, using the results, they proved a weak convergence theorem for the mapping which is connected with the problem of image recovery.

Theorem 5.2 ([11]). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $C_1, C_2, \dots, C_r$ be sunny nonexpansive retracts of C such that $\bigcap_{i=1}^r C_i \neq \emptyset$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \dots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto C_i. Then, $F(T) = \bigcap_{i=1}^r C_i$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^r C_i$.

Theorem 5.2 was extended to nonexpansive retracts by Takahashi and Tamura [25]. They dealt with the problem by convex combinations of nonexpansive retractions. They proved the following lemma.

Lemma 5.3 ([25]). Let C be a nonempty convex subset of a Banach space E and let S be a mapping of C into itself given by $S = \alpha_0 I + \sum_{i=1}^r \alpha_i S_i$, $0 < \alpha_i < 1$, $i = 0, \dots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, S_i is a nonexpansive mapping of C into itself and $\bigcap_{i=1}^r F(S_i) \neq \emptyset$. Then, S is asymptotically regular.

Using the lemma, Takahashi and Tamura [25] proved a weak convergence theorem for a mapping given by convex combinations of nonexpansive retractions.

Theorem 5.4 ([25]). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $C_1, C_2, \ldots, C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto C_i. Then, $F(T) = \bigcap_{i=1}^{r} C_i$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$.

Takahashi and Tamura [25] proved another weak convergence theorem. Before stating it, we give a definition. We say that E satisfies Opial's condition [14] if for any sequence $\{z_n\} \subset E$ with $z_n \rightharpoonup x \in E$, the inequality

$$\varliminf_{n \to \infty} \|z_n - x\| < \varliminf_{n \to \infty} \|z_n - y\|$$

holds for every $y \in E$ with $y \neq x$.

Theorem 5.5 ([25]). Let E be a reflexive and strictly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $C_1, C_2, \ldots, C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto C_i. Then, $F(T) = \bigcap_{i=1}^{r} C_i$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} C_i$.

On the other hand, we consider the problem by using an iteration scheme which is different from [8, 11, 23, 25, 24]. Using Theorem 4.5, we obtain a *strong convergence* theorem which is connected with the problem of image recovery in a Banach space setting. Before stating it, we give the following lemma.

Lemma 5.6 ([2]). Let C be a nonempty closed convex subset of a Banach space E and let $C_1, C_2, \ldots, C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Let W be the W-mapping generated by $P_1, \ldots, P_r$ and $\alpha_1, \ldots, \alpha_r$, where P_i is a nonexpansive retraction from C onto C_i and $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \ldots, r$. Then,

$$F(W) \supset \bigcap_{i=1}^{r} C_i.$$

If E is strictly convex and $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_r \leq 1$, then

$$F(W) = \bigcap_{i=1}^{r} C_i.$$

As a direct result of Theorem 4.5 and Lemma 5.6, we have the following theorem.

Theorem 5.7 ([2]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let $C_1, C_2, \dots, C_r$ be nonexpansive retracts of C such that $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Let W be the W-mapping of C into itself generated by $P_1, P_2, \dots, P_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$, where $P_i (i = 1, 2, \dots, r)$ is a nonexpansive retraction from C onto C_i and $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_r \leq 1$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \dots$, $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{y_n\}$ is given by $y_1 = x \in C$ and

$$y_{n+1} = \beta_n x + (1 - \beta_n) W y_n$$

for every $n = 1, 2, \dots$, then $\{y_n\}$ converges strongly to $Px \in F(W) = \bigcap_{i=1}^{r} C_i$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} C_i$.

In this section, we finally consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings. Kitahara and Takahashi [11] dealt with the problem by convex combinations of sunny nonexpansive retractions in a uniformly convex and uniformly smooth Banach space and they proved a weak convergence theorem.

Theorem 5.8 ([11]). Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, \dots, S_r\}$ be a commuting family of nonexpansive mappings of C into itself with $F(S_i) \neq \emptyset$ for every $i = 1, 2, \dots, r$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \dots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto $F(S_i)$. Then, $F(T) = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. Further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$.

Takahashi and Tamura [25] dealt with the problem by convex combinations of nonexpansive retractions. They obtained weak convergence theorems which are generalizations of Kitahara and Takahashi's result [11].

Theorem 5.9 ([25]). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, \dots, S_r\}$ be a commuting family of nonexpansive mappings of C into itself such that $F(S_i) \neq \emptyset$ for every $i = 1, 2, \ldots, r$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto $F(S_i)$. Then, $F(T) = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. Further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$.

Theorem 5.10 ([25]). Let E be a reflexive and strictly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, \ldots, S_r\}$ be a commuting family of nonexpansive mappings of C into itself such that $F(S_i) \neq \emptyset$ for every $i = 1, 2, \ldots, r$. Let T be a mapping of C into itself given by $T = \sum_{i=1}^{r} \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i) I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a nonexpansive retraction of C onto $F(S_i)$. Then, $F(T) = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $\bigcap_{i=1}^{r} F(S_i)$.

We consider the problem in a Banach space setting by using an iteration scheme which is different from [11, 25]. Using Theorem 4.5, we obtain a *strong convergence* theorem which is connected with the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings.

Theorem 5.11 ([2]). Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, \ldots, S_r\}$ be a commuting finite family of nonexpansive mappings of C into itself such that $F(S_i) \neq \emptyset$ for every $i = 1, 2, \ldots, r$. Let W be the W-mapping generated by $P_1, P_2, \ldots, P_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$, where $P_i (i = 1, 2, \ldots, r)$ is a unique sunny nonexpansive retraction from C onto $F(S_i)$ and $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots, r-1$ and $0 < \alpha_r \leq 1$. Then, $F(W) = \bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for every $n = 1, 2, \ldots$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. If $\{y_n\}$ is given by $y_1 = x \in C$ and

$$y_{n+1} = \beta_n x + (1 - \beta_n) W y_n$$

for every $n = 1, 2, \ldots$, then $\{y_n\}$ converges strongly to $Px \in F(W) = \bigcap_{i=1}^{r} F(S_i)$, where P is a unique sunny nonexpansive retraction from C onto $F(W) = \bigcap_{i=1}^{r} F(S_i)$.

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