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REMARKS ON HILLE'S UNIFORM AND STRONG ERGODIC THEOREMS

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ABSTRACT. We give a full answer to the converse problem of Hille in the uniform and strong ergodic theorems with a view to relating the Cesàro (C,α) ergodic theorems for a linear operator T on a Banach space and the properties of the resolvent $R(\lambda;T)$.

My purpose in this exposé is to give a brief summary of some recent research on the uniform and strong convergence of Cesàro (C,α) and Abelian ergodic operator averages. The exposé is mainly a report on the author's personal work on the subject, by a general survey. Most of the results mentioned below were discussed in [6].

Let $(X,\|\cdot\|)$ be a complex Banach space and let $B[X,X]$ denote the Banach algebra of bounded linear operators on X to itself. For a real number $\alpha > -1$ and each integer $n \geq 0$, let $A_n^{(\alpha)}$ be the (C,α) coefficient of order α which is defined by the generating function $(1-\mu)^{-(\alpha+1)} = \Sigma_{n=0}^{\infty} A_n^{(\alpha)} \mu^n$ $(0 < \mu < 1)$. In particular, we have

$$A_0^{(\alpha)} = A_n^{(0)} = 1, \quad A_n^{(\alpha)} > 0, \quad A_n^{(\alpha)} - A_{n-1}^{(\alpha)} = A_n^{(\alpha-1)}, \quad n \geq 1,$$

$$A_n^{(\alpha)} = \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \sim \frac{n^{\alpha}}{\Gamma(\alpha+1)}.$$

Furthermore, $A_n^{(\alpha)}$ is an increasing function of $n \geq 0$ for $\alpha > 0$ and is a decreasing function of $n \geq 0$ for $-1 < \alpha < 0$. If λ is such that $|\lambda| > \gamma(T)$, where $\gamma(T)$ stands for the spectral radius of $T \in B[X,X]$, then two series $I - \Sigma_{n=1}^{\infty}(T^{n+1}-T^n)/\lambda^n$ and $\Sigma_{n=0}^{\infty} T^n/\lambda^{n+1}$ converge in the uniform operator topology and

$$(\lambda-1)R(\lambda;T) = (\lambda-1)\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} = I - (I-T)\sum_{n=1}^{\infty} \frac{T^{n-1}}{\lambda^n}.$$

For each $\lambda > \max(1, \gamma(T))$, $(\lambda-1)R(\lambda;T)$ is said to be the Abel average for T. The resolvent $R(\lambda;T)$ is analytic in the resolvent set $\rho(T)$. Let D be an open set containing $\rho(T)$ and let ∂D denote the boundary of D which is assumed to consist of a finite number of rectifiable Jordan curves, oriented in the usual sense. The Cesàro (C,α) averages of order $\alpha > 0$ for $T \in B[X,X]$ are defined by

$$C_n^{(\alpha)}[T] = \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} T^k = \frac{1}{2\pi i} \int_{\partial D} C_n^{(\alpha)}(\lambda) R(\lambda;T)\, d\lambda, \quad n \geq 0,$$

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where $C_n^{(\alpha)}(\lambda) = (A_n^{(\alpha)})^{-1} \Sigma_{k=0}^{n} A_{n-k}^{(\alpha-1)} \lambda^k$. In 1945, E. Hille obtained the following theorem.

THEOREM 1 ([3], Theorems 6, 7). A necessary condition for the existence of an operator $E \varepsilon B[X,X]$ such that for some fixed $\alpha > 0$

(1) $\quad$ (uo)(resp. (so)) $\lim_{n\to\infty} C_n^{(\alpha)}[T] = E$

is that

(2) $\quad$ (uo)(resp. (so)) $\lim_{n\to\infty} T^n/n^\alpha = \theta$ (the null operator) and

(3) $\quad$ (uo)(resp. (so)) $\lim_{\lambda\to 1+0} (\lambda-1)R(\lambda;T) = E$.

Conversely, if (2) is replaced by the power-boundedness of T, then (3) implies (1) for every $\alpha > 0$.

Our particular interest consists in the converse statement of the above theorem, when the operators in question are not necessarily power-bounded. Such a case seems to have not been considered by Hille. More precisely, the question is whether the power-boundedness of the operators in question is indispensable to deduce (1) from (3). A partial negative answer to this question was first given by M. Lin in the case $\alpha = 1$, proving the following theorem.

THEOREM 2 ([4], Proposition). Let $T \varepsilon B[X, X]$ satisfy the condition $(uo)\lim_{n\to\infty} T^n/n = \theta$. Then the following conditions are equivalent :

(1) $\quad$ (uo) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k$ exists in $B[X,X]$.

(2) $\quad$ (uo) $\lim_{\lambda\to 1+0} (\lambda-1)R(\lambda;T) = E$ for some operator $E \varepsilon B[X,X]$.

Our purpose is to answer the question negatively for any real order $\alpha > 0$. The main results are stated as follows :

THEOREM 3 ([6], Theorem 1). For any fixed real $\alpha > 0$ let $T \varepsilon B[X,X]$ satisfy the condition $(uo)\lim_{n\to\infty} T^n/n^\omega = \theta$, where $\omega = \min(1,\alpha)$. Then there exists an operator $E \varepsilon B[X,X]$ such that

$$(uo) \lim_{n\to\infty} C_n^{(\alpha)}[T] = E \text{ if and only if } (uo) \lim_{\lambda\to 1+0} (\lambda-1)R(\lambda;T) = E.$$

THEOREM 4 ([6], Theorem 2). For any fixed real $\alpha > 0$ let $T \varepsilon B[X,X]$ satisfy the condition $(so)\lim_{n\to\infty} T^n/n^\omega = \theta$, where $\omega = \min(1,\alpha)$. Suppose that

$$\sup_n \| C_n^{(\alpha)}[T]x \| < \infty \text{ for all } x \varepsilon \overline{\text{Range}(I-T)}.$$

Then there exists an operator $E \varepsilon B[X,X]$ such that

$$(so) \lim_{n\to\infty} C_n^{(\alpha)}[T] = E \text{ if and only if } (so) \lim_{\lambda\to 1+0} (\lambda-1)R(\lambda;T) = E.$$

Condition (2) appearing in the Hille theorem (Theorem 1) is very important in ergodic theory. In fact, this condition plays an essential role in relating Tauberian theorem and ergodic theorems. Hille himself stated only the fact of implication (1)$\Longrightarrow$(2) without proof. Since then, its proof has never been given so far. So it seems to be worth while to give the proof of implication (1)$\Longrightarrow$(2). We prove the following (new) theorem which is the key of the proof.

THEOREM 5. Suppose that for any fixed real $\alpha > 0$ there exists an operator $E \in B[X,X]$ such that (uo)(resp. (so))$\lim_{n\to\infty} C_n^{(\alpha)}[T] = E$. Then we have

$$\text{(uo)(resp. (so))}\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{k=0}^{n} A_{n-k}^{(\beta-1)} T^k = \theta \quad \text{for } -1 < \beta < \min(1,\alpha).$$

Proof. Here we consider only the case of uniform operator topology. For notational covenience we write

$$u_n[T] = T^n, \quad S_n^{(\alpha)}[T] = \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} T^k, \quad n \geq 0.$$

Then it follows that

$$\sum_{n=0}^{\infty} u_n[T] r^n = (1-r)^\alpha \sum_{n=0}^{\infty} S_n^{(\alpha)}[T] r^n, \quad 0 < r < 1, \quad 1/r > \gamma(T).$$

Now fix a β such that $-1 < \beta < \min(1,\alpha)$. We obtain

$$\begin{aligned}\sum_{n=0}^{\infty} S_n^{(\beta)}[T] r^n &= (1-r)^{-\beta} \sum_{n=0}^{\infty} u_n[T] r^n \\ &= (1-r)^{\alpha-\beta} (1-r)^{-\alpha} \sum_{n=0}^{\infty} u_n[T] r^n \\ &= (1-r)^{\alpha-\beta} \sum_{n=0}^{\infty} S_n^{(\alpha)}[T] r^n,\end{aligned}$$

so that

$$S_n^{(\beta)}[T] = \sum_{k=0}^{n} (-1)^k \binom{\alpha-\beta}{k} S_{n-k}^{(\alpha)}[T], \quad n \geq 0,$$

$$A_n^{(\beta)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha-\beta}{k} A_{n-k}^{(\alpha)}, \quad n \geq 0.$$

There exist constants $K_1 > 0$ and $K_2 > 0$ independent of n and k, such that

$$\frac{A_{n-k}^{(\alpha)}}{n^\alpha} < K_1 \left(2 - \frac{k}{n}\right)^\alpha, \quad 0 \leq k \leq n, \quad n \geq 1,$$

$$\left| \binom{\alpha-\beta}{k} \right| < \frac{K_2}{k^{\alpha-\beta+1}}, \quad k \geq 1.$$

Put $m = [\frac{n}{2}]$. Given $\varepsilon > 0$, choose an integer $N_1 = N_1(\varepsilon)$ say, large enough so that

$$\frac{A_n^{(\alpha)}}{n^\alpha} \| C_n^{(\alpha)}[T] - E \| < \frac{\varepsilon}{3}, \quad n \geq N_1 + 1,$$

$$\| C_{n-k}^{(\alpha)}[T] - E \| < \frac{\varepsilon}{6\sigma K_1 K_2}, \quad 1 \le k \le m, \quad n \ge N_1 + 1,$$

where $\sigma = \Sigma_{k=1}^{\infty}\{1/k^{\alpha-\beta+1}\}$ which converges since $\alpha-\beta+1 > 1$. Then we have

$$K_1 K_2 \sum_{k=1}^{m} (2 - \frac{k}{n})^{\alpha} \frac{1}{k^{\alpha-\beta+1}} \| C_{n-k}^{(\alpha)}[T] - E \| < \frac{\varepsilon}{3}, \quad n \ge N_1 + 1.$$

On the other hand, since $\{\| C_n^{(\alpha)}[T] - E \|\}$ is bounded, there exists an integer $N_2 = N_2(\varepsilon)$ say, large enough so that

$$\begin{aligned} K_1 K_2 \sum_{k=m+1}^{n} (2 - \frac{k}{n})^{\alpha} \frac{1}{k^{\alpha-\beta+1}} \| C_{n-k}^{(\alpha)}[T] - E \| &\le K_1 K_2 K_3 \sum_{k=m+1}^{n} (2 - \frac{k}{n})^{\alpha} \\ &\le \frac{K_1 K_2 K_3}{n^{\alpha-\beta}} \int_{\frac{1}{2}}^{1} \frac{(2-t)^{\alpha}}{t^{\alpha-\beta+1}}\, dt \\ &< \frac{\varepsilon}{3}, \qquad n \ge N_2 + 1 \end{aligned}$$

with some constant $K_3 > 0$. Summing up the estimates obtained above, we have for all $n > \max(N_1, N_2)$

$$\begin{aligned} \left\| \frac{S_n^{(\alpha)}[T]}{n^{\alpha}} - \frac{A_n^{(\beta)} E}{n^{\alpha}} \right\| &\le \frac{A_n^{(\alpha)}}{n^{\alpha}} \| C_n^{(\alpha)}[T] - E \| \\ &+ K_1 K_2 \sum_{k=1}^{m} (2 - \frac{k}{n})^{\alpha} \frac{1}{k^{\alpha-\beta+1}} \| C_{n-k}^{(\alpha)}[T] - E \| \\ &+ K_1 K_2 \sum_{k=m+1}^{n} (2 - \frac{k}{n})^{\alpha} \frac{1}{k^{\alpha-\beta+1}} \| C_{n-k}^{(\alpha)}[T] - E \| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since $\|A_n^{(\alpha)} E\| = o(n^{\alpha})$, we therefore have $\| S_n^{(\beta)}[T] \| = o(n^{\alpha})$ and the theorem follows.

Using Theorem 5 we see that implication $(1) \Longrightarrow (2)$ in Theorem 1 follows from a modification of the proof given by S. Chapman [1] in the case of numerically valued sequences.

REMARK. Following Hille [3], we take X to be the space $C_0[0,1]$ of functions $f(x)$ continuous for $0 \le x \le 1$ which vanish at 0, with $\| f \| = \max | f(x) |$. Let $\beta > 0$ be fixed and define

$$Q_{\beta} f = (I - J_{\beta}) f, \quad (J_{\beta} f)(x) = \int_0^x [\Gamma(\beta)]^{-1} (x-u)^{\beta-1} f(u)\, du, \quad 0 \le x \le 1$$

for $f \in X$. Then the iterate Q_{β}^n, for each n, has the form

$$(Q_\beta^n f)(x) = f(x) - \int_0^x P_n(x-u,\beta) f(u)\,du,$$

where

$$P_n(x-u,\beta) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} [\Gamma(k\beta)]^{-1} (x-u)^{k\beta-1}.$$

E. Hille proved that (i) $\| Q_1^n \| = O(n^{1/4})$, $\lim_{n\to\infty} \| Q_1^n \| = \infty$, and (ii) Q_1 is strongly (C,α) ergodic for $\alpha > 1/2$. But we see that Q_1 fails to be uniformly (C,α) ergodic when $1/2 < \alpha < 1$ (see [6]). Let $T_\beta = \Gamma(\beta+1) Q_1 J_\beta$ for $\beta > 1$ with the operators Q_1 and J_β. Since $\|J_\beta\| \le 1/\Gamma(\beta+1)$, we have $\|T_\beta^n\| = O(n^{1/4})$. Note that T_β is compact. Then by Theorem 3.1 of [5] T_β is uniformly (C,α) ergodic for $\alpha > 1/4$.

CONJECTURE. $\lim_{n\to\infty} \| T_\beta^n \| = \infty$ for some $\beta > 1$.

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