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A note on Feynman path integrals for Schrödinger equations

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ABSTRACT. In this paper, we shall introduce *generalized* vector measures for the path integrals for Schrödinger equations. We prove that $[\Phi(V)](\gamma) \equiv e^{i\int_0^t V(\gamma(s))ds}$ is *integrable* with respect to our *generalized* measure for any real function $V(\vec{x})$ which continuous in $\vec{x} \in \mathbb{R}^N \backslash \{0\}$

INTRODUCTION

It is well known that for the case of space-dimension ≥ 2,Feynman path integrals for Schrödinger or Dirac equations are not represented by (scalar-valued) measures.

In this paper, we shall introduce *generalized* vector measures and prove that $[\Phi(V)](\gamma) \equiv e^{i\int_0^t V(\gamma(s))ds}$ in the following equation(2) is *integrable* with respect to our *generalized* measure for any real function $V(\vec{x})$ which continuous in $\vec{x} \in \mathbb{R}^N \backslash \{0\}$ (see Corollary 3).

For the Schrödinger equation in $\mathbb{R}^N$

$$(1) \quad \frac{\partial}{\partial t}u(t,\vec{x}) = -i\triangle u(t,\vec{x}) + iV(\vec{x})u(t,\vec{x}), \quad u(0,\vec{x}) = \varphi(\vec{x}), \quad \varphi \in L^2(\mathbb{R}^N;\mathbb{C}),$$

we would write the associated semigroup $\{S_t\}$:

$$S_t\varphi(\vec{x}) = u(t,\vec{x}) = \int K(t;\vec{x},\vec{y})\varphi(\vec{y})dy$$

(where $K(t;\vec{x},\vec{y})$ is the elementary solution of (1)) as the path integral

$$(2) \qquad S_t\varphi(\vec{x}) = \int_{\Omega_{[0,t]}} e^{i\int_0^t V(\gamma(s))ds}\varphi(\gamma(0))d\mu^Q(\gamma) \quad S_t\varphi \in L^2(\mathbb{R}^N;\mathbb{C}).$$

where every γ is a path on $\mathbb{R}^N$, or $\gamma \in \Omega_{[0,t]} \equiv \Pi_{\alpha\in[0,t]}\mathbb{R}^N_\alpha$, ($\mathbb{R}^N_\alpha$ = a copy of $\mathbb{R}^N$). If $V \equiv 0$ we denote the semigroup by $\{T_t\}$ and the elementary solution by $K_0(t;x,y)$.

F. Takeo [13],[14] introduced the notion of generalized vector measures similar to ours for the path integrals for Dirac and Schrödinger equations. For the necessity and the more detailed discussion of generalized measures associated with path integrals, see [8].

1. Path Integrals and generalized measure

Using L^2-wellposedness of $\frac{\partial}{\partial t}u(t,\vec{x}) = -i\triangle u(t,\vec{x})$, we shall try to represent path integrals for Schrödinger equations as a kind of Stieltjes integrals. Though this is similar to Takeo's measure, our generalized measure is easy to understand.

For an L^∞-valued measure $g = g(\vec{x},\tau), \vec{x} \in \mathbb{R}^N, \tau \in I = [0,t]$, let

$$[\Phi(g)_{\tau_0,\tau}](\vec{x}) = e^{\int_{\tau_0}^{\tau} g(\vec{x},s)ds}, \quad [\Phi(g)_{\tau_0,\tau}](\gamma) \equiv [\Phi(V)_{\tau_0,\tau}]((\vec{x}_\alpha)) = e^{\int_{\tau_0}^{\tau} g(\gamma(s),s)ds}$$

$$\text{for } \gamma = (\vec{x}_\alpha \in \mathbb{R}^N) \text{ (or } \gamma(\alpha) = \vec{x}_\alpha), \ \alpha \in [\tau_0,\tau],$$

be a function $\mathbb{R}^{N\times[\tau_0,\tau]} \longrightarrow \mathbb{C}$. For $\tau_0 = 0$ we denote $\Phi(g)_{0,\tau}$ simply by $\Phi(g)_\tau$ and $[\Phi(g)_{0,t}](\vec{x})$ by $[\Phi(g)](\vec{x},t)$. Note that

$$\|\Phi(g)_{\tau_0,\tau}\|_\infty \leq e^{\int_{\tau_0}^{\tau} \sup_{\vec{x}} \Re g(\vec{x},s)ds}.$$

Definition 1.

$$var(F \circ \gamma) = \sup_{\triangle} \Big\{ \sum_j |F(\gamma(\tau_j)) - F(\gamma(\tau_{j-1}))| \Big\},$$

$$var(F) = \sup_{\triangle} \Big\{ \sum_j \|F_{\tau_j} - F_{\tau_{j-1}}\|_\infty \Big\} = \sup_{\gamma} var(F \circ \gamma),$$

where $\triangle$ runs over all divisions of $[0,t]$, $\triangle : 0 = \tau_0 < \cdots < \tau_L = t$.

For any fixed γ, $(F\circ\gamma)(\tau) \equiv F_\tau(\gamma)$ is a function with bounded variation if $var(F) < \infty$, and we have $var(F) = \sup_\gamma var(F \circ \gamma)$. We shall define a norm of the space X of the functions with bounded variation. The unit ball B and the Banach space X associated with the norm $\|\cdot\|_{t,v}$ are defined as usual:

$$B = \textit{the convex hull of } \Big\{\Phi(g) \ \Big| \ e^{\int_0^t \sup_{\vec{x}} |g(\vec{x},s)|ds} \leq e\Big\},$$

$$X = \textit{the completion of the normed space with unit ball } B.$$

Definition 2. *We denote by L^{bv} the space of functions with bounded variation. We denote the space of F above by $L^{bv}([0,t], L^\infty(\mathbb{R}^N))$ if F is considered as a function of $(t,\vec{x})$, and by $L^{bv}(\Omega_{[0,t]})$ if F is considered as a function of γ.*

Evidently we have

$$\|F\|_{t,v} = 1 \iff var(G) = e, \quad \forall F(\vec{x}) = e^{\int_0^t \sup_{\vec{x}} g(\vec{x},s)ds}, \tag{3}$$

where $G(t) = \int_0^t g(s)ds$. Note that

$$\frac{\partial}{\partial t}\|F\|_{t,v} = \|g(t)\|_\infty e^{\int_0^t \|g(s)\|_\infty ds},$$

$$C_1 \int_0^t \|g(s)\|_\infty ds \leq var(\Phi(g)) \leq C_2 \int_0^t \|g(s)\|_\infty ds.$$

We define a finitely additive measure

$$(\mu^Q_{[s_j,s_{j+1}]}(B_j)f)(\vec{x}) = T_h(\chi_{B_j}f) = \int_{B_j} K_0(h,\vec{x},\vec{y})f(\vec{y})d\vec{y},$$
$$(\mu^Q_{0,t}(B)f)(\vec{x}) = (\prod(\mu^Q_{[s_j,s_{j+1}]}(B_j))f)(\vec{x}),$$

where χ_{B_j} be the characteristic function of B_j, that is, $\chi_{B_j}(\vec{x}) = 1$ for $\vec{x} \in B_j, \chi_{B_j}(\vec{x}) = 0$ for $\vec{x} \notin B_j$. The solution $u(t) = S_t u(0)$ to the equation

$$\frac{d}{dt}u = (A+g)u, \quad u(0,\vec{x}) = f(\vec{x})$$

where $A = -i\triangle$ is formally written as

$$u(t) = \lim_{h\downarrow 0}\prod_{j=1}^{t/h} T_h e^{g(\tau_j)h} f = \int_{\Omega_{[0,t]}} e^{\int_0^t V(\gamma(\tau))d\tau} f d\mu^Q.$$

On the other hand,

$$u(t) = T_t f + \int_0^t T_{t-s}g(s)fds = T_t f + G(t) - T_t G(0) - \int_0^t G(t-s)dT_s f. \tag{4}$$

Lemma 1. *The equation* (4) *has a unique solution. More precisely,*

$$\|u(t)\| \le \Big(1 + \int_0^t \|g(s)\|_\infty ds\Big)\|u(0)\|.$$

Proof. Since $T_{t-s}f$ is an L^2-valued continuos function and $g(s)$ is an L^∞-valued measure, $T_{t-s}g(s)f$ is an L^∞-valued measure and hence integrable.

Definition 3. *An element $\mu \in (L^{bv}(\Omega_{[0,t]}); X)$ is called an X-valued generalized measure. For $f, f' \in L^2$ we formally denote*

$$\Big\langle W_t(\Phi(g))f, f'\Big\rangle = \Big\langle\Big(\int_0^t \Phi(g)d\mu^Q\Big)f, f'\Big\rangle = \Big\langle\int e^{\int g}d\mu^Q f, f'\Big\rangle.$$

Lemma 2. *For $\Phi(g) = e^{\int g} \in L^{bv}(\Omega_{[0,t]})$, $f \in L^2$, the bilinear operator W_t*

$$W_t : L^2(\mathbb{R}^N) \times L^{bv}(\Omega_{[0,t]}) \longrightarrow L^2(\mathbb{R}^N) \tag{5}$$

defines a bounded linear operator in $L(L^{bv}(\Omega_{[0,t]};\mathbb{C}); L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})))$.

Proof. By Lemma 1, W_t is defined and $\sup_{\|F\|\le 1}\|W_t(F)\| < C(t)$. $\|W_t\| \le 1$. W_t is uniquely extended to a bounded linear operator : $L^2(\mathbb{R}^N) \times L^{bv}(\Omega_{[0,t]}) \ni f_0 F \longrightarrow W(F)(f_0) \in L^2(\mathbb{R}^N)$ which is denoted by the same notation W_t. By Lemma 2 we get

Theorem 1. *The path integral for Schrödinger equations is expressed by the generalized measure $\mu^Q \in L(L^{bv}(\Omega_{[0,t]};\mathbb{C}); L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})))$.*

Corollary 1. *If exp* $(\int_0^\tau iV(\gamma(s))ds)$ *is continuous in* τ *uniformly on the set* $\{\gamma\}$, *it is integrable by our measure.*

Theorem 2. *For some* $\phi \in C(\Omega_{[0,t]}; L_s(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})))$

$$\mu^Q(\gamma) = \frac{\partial}{\partial \tau}\phi_\tau(\gamma),$$

where $L_s(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$ *means* $L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C}))$ *with the topology of simple convergence (= the weakest topology such that* $L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})) \ni T : L^2 \ni f \longrightarrow Tf \in L^2$ *is continuous).*

Proof. By Theorem 1 we have $\phi_\tau \in L^\infty(\Omega_{[0,t]}, L_s(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})))$. It suffices to show ϕ_τ is continuous in τ. Suppose $\phi(\tau)$ is not continuous at τ_0. Then $\lim_{\varepsilon\downarrow 0}\|\mu^Q([\tau_0-\varepsilon,\tau_0+\varepsilon])f\|$ is not zero for some $f \in L^2$. This contradicts the continuity of $T_\tau = \mu^Q([0,\tau])$ in L^2 with respect to τ.

Remark 1. *Since* G *is a vector-valued function of bounded variation and* $T_t f$ *is a vector-valued continuous function, the integral* $\int_0^t G(t-s)dT_s f$ *in* (4) *is a vector-valued Stieltjes integral. Our measure* μ^Q *corresponds to the Stieltjes measure* $dT_t = AT_t dt$, *the differential of the continuous operator-valued function* T_t.

2. Family of μ^Q-integrable functions

Associated with the restriction of a measure to $B^L = \{\vec{x} \in \mathbb{R}^N | \|\vec{x}\| \leq L\}$, we consider the Schrödinger equation with Dirichlet boundary condition: a path γ is excluded if $0 \leq \exists s < t : \gamma(s) \notin B^L$.

We shall express the generator with Dirichlet boundary condition in a form of the penalty method. Let χ^L be the characteristic function of B^L, that is, $\chi^L(\vec{x}) = 1$ for $\vec{x} \in B^L, \chi_{B_j}(\vec{x}) = 0$ for $\vec{x} \notin B^L$.

$$\frac{d}{dt}u(t) = Au(t), \quad u(0,\vec{x}) = \varphi(\vec{x}), \quad \varphi \in L^2(\mathbb{R}^N), \tag{6}$$

$$\frac{d}{dt}u^L(t) = A^L u(t), \quad u(0,\vec{x}) = \varphi(\vec{x}), \quad \varphi \in L^2(\mathbb{R}^N), \tag{7}$$

where $A = -i\triangle$, $A^L = A - \lim_{\lambda\uparrow\infty}\lambda\cdot\chi^L = A + \log\chi^L$.

Lemma 3.

$$(I - A^L)f(\vec{x}) = \begin{cases} (I-A)f(\vec{x}), & \textit{for } \|\vec{x}\| \leq L, \\ \infty, & \textit{for } f(\vec{x}) \neq 0, \textit{ and } \|\vec{x}\| > L, \\ 0, & \textit{for } f(\vec{x}) = 0, \textit{ and } \|\vec{x}\| > L. \end{cases} \tag{8}$$

Hence

$$(I - A^L)^{-1}g(\vec{x}) = \chi^L(\vec{x}) \cdot (I-A)^{-1}g(\vec{x}),\ g \in L^2(B^L) \subset L^2(\mathbb{R}^N). \tag{9}$$

Note that $-A^L$ is accretive:

$$\|(I-A^L)^{-1}g\| = \|\chi^L(I-A)^{-1}g\| \le \|(I-A)^{-1}g\| \le \|g\| \quad \forall g \in L^2(B^L).$$

$-A^L$ is maximal accretive since $-A$ is maximal accretive.

We use the following notation:

$$\begin{aligned}
\Omega^L_{[0,t]} &= \{\gamma \in \Omega_{[0,t]} \mid \gamma(s) \in B^L \quad \forall s \in [0,t]\},\\
\Omega^\infty_{[0,t]} &= \bigcup_{L>0} \Omega^L_{[0,t]}.\\
\Omega^U_{[0,t]} &= \{\gamma \in \Omega_{[0,t]} \mid \gamma(s) \in U \quad \forall s \in [0,t]\}.\\
\Omega^{r,C}_{[0,t]} &= \{\gamma \in \Omega_{[0,t]} \mid |\gamma(s)| > r \quad \forall s \in [0,t]\}, \quad \forall r \ge 0.
\end{aligned}$$

Definition 4. *Let $\{T^L_t\}$ be the semigroup generated by A^L. We define*

$$\begin{aligned}
&\mu^{QL}(\Omega_{[0,t]})\varphi\Big(\equiv \int_{\Omega_{[0,t]}} \varphi(\gamma(0))d\mu^{QL}(\gamma)\Big)\\
&= \mu^Q(\Omega^L_{[0,t]})\varphi\Big(\equiv \int_{\Omega^L_{[0,t]}} \varphi(\gamma(0))d\mu^{Q}(\gamma)\Big) = T^L_t\varphi.
\end{aligned}$$

Theorem 3. *The support of μ^Q is contained in $\Omega^\infty_{[0,t]}$. That is,*

$$T_t\varphi(\gamma(0)) = \int_{\Omega^\infty_{[0,t]}} \varphi(\gamma(0))d\mu^Q(\gamma), \quad \varphi \in L^2(\mathbb{R}^N). \tag{10}$$

Proof. By (9) we have

$$\lim_{L\to\infty}(I-A^L)^{-1}g(\vec{x}) = (I-A)^{-1}g(\vec{x}) \quad \forall g \in L^2(\mathbb{R}^N).$$

Thus a modified Trotter-Kato Theorem (see Lemma 4 below) implies

$$\lim_{L\to\infty} T^L_t\varphi = T_t\varphi, \quad \forall \varphi \in L^2(\mathbb{R}^N).$$

That is, for a family of open sets $\{S \subset \mathbb{R}^N\}$,

$$\lim_{S\uparrow\mathbf{R}^N}\int_{\Omega^S_{[0,t]}} \varphi(\gamma(0))d\mu^Q(\gamma) = \int_{\Omega_{[0,t]}} \varphi(\gamma(0))d\mu^Q(\gamma), \quad \forall \varphi \in L^2(\mathbb{R}^N).$$

On the other hand,

$$\int_{\Omega^\infty_{[0,t]}} \varphi(\gamma(0))d\mu^Q(\gamma) = \lim_{S\uparrow\mathbf{R}^N}\int_{\Omega^S_{[0,t]}} \varphi(\gamma(0))d\mu^Q(\gamma).$$

This means the support of the measure μ^Q is contained in the set of bounded paths in $\Omega^\infty_{[0,t]}$.

Corollary 2. *The set of unbounded paths $\Omega_{[0,t]} \setminus \Omega^{\infty}_{[0,t]}$ is of measure zero.*

We give a modified Trotter-Kato Theorem:

Lemma 4. *Let $\{X_r \mid r > 0\}$ be a decreasing family of closed subspaces of a Banach space X_0 satisfying $\overline{\bigcup_{r>0} X_r} = X_0$. Let T^r_t be a C_0-semigroup such that $T^r_t X \subset X_r$ and A^r its generator. We assume*

$$\forall f \in X_0, \ \forall \varepsilon > 0, \ \exists r > 0,$$

$$\exists f_r \in X_r : \|f - f_r\| < \varepsilon, \ \|(I - A)^{-1} f - (I - A^r)^{-1} f_r\| < \varepsilon \|(I - A)^{-1} f\|. \tag{11}$$

Then we have

$$T^r_t f \longrightarrow T_t f, \quad \forall f \in \bigcup_{r>0} X_r, \ \varepsilon \downarrow 0, \ r = r(\varepsilon) \downarrow 0.$$

Proof. See [4]

Theorem 4. *Let the dimension $N \geq 3$. Then the outer measure of the set $\Omega_{[0,t]} \setminus \Omega^{r,C}_{[0,t]}$ can be arbitrarily small:*

$$\forall \varepsilon > 0, \forall \varphi \in L^2(\mathbb{R}^N), \exists r > 0 : \|\mu^Q(\Omega^{r,C}_{[0,t]})\varphi\| > (1 - \varepsilon)\|\varphi\|.$$

Proof. By virtue of Lemma 4, it suffices to show

$$\lim_{r \to 0} (I - A^{r,C})^{-1} g(\vec{x}) = (I - A)^{-1} g(\vec{x}), \quad \forall g \in L^2(\mathbb{R}^N). \tag{12}$$

For $N \geq 4$ the proof is easy since the H^2-closure of $\bigcup_{r>0} D(A^r)$ contains $D(A^0)$. We shall prove our theorem for $N = 3$. Since $\|(I - A^{r,C})^{-1}\| \leq 1$ and since H^k is dense in L^2, we may assume $g \in H^k$.

For $f(\vec{x}) = (I - A)^{-1} g(\vec{x})$, put

$$f_r(\vec{x}) = \begin{cases} (1 - \frac{r}{\|\vec{x}\|}) f(\vec{x}), & \text{for } \|\vec{x}\| > r, \\ 0, & \text{for } \|\vec{x}\| \leq r, \end{cases}$$
$$g_r(\vec{x}) = (I - A^{r,C}) f_r(\vec{x}).$$

Since $f_r|_{\partial S^r} = 0$, $f_r \in D(A^{r,C})$. Note that $A \frac{1}{\|\vec{x}\|} = 0$ for $\vec{x} \neq 0$, we obtain

$$\begin{aligned} A^{r,C} f_r(\vec{x}) &= (1 - \frac{r}{\|\vec{x}\|}) A f(\vec{x}) - r(A \frac{1}{\|\vec{x}\|}) f(\vec{x}) - 2r(\partial \frac{1}{\|\vec{x}\|}) \partial f(\vec{x}) \\ &= (1 - \frac{r}{\|\vec{x}\|}) A f(\vec{x}) - 2r(\partial \frac{1}{\|\vec{x}\|}) \partial f(\vec{x}), \end{aligned}$$

Hence we have

$$g(\vec{x}) - g_r(\vec{x}) = (I-A)f(\vec{x}) - (I-A^{r,C})f_r(\vec{x})$$
$$= f(\vec{x}) - f_r(\vec{x}) + A^{r,C}f_r(\vec{x}) - Af(\vec{x})$$
$$= \frac{r}{\|\vec{x}\|}f(\vec{x}) - \frac{r}{\|\vec{x}\|}Af(\vec{x}) - 2r\partial\frac{1}{\|\vec{x}\|}\partial f(\vec{x}),\ \ for\ \|\vec{x}\| > r,$$

$g \in H^k,\ k \geq 2$ implies $f \in H_{k+2}$, hence$\|Af\|_\infty, \|\partial f\|_\infty < M$.

$$\|r\partial\frac{1}{\|\vec{x}\|}\partial f(\vec{x})\|_{L^2(B^R)} \leq \|r\partial\frac{1}{\|\vec{x}\|}\partial f(\vec{x})\|_{L^2(B^R\cap B^{r,C})} + \|r\partial\frac{1}{\|\vec{x}\|}\partial f(\vec{x})\|_{L^2(B^r)}$$
$$\leq \|r\partial\frac{1}{\|\vec{x}\|}\|_{L^2(B^R\cap B^{r,C})}\|\partial f(\vec{x})\|_\infty + \|r\partial\frac{1}{\|\vec{x}\|}\|_{L^2(B^r)}\|\partial f(\vec{x})\|_{L^\infty(B^r)}$$
$$< C_1 rRM + rC_2M \longrightarrow 0 \qquad as\ R\to\infty,\quad rR\to 0.$$

Hence

$$\|g(\vec{x}) - g_r(\vec{x})\| \leq \|\frac{r}{\|\vec{x}\|}\|_{L^2(B^{r,C})}(\|f(\vec{x})\|_\infty + \|Af(\vec{x})\|_\infty) + 2\|r\partial\frac{1}{\|\vec{x}\|}\partial f(\vec{x})\|_{L^2(B^R)}$$
$$\longrightarrow 0, \qquad as\ r\to 0\quad and\quad rR\to 0.$$

Thus

$$\|(I-A^{r,C})^{-1}g(\vec{x}) - (I-A)^{-1}g(\vec{x})\|$$
$$\leq \|(I-A^{r,C})^{-1}g(\vec{x}) - (I-A^{r,C})^{-1}g_r(\vec{x})\| + \|(I-A^{r,C})^{-1}g_r(\vec{x}) - (I-A)^{-1}g(\vec{x})\|$$
$$\leq \|g(\vec{x}) - g_r(\vec{x})\| + \|f_r(\vec{x}) - f(\vec{x})\|$$
$$\leq \|g(\vec{x}) - g_r(\vec{x})\| + \|\frac{r}{\|\vec{x}\|}f(\vec{x})\|_{L^2(B^{r,C})} + \|f(\vec{x})\|_{L^2(B^r)} \longrightarrow 0, \qquad as\ r\to 0.$$

Theorem 5. *Let $V(\vec{x})$ be a real continuous function. $[\Phi(V)](\gamma) = e^{i\int_0^\tau V(\gamma(s))ds}$ is μ^Q-integrable, that is, the fundamental solution in (2) is given by*

$$S_t = \int_{\Omega_{[0,t]}} e^{i\int_0^t V(\gamma(s))ds}d\mu^Q(\gamma) \in L(L^2(\mathbb{R}^N;\mathbb{C}), L^2(\mathbb{R}^N;\mathbb{C})). \tag{13}$$

Proof. Formally the solution to the inhomogeneous equation is given by

$$S_t^L f = T_t^L f + \int_0^t T_{t-s}^L V f ds = \int_{\Omega_{[0,t]}^L} e^{i\int_0^t V(\gamma(s))ds} f(\gamma(0))d\mu^Q(\gamma).$$

by Definition 4. This is easily justified since it holds for a bounded sufficiently smooth V.

If γ is bounded and measurable, the function $V(\gamma(s))$ is bounded and measurable. In this case $[\Phi(V)](\gamma) = e^{i\int_0^\tau V(\gamma(s))ds}$ is well defined for a.e. γ.

Put $F^L(\gamma) = \chi^L(\gamma)F(\gamma)$. F^L is in $L^{bv}(\Omega)$ and hence integrable. By Theorem 3, we have

$$\int_{\Omega_{[0,t]}} F(\gamma(s))d\mu^Q(\gamma) = \lim_{L\to\infty}\int_{\Omega_{[0,t]}} F^L(\gamma(s))d\mu^Q(\gamma) = \lim_{L\to\infty}\int_{\Omega_{[0,t]}} F(\gamma(s))d\mu^{QL}(\gamma)$$

exists. For the measurability of a path γ we cite the Axiom of Determinacy [5]: *Any $\mathbb{R}^N$-valued function is Lebegue-measurable.* (Another justification will be given in [8].) By Theorem 4, 5, we have

Corollary 3. *Let $N \geq 3$. If $V(\vec{x})$ is real and continuous in $\vec{x} \in \mathbb{R}^N\backslash\{0\}$ (for instance, $V(\vec{x}) = U(\vec{x}) + 1/\|\vec{x}\|^m, U \in C(\mathbb{R}^N)$) , then $[\Phi(V)](\gamma) \equiv e^{i\int V(\gamma(s))ds}$ is μ^Q-integrable.*

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