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Besov spaces on fractal sets

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Abstract. We define two Besov spaces on the boundary of a bounded domain with fractal boundary and show that two operators with respect to the Dirichlet problem is bounded from one of the two Besov spaces to the other.

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1. Introduction

Let D be a bounded domain in $\mathbf{R}^d$ $(d \geq 2)$ such that the boundary ∂D of D is a β-set for β satisfying $d-1 \leq \beta < d$. According to A. Jonsson and H. Wallin we say that a closed set F is a β-set if there exist a positive Radon measure μ on F and positive real numbers b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap F) \leq b_2 r^\beta \tag{1.2}$$

for all $z \in F$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball with center z and radius r in $\mathbf{R}^d$. Such a measure μ is called a β-measure.

We give some examples.

1. If D is a bounded Lipschitz domain in $\mathbf{R}^d$, then ∂D is a $(d-1)$-set and the surface measure is a $(d-1)$-measure.

2. If ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set and the β-dimensional Hausdorff measure restricted to ∂D is a β-measure (cf. [**Hu**]). A typical example is the Von Koch snowflake.

For such a domain the space of all α-Hölder continuous functions on ∂D $(\alpha > \beta - (d-1) \geq 0)$ is more useful than the space of all continuous functions on ∂D.

In 1991 J. Harrisson and A. Norton showed the following facts. For such a domain in $\mathbf{R}^2$, the line integral

$$\int_{\partial D} f dx + g dy$$

is not defined for continuous functions f, g on ∂D. But, if α satisfies $1 \geq \alpha > \beta - (d-1)$, one extends α-Hölder functions defined on ∂D to $\mathcal{E}(f)$ on $\mathbf{R}^2$ by using a Whitney decomposition, and can define the 'line integral' by

$$\int_D (-\frac{\partial \mathcal{E}(f)}{\partial y} + \frac{\partial \mathcal{E}(g)}{\partial x}) dxdy.$$

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Because of a similar reason we consider Besov spaces on ∂D instead of the L^p-space with respect to a β-measure on ∂D.

To do so, we shall fix a β-measure μ on ∂D and suppose $\overline{D} \subset B(0, R/2)$ with $R \geq 1$. We denote by $\mathcal{V}(G)$ a Whitney decomposition of an open set G and set $\mathcal{V} = \mathcal{V}(D) \cup \mathcal{V}(\mathbf{R}^d \setminus \overline{D})$ (cf. [S]).

In [W3] we constructed an extension operator $\mathcal{E}$ having the following properties.

Proposition A. *Assume that $\overline{D} \subset B(0, R/2)$. Then there exists a linear operator $\mathcal{E}$ from $L^p(\mu)$ to $L^p(\mathbf{R}^d)$ having the properties (i)-(v).*

(i) $\mathcal{E}(f)$ is a C^∞-function in $\mathbf{R}^d \setminus \partial D$,

(ii) $\mathcal{E}(f) = f$ on ∂D,

(iii) supp $\mathcal{E}(f) \subset B(0, 2R)$,

(iv)

$$\int |\mathcal{E}(f)|^p dy \leq c \int |f|^p d\mu,$$

where c is a constant independent f.

(v) Let $Q \in \mathcal{V}$ be a cube with common side-length l. Then , for each $y \in Q \cap B(0, R)$,

$$\left|\frac{\partial}{\partial y_i} \mathcal{E}(f)(y)\right| \leq c l^{-1-\beta} \int_{B(a,sl)} |f(z)| d\mu(z) \quad (i = 1, \cdots, d),$$

where a is a boundary point satisfying $\mathrm{dist}(\partial D, Q) = \mathrm{dist}(a, Q)$ *and c, s are constants independent of l, y and f.*

We note that $\mathrm{dist}(A, B)$ stands for the distance between a set A and a set B. Using this extension operator $\mathcal{E}$, we introduce the following function spaces $\mathcal{B}^+_{\alpha,p}$ and $\mathcal{B}^-_{\alpha,p}$. Let $p > 1$ and $p - p\alpha - d + \beta > 0$. The space $\mathcal{B}^+_{\alpha,p}$ (resp. $\mathcal{B}^-_{\alpha,p}$) is defined to be the family of all $f \in L^p(\mu)$ satisfying

$$\int_D |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} < \infty$$

$$(resp. \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} < \infty),$$

where $\delta(y)$ stands for the distance of y from ∂D.

It is easy to show the following proposition.

Proposition 1.1. *Assume that $p > 1$ and $p - p\alpha - d + \beta > 0$. Both of spaces $\mathcal{B}^+_{\alpha,p}$ and $\mathcal{B}^-_{\alpha,p}$ are Banach spaces with norms*

$$\|f\|_{\mathcal{B}^+_{\alpha,p}} := \left(\int |f|^p d\mu\right)^{1/p} + \left(\int_D |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy\right)^{1/p}$$

and

$$\|f\|_{\mathcal{B}^-_{\alpha,p}} := \left(\int |f|^p d\mu\right)^{1/p} + \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p-p\alpha-d+\beta} dy\right)^{1/p},$$

respectively.

In these Besov spaces we shall discuss the boundedness of operators K_1 and K_2, which are important to solve the Dirichlet problems for D and $\mathbf{R}^d \setminus \overline{D}$ by layer potential method.

Hereafter we assume that $p > 1$ and $0 \leq \beta - (d-1) < \alpha < 1 - (d-\beta)/p$. It is possible that our domain does not have outward units normal to D and a surface measure. But double layer potentials can be defined as follows: Define, for $f \in \mathcal{B}^-_{\alpha,p}$,

$$\Phi f(x) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy$$

for $x \in D$. We also define, for $f \in \mathcal{B}^+_{\alpha,p}$,

$$\Phi f(x) = -\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(x-y) \rangle dy$$

for $x \in \mathbf{R}^d \setminus \overline{D}$, where

$$N(x-y) = \begin{cases} \frac{1}{\omega_d (d-2)|x-y|^{d-2}} & \text{if } d \geq 3 \\ -\frac{3R}{2\pi} \log \frac{|x-y|}{3R} & \text{if } d = 2 \end{cases}$$

and ω_d stands for the surface area of the unit ball in $\mathbf{R}^d$.

Let ρ, $h > 0$. An open set G is said to satisfy condition (c) with (ρ, h), or simply to satisfy condition (c) if for every $z \in \partial G$ and every positive number $r \leq \rho$, there exists $x_1 \in G$ such that

$$B(x_1, hr) \subset G \cap B(z, r).$$

The oprators K_1 and K_2 are defined as follows: For $f \in \mathcal{B}^-_{\alpha,p}$ and $z \in \partial D$

$$K_1 f(z) = \int_{\mathbf{R}^d \setminus \overline{D}} \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy$$

if it is well-defined and $K_1 f(z) = 0$ otherwise, and, for $f \in \mathcal{B}^+_{\alpha,p}$ and $z \in \partial D$,

$$K_2 f(z) = -\int_D \langle \nabla_y \mathcal{E}(f)(y), \nabla_y N(z-y) \rangle dy$$

if it is well-defined and $K_2 f(z) = 0$ otherwise.

We have the following theorem.

Theorem. *Assume that D is a bounded domain in $\mathbf{R}^d$ such that $\mathbf{R}^d \setminus D$ is connected and ∂D is a β-set $(d-1 < \beta < d)$. Let $p > 1$ and $\beta - (d-1) < \alpha < 1 - (d-\beta)/p$.*

(i) If $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h), then K_1 is a bounded operator from $\mathcal{B}^-_{\alpha,p}$ to $\mathcal{B}^+_{\alpha,p}$.

(ii) If D satisfies condition (c) with (ρ, h), then K_2 is a bounded operator from $\mathcal{B}^+_{\alpha,p}$ to $\mathcal{B}^-_{\alpha,p}$.

2. Some lemmas

Hereafter we assume that a bounded domain D has a boundary which is a β-set $(d-1 \leq \beta < d)$. To prove our theorem, we prepare some lemmas.

We shall often use the following lemma.

Lemma B. *([W1, Lemma 2.3]) Let λ, k be real numbers. If $d-\beta > \lambda$ and $d-\lambda+k > 0$, then*

$$\int_{B(z,r)} \delta(y)^{-\lambda}|y-z|^k dy \leq cr^{d-\lambda+k}$$

for every $z \in \partial D$ and $0 < r \leq 3R$.

The following lemma is fundamental.

Lemma 2.1. *Assume that $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h). Let $b > 0$, $0 < \epsilon \leq \min\{\rho, br\}$, $z \in \partial D$ and put*

$$E_\epsilon = \{x \in \mathbf{R}^d \setminus \overline{D}; \delta(x) < \epsilon\}.$$

Then

$$c_1 \epsilon^{d-\beta} r^\beta \leq \int_{E_\epsilon \cap B(z,r)} dx \leq c_2 \epsilon^{d-\beta} r^\beta, \tag{2.1}$$

where c_1 and c_2 are constants independent of ϵ, r and z.

Proof. In [W1, Lemma 2.1] we saw that the second inequality holds without condition (c). Let us prove the first inequality. We first assume that $\epsilon \leq r/4$. We can pick up $z_1, \cdots, z_m \in \partial D$ such that

$$\partial D \cap \overline{B(z, r/4)} \subset \cup_{j=1}^m B(z_j, \epsilon) \subset \{y; \delta(y) < \epsilon\}$$

and

$$\overline{\partial D \cap B(z, r/4)} \cap B(z_j, \epsilon) \neq \emptyset.$$

By virtue of the covering lemma we can choose a subfamily $\{B(z_{n_k}, \epsilon)\}_{k=1}^l$ of $\{B(z_j, \epsilon)\}_{j=1}^m$ such that

$$\cup_{j=1}^m B(z_j, \epsilon) \subset \cup_{k=1}^l B(z_{j_k}, 5\epsilon)$$

and $\{B(z_{j_k}, \epsilon)\}_{k=1}^l$ are disjoint each other.

Note that $B(z_j, \epsilon) \cap (\mathbf{R}^d \setminus \overline{D}) \subset E_\epsilon \cap B(z, r)$. Since ∂D is a β-set and $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h), we get

$$\begin{aligned}
\epsilon^{d-\beta} r^\beta &\leq c_1 \epsilon^{d-\beta} \int_{\partial D \cap B(z, r/4)} d\mu(y) \leq c_1 \epsilon^{d-\beta} \sum_{k=1}^l \int_{\partial D \cap B(z_{j_k}, 5\epsilon)} d\mu(y) \\
&\leq c_2 \epsilon^{d-\beta} (5\epsilon)^\beta l = c_3 \epsilon^d l \leq c_4 |\cup B(y_k, h\epsilon)| \\
&\leq c_5 \int_{E_\epsilon \cap B(z,r)} dy,
\end{aligned}$$

where $B(y_k, h\epsilon)$ is a ball included in $B(z_{j_k}, \epsilon) \cap (\mathbf{R}^d \setminus \overline{D})$ and $|A|$ for a set A stands for the d-dimensional volume of A.

We next assume that $r/4 < \epsilon \leq br$. Then $4b > 1$. Using the above consideration, we get

$$\epsilon^{d-\beta} r^{\beta} = (4b)^{d-\beta} \left(\frac{\epsilon}{4b}\right)^{d-\beta} r^{\beta} \leq c_6 \int_{E_{\epsilon/4b} \cap B(z,r)} dy$$
$$\leq c_6 \int_{E_\epsilon \cap B(z,r)} dy.$$

This comletes the proof. □

Lemma 2.2. *Assume that $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h) and let $b > 1$. Then there exists a positive real number k such that, for each $x_0 \in D$ and each $r_0 \geq b\delta(x_0)$, the set*

$$\{x \in D; \delta(x) \leq \rho/(b-1),\ B(x, b\delta(x)) \cap (\mathbf{R}^d \setminus \overline{D}) \subset B(x_0, r_0)\} \tag{2.2}$$

is included in $B(x_0, kr_0)$.

Proof. Denote by F the set of (2.2) and choose $x' \in \partial D$ satisfying $|x' - x| = \delta(x)$. Then

$$B(x', (b-1)\delta(x)) \cap (\mathbf{R}^d \setminus \overline{D}) \subset B(x, b\delta(x)) \cap (\mathbf{R}^d \setminus \overline{D}) \subset B(x_0, r_0). \tag{2.3}$$

Note that $(b-1)\delta(x) \leq \rho$. Since $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c), there exists $h > 0$, independent of x, such that

$$B(y, h(b-1)\delta(x)) \subset B(x', (b-1)\delta(x)) \cap (\mathbf{R}^d \setminus \overline{D})$$

for some $y \in \mathbf{R}^d \setminus \overline{D}$. By virtue of (2.3) it must hold $h(b-1)\delta(x) \leq r_0$. Then, by (2.3),

$$|x_0 - x| \leq |x_0 - y| + |y - x| \leq r_0 + b\delta(x) < r_0\left(1 + \frac{b}{h(b-1)}\right).$$

Therefore $k = 1 + \dfrac{b}{(b-1)h}$ is a desired number. □

Lemma 2.3. *Assume that $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h). Let $\lambda > 0$, $d - \beta - \lambda > 0$, $b > 1$ and set*

$$D_0 = \{x \in D;\ \delta(x) \leq \frac{\rho}{b-1}\}. \tag{2.4}$$

Further set, for $x_0 \in D_0$ and r_0 satisfying $b\delta(x_0) \leq r_0 \leq b\rho/(b-1)$,

$$F = \{x \in D_0; B(x, b\delta(x)) \cap (\mathbf{R}^d \setminus \overline{D}) \subset B(x_0, r_0)\}.$$

Then

$$c_1 \int_F \delta(x)^{-\lambda} dx \leq r_0^{-\lambda+d} \leq c_2 \int_{B(x_0,r_0)\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{-\lambda} dy, \tag{2.5}$$

where c_1 *and* c_2 *are constants independent of* x_0, r_0.

Proof. By Lemma 2.2 we can choose $k > 0$ such that $F \subset B(x_0, kr_0)$. Then

$$\int_F \delta(x)^{-\lambda} dx \leq \int_{B(x_0,kr_0)\cap D} \delta(x)^{-\lambda} dx \leq \int_{B(x_0',(k+1)r_0)\cap D} \delta(x)^{-\lambda} dx,$$

where x_0' is a point of ∂D satisfying $\delta(x) = |x_0 - x_0'|$. Hence, by Lemma B,

$$\int_F \delta(x)^{-\lambda} dx \leq c_1 \left((k+1)r_0\right)^{-\lambda+d} = c_2 r_0^{-\lambda+d},$$

which gives the first inequality of (2.5).

We next prove the second inequality of (2.5). Put

$$E_j = \{y \in \mathbf{R}^d \setminus \overline{D}; \delta(y)^{-\lambda} > 2^j\}.$$

Then $\delta(y) < 2^{-j/\lambda}$. Noting that $r_0(1-1/b)) \leq r_0 - \delta(x_0)$, we get

$$\begin{aligned} I \equiv \int_{B(x_0,r_0)\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{-\lambda} dy &\geq \int_{B(x_0',r_0(1-1/b))\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{-\lambda} dy \\ &\geq c_3 \sum_{j=j_0}^{\infty} 2^j \int_{B(x_0',r_0(1-1/b))\cap E_j} dy, \end{aligned}$$

where j_0 is the integer satisfying $\left(2^{-1/\lambda}\right)^{j_0} \leq r_0(1-1/b) < \left(2^{-1/\lambda}\right)^{j_0-1}$. Noting that $2^{-j/\lambda} \leq r_o(1-1/b) \leq \rho$ for every $j \geq j_0$, Lemma 2.1 yields

$$I \geq c_4 \sum_{j=j_0}^{\infty} 2^j r_0^{\beta}(1-1/b)^{\beta}(2^{-j/\lambda})^{d-\beta} = c_5 \sum_{j=j_0}^{\infty} r_0^{\beta} 2^{(1-(d-\beta)/\lambda)j}.$$

Noting that $d - \beta - \lambda > 0$ and

$$2^{(1-(d-\beta)/\lambda)j_0} = \left(2^{-j_0/\lambda}\right)^{d-\beta-\lambda} \geq \left(r_0(1-1/b)\right)^{d-\beta-\lambda},$$

we get

$$I \geq c_6 r_0^{\beta} r_0^{d-\beta-\lambda} = c_6 r_0^{d-\lambda}.$$

This gives the second inequality of (2.5). □

Remark 2.1. Lemmas 2.1, 2.2 and 2.3 are also true even if one exchanges $\mathbf{R}^d \setminus \overline{D}$ for D in their assumptions and conclusions.

3. Maximal functions of Hörmander's type

In this section we introduce maximal functions of Hörmander's type. We first define two measures on D and $\mathbf{R}^d \setminus \overline{D}$.

Let $p > 1$ and α be a positive real number satisfying $\beta - (d-1) < \alpha < 1 - (d-\beta)/p$. We define measures $\nu^+_{\alpha,p}$ and $\nu^-_{\alpha,p}$ by

$$\nu^+_{\alpha,p}(E) = \int_{E\cap D} \delta(x)^{(-1+\alpha+(d-\beta)/p)q} dx$$

and

$$\nu^-_{\alpha,p}(E) = \int_{E\cap(\mathbf{R}^d\setminus D)} \delta(y)^{(-1+\alpha+(d-\beta)/p)q} dy$$

for a Borel measurable set E, where $q = p/(p-1)$.

Fix $b > 1$ and define D_0 by (2.4). Let $x \in D_0$ and $t \geq b\delta(x)$. We denote by $B_{(x,t)}$ the set $B(x,t) \cap (\mathbf{R}^d \setminus \overline{D})$.

Lemma 3.1. *Let $p > 1$ and α be a positive real number satisfying $\beta - (d-1) < \alpha < 1 - (d-\beta)/p$. If $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h) then measures $\nu^+_{\alpha,p}$ restricted to D_0 and $\nu^-_{\alpha,p}$ have the following Hörmander's conditions.*

(i) If $x \in D_0$ and $t \geq b\delta(x)$, then

$$\int_{B_{(x,t)}} d\nu^-_{\alpha,p} < \infty,$$

(ii) There are positive numbers k and K with $1 < k \leq K$ such that

$$s \leq kt,\ x,\ y \in D_0 \text{ and } B_{(x,s)} \cap B_{(y,t)} \neq \emptyset \text{ imply } B_{(x,s)} \subset B_{(y,Kt)},$$

(iii) There exists a constant c_1 such that

$$\nu^-_{\alpha,p}(B_{(x,Kt)}) \leq c_1 \nu^-_{\alpha,p}(B_{(x,t)}) \quad \text{for all } x \in D_0 \text{ and } 0 < t \leq \frac{b}{b-1}\rho,$$

(iv) There exists a constant c_2 such that

$$\nu^+_{\alpha,p}\left(\{x \in D_0; B_{(x,b\delta(x))} \subset B_{(y,t)}\}\right) \leq c_2 \nu^-_{\alpha,p}(B_{(y,t)})$$

for every $y \in D_0$ and evry t satisfying $b\delta(y) \leq t \leq \frac{b}{b-1}\rho$.

Proof. (i) Let $x \in D$ and $t \geq b\delta(x)$. Choose $x' \in \partial D$ satisfying $\delta(x) = |x - x'|$. Noting that $(\alpha - 1 + (d-\beta)/p)q + d - \beta > 0$, we get, by Lemma B,

$$\int_{B_{(x,t)}} d\nu^-_{\alpha,p} = \int_{B_{(x,t)}} \delta(y)^{(\alpha-1+(d-\beta)/p)q} dy$$
$$\leq \int_{B(x',2t)\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{(\alpha-1+(d-\beta)/p)q} dy \leq c_3(2t)^{(\alpha-1+(d-\beta)/p)q+d} < \infty.$$

(ii) Pick $y_0 \in B_{(x,s)} \cap B_{(y,s)}$. Then, for each $z \in B_{(x,s)}$,

$$|y - z| \le |y - y_0| + |y_0 - x| + |x - z| \le t + s + s \le (2k+1)t.$$

Therefore, putting $K = 2k + 1$, we have $z \in B_{(y,Kt)}$.

(iii) Let $x \in D_0$ and pick $x' \in \partial D$ satisfying $\delta(x) = |x - x'|$. Noting that $(\alpha - 1 + (d-\beta)/p)q + d - \beta > 0$, we get, by Lemma B,

$$\begin{aligned}
\int_{B_{(x,Kt)}} d\nu^-_{\alpha,p} &= \int_{B_{(x,Kt)}} \delta(y)^{(\alpha-1+(d-\beta)/p)q} dy \\
&\le \int_{B(x',2Kt)\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{(\alpha-1+(d-\beta)/p)q} dy \\
&\le c_4 (2Kt)^{(\alpha-1+(d-\beta)/p)q+d} \le c_5 t^{(\alpha-1+(d-\beta)/p)q+d}.
\end{aligned}$$

Since $t \le b\rho/(b-1)$, Lemma 2.3 leads to (iii).

(iv) The assertion (iv) follows from Lemma 2.3. □

Remark 3.1. Putting $b = 3/4$, $k = 3/2$ and $K = 4$, we use this lemma to prove our theorem.

Remark 3.2. If D satisfies condition (c) with (ρ, h), then we obtain a lemma, corresponding to Lemma 3.1, in which $\mathbf{R}^d \setminus \overline{D}$ is replaced with D.

Let $b > 1$ and $f \in L^1_{loc}(\nu^-_{\alpha,p})$. We define a maximal function $\tilde{f}$ of f by

$$\tilde{f}(x) := \sup\{\frac{1}{\nu^-_{\alpha,p}(B_{(x,r)})} \int_{B_{(x,r)}} |f| d\nu^-_{\alpha,p};\ b\delta(x) \le r \le \frac{b}{b-1}\rho\}$$

for $x \in D_0$.

By [He, Theorem 2.4] and Lemma 3.1 we have

Lemma 3.2. *Let $p > 1$ and $0 \le \beta - (d-1) < \alpha < 1 - (d-\beta)/p$, and assume that $\mathbf{R}^d \setminus \overline{D}$ satisfies condition (c) with (ρ, h). Then*

$$\int_{D_0} \tilde{f}^p d\nu^+_{\alpha,p} \le c \int |f|^p d\nu^-_{\alpha,p}$$

for every $f \in L^p(\nu^-_{\alpha,p})$.

Similarly we define another maximal function $\hat{f}$ of f by

$$\hat{f}(y) := \sup\{\frac{1}{\nu^+_{\alpha,p}(B(y,r)\cap D)} \int_{B(y,r)\cap D} |f| d\nu^+_{\alpha,p};\ b\delta(y) \le r \le \frac{b}{b-1}\rho\}$$

for $y \in (\mathbf{R}^d \setminus \overline{D}) \cap \{x; \delta(x) \le \rho/(b-1)\}$.

Similarly we can prove the following theorem.

Lemma 3.3. *Let $p > 1$ and $0 \le \beta - (d-1) < \alpha < 1 - (d-\beta)/p$, and D satisfies condition (c) with (ρ, h). Then*

$$\int_{(\mathbf{R}^d \setminus \overline{D}) \cap \{y; \delta(y) \le \rho/(b-1)\}} \hat{f}^p d\nu^-_{\alpha,p} \le c \int |f|^p d\nu^+_{\alpha,p}$$

for every $f \in L^p(\nu^+_{\alpha,p})$.

4. Proof of Theorem

In this section we prove our theorem. To do so, let us recall the construction of the extension operator $\mathcal{E}$ in [W3].

Fix a positive real number $\eta < 1/4$ and choose a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi = 1 \text{ on } Q_0, \quad \operatorname{supp} \phi \subset (1+\eta)Q_0, \quad 0 \le \phi \le 1,$$

where Q_0 is the closed cube of unit length centered at the origin and $(1+\eta)Q_0$ stands for the set $\{(1+\eta)x; x \in Q_0\}$.

Let $\mathcal{V}(D)$ and $\mathcal{V}(\mathbf{R}^d \setminus \overline{D})$ be Whitney decompotitions of D and $\mathbf{R}^d \setminus \overline{D}$, respectively. Set

$$\mathcal{V} = \{Q_j\} = \mathcal{V}(D) \cup \mathcal{V}(\mathbf{R}^d \setminus \overline{D})$$

and let $q^{(j)}$, l_j be the center of Q_j and the common length of its sides, respectively. For each j pick a point $a^{(j)} \in \partial D$ satisfying $\operatorname{dist}(a^{(j)}, Q_j) = \operatorname{dist}(\partial D, Q_j)$ and fix it. Set

$$t(x) = \sum_j \phi\left(\frac{x - q^{(j)}}{l_j}\right)$$

and

$$\phi_j^*(x) = \frac{\phi((x - q^{(j)})/l_j)}{t(x)}.$$

Let $p \ge 1$ and $f \in L^p(\mu)$. We define

$$\mathcal{E}_0(f)(x) = \sum_j \frac{1}{\mu(B(a^{(j)}, \eta l_j) \cap \partial D)} \left(\int_{B(a^{(j)}, \eta l_j) \cap \partial D} f(z) d\mu(z)\right) \phi_j^*(x)$$

if $x \in \mathbf{R}^d \setminus \partial D$ and $\mathcal{E}_0(f)(x) = f(x)$ if $x \in \partial D$. Choose a C^∞-function ϕ_0 satisfying

$$\phi_0 = 1 \text{ on } \overline{B(0,R)}, \quad \operatorname{supp} \phi_0 \subset B(0, 2R), \quad 0 \le \phi_0 \le 1$$

and define

$$\mathcal{E}(f)(x) = \mathcal{E}_0(f)(x)\phi_0(x).$$

Proof of Theorem. (i) We first show that

$$\left(\int |K_1 f(z)|^p d\mu(z)\right)^{1/p} \leq c_1 \|f\|_{\mathcal{B}^-_{\alpha,p}}. \tag{4.1}$$

Set $q = p/(p-1)$. Choosing $\epsilon > 0$ satisfying $\epsilon < \alpha$, we have, for $z \in \partial D$,

$$|K_1 f(z)| \leq c_2 \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} |z-y|^{-\beta+\epsilon p} dy\right)^{1/p}$$
$$\times \left(\int_{\mathbf{R}^d \setminus \overline{D}} \delta(y)^{-q(1-\alpha-(d-\beta)/p)} |z-y|^{q(1-d+\beta/p-\epsilon)} dy\right)^{1/q}.$$

Noting that $-q(1-\alpha-(d-\beta)/p)+d-\beta > 0$ and $-q(1-\alpha-(d-\beta)/p)+q(1-d+\beta/p-\epsilon) = q(\alpha-\epsilon) > 0$ and using Lemma B, we get

$$|K_1 f(z)| \leq c_3 \left(\int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} |z-y|^{-\beta+\epsilon p} dy\right)^{1/p}.$$

Hence

$$\begin{aligned}
&\int |K_1 f(z)|^p d\mu(z) \\
&\leq c_4 \int_{\mathbf{R}^d \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)|^p \delta(y)^{p(1-\alpha-(d-\beta)/p)} dy \int |z-y|^{-\beta+\epsilon p} d\mu(z) \\
&\leq c_5 \|f\|^p_{\mathcal{B}^-_{\alpha,p}}.
\end{aligned}$$

This shows (4.1).

We next prove that there exists $t_0 > 0$ such that

$$\left(\int_{D \cap \{\delta(x) \leq t_0 \rho\}} |\nabla \mathcal{E}(K_1 f)(x)|^p \delta(x)^{p-p\alpha-d+\beta} dx\right)^{1/p} \leq c_6 \|f\|_{\mathcal{B}^-_{\alpha,p}} \tag{4.2.}$$

for every $f \in \mathcal{B}^-_{\alpha,p}$.

To do so, let $Q \in \mathcal{V}$, $Q \subset D$ and a be a boundary point satisfying $\operatorname{dist}(D,Q) = \operatorname{dist}(a,Q)$. Put $\lambda = q(1-\alpha-(d-\beta)/p)$. Note that $d-\beta-\lambda = q(d-\beta-1+\alpha) > 0$. We set $F(y) = |\nabla \mathcal{E}(f)(y)| \delta(y)^\lambda$ and denote by x_0 and l the center and the common side length of Q, respectively. Let $x \in Q$. We write

$$\begin{aligned}
I &\equiv \left|\frac{\partial \mathcal{E}(K_1 f - \Phi f(x_0))}{\partial x_i}(x)\right| \delta(x)^\lambda \\
&\leq c_7 \delta(x)^{\lambda-1-\beta} \int_{B(a, s\delta(x))} d\mu(z) \\
&\quad \int_{B(0,2R) \setminus \overline{D}} F(y) \delta(y)^{-\lambda} |\nabla_y N(z-y) - \nabla_y N(x_0-y)| dy,
\end{aligned}$$

where $s \geq 4/3$ is a constant independent of Q and x. Further we write

$$\begin{aligned} I &\leq c_8 \delta(x)^{\lambda-1-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{|z-y|\leq\delta(x)} F(y)\delta(y)^{-\lambda} \frac{1}{|z-y|^{d-1}} dy \\ &+ c_8 \delta(x)^{\lambda-1-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{|x_0-y|\leq\delta(x)} F(y)\delta(y)^{-\lambda} \frac{1}{|x_0-y|^{d-1}} dy \\ &+ c_8 \delta(x)^{\lambda-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{|z-y|>\delta(x)} F(y)\delta(y)^{-\lambda} \frac{1}{|z-y|^{d}} dy \\ &+ c_8 \delta(x)^{\lambda-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{|x_0-y|>\delta(x)} F(y)\delta(y)^{-\lambda} \frac{1}{|x_0-y|^{d}} dy \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us estimate I_1. If $|y-z| \leq \delta(x)$, then

$$\begin{aligned} |x-y| &\leq |x-a| + |a-z| + |z-y| \\ &\leq \delta(x) + s\delta(x) + \delta(x) \leq 2^{k_0+1}\delta(x), \end{aligned}$$

where k_0 is the integer satisfying $2^{k_0-1} < 1+s \leq 2^{k_0}$. Hence, by Lemma B,

$$\begin{aligned} I_1 &\leq c_9 \delta(x)^{\lambda-1-\beta} \int_{\{|x-y|\leq 2^{k_0+1}\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F(y)\delta(y)^{-\lambda} dy \\ &\quad \int_{|z-y|\leq\delta(x)} \frac{1}{|z-y|^{d-1}} d\mu(z) \\ &\leq c_{10} \delta(x)^{\lambda-d} \int_{\{|x-y|\leq 2^{k_0+1}\delta(x)\}\cap(\mathbf{R}^d\setminus\overline{D})} F(y)\delta(y)^{-\lambda} dy \end{aligned}$$

We set $b = 4/3$ in the definition of the maximal function $\tilde{F}$. Then $\frac{b}{b-1} = 4$. Further set $t_0 = \frac{4}{5}2^{-k_0-1}$ and

$$D_1 = \{x \in D; \delta(x) \leq t_0\rho\}.$$

We note that a cube $Q \in \mathcal{V}$ with the common side length l has the following property.

$$l\sqrt{d} \leq \operatorname{dist}(Q, \partial D) \leq 4l\sqrt{d}.$$

Let $x \in Q$ and $Q \cap D_1 \neq \emptyset$ and $x_1 \in Q \cap D_1$. Then $\delta(x) \leq 5\sqrt{d}l \leq 5\delta(x_1) \leq 5t_0\rho$. Hence $2^{k_0+1}\delta(x) \leq 4\rho$ and $2^{k_0+1} \geq 2s > \frac{4}{3}$. Since, by Lemma 2.3,

$$\nu^-_{\alpha,p}(B_{(x,r)}) = \int_{\{|x-y|\leq r\}\cap(\mathbf{R}^d\setminus\overline{D})} \delta(y)^{-\lambda} dy \geq c_{11} r^{d-\lambda}$$

for every r satisfying $\frac{4}{3}\delta(x) \leq r \leq 4\rho$, we have

$$I_1 \leq c_{12}\tilde{F}(x).$$

We next estimate I_2. The inequalities

$$\delta(x_0) \geq \delta(x) - |x - x_0| \geq \frac{\sqrt{d}}{2} l \quad \text{and} \quad \delta(x) \leq 5\sqrt{d} l$$

imply $\delta(x_0) \geq \frac{\delta(x)}{10}$. Hence

$$\begin{aligned} I_2 &\leq c_{13} \delta(x)^{\lambda-1-\beta} \delta(x)^{\beta} \delta(x)^{1-d} \int_{\{|x_0-y| \leq \delta(x)\} \cap (\mathbf{R}^d \setminus \overline{D})} F(y) \delta(y)^{-\lambda} dy \\ &\leq c_{14} \delta(x)^{\lambda-d} \int_{\{|x-y| \leq 2\delta(x)\} \cap (\mathbf{R}^d \setminus \overline{D})} F(y) \delta(y)^{-\lambda} dy. \end{aligned}$$

Noting that $2\delta(x) \leq 4\rho$, we also get

$$I_2 \leq c_{15} \tilde{F}(x).$$

We next consider I_3. We write

$$\begin{aligned} I_3 &\leq c_{16} \sum_{k=1}^{m} \delta(x)^{\lambda-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{2^{k-1}\delta(x) < |z-y| \leq 2^k \delta(x)} F(y) \delta(y)^{-\lambda} \frac{1}{|z-y|^d} dy \\ &+ c_{16} \delta(x)^{\lambda-\beta} \int_{B(a,s\delta(x))} d\mu(z) \int_{|z-y| > 2^m \delta(x)} F(y) \delta(y)^{-\lambda} \frac{1}{|z-y|^d} dy \\ &\equiv I_{31} + I_{32}, \end{aligned}$$

where m is the greatest integer satisfying $2^{k_0+m}\delta(x) \leq 4\rho$.

If $1 \leq k \leq m$ and $|z-y| \leq 2^k \delta(x)$, then $|x-y| \leq 2^{k_0+k}\delta(x) \leq 4\rho$ and $2^{k_0+k} \geq 2^{k_0+1} \geq 2$. Hence, by Lemma B,

$$\begin{aligned} I_{31} &\leq c_{17} \sum_{k=1}^{m} \delta(x)^{\lambda} 2^{-(k-1)d} \delta(x)^{-d} \int_{|x-y| \leq 2^{k_0+k}\delta(x)} F(y) \delta(y)^{-\lambda} dy \\ &\leq c_{18} \sum_{k=1}^{m} (2^{-\lambda})^k \left(2^{k_0+k}\delta(x)\right)^{\lambda-d} \int_{|x-y| \leq 2^{k_0+k}\delta(x)} F(y) \delta(y)^{-\lambda} dy \\ &\leq c_{19} \left(\sum_{k=1}^{m} (2^{-\lambda})^k \right) \tilde{F}(x) \leq c_{20} \tilde{F}(x). \end{aligned}$$

We next estimate I_{32}. Since

$$\begin{aligned} I_{32} &\leq c_{21} \delta(x)^{\lambda-\beta} \left(2^m \delta(x)\right)^{-d} \delta(x)^{\beta} \int_{B(0,2R) \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)| dy \\ &= c_{21} \delta(x)^{\lambda-d} (2^m)^{-d} \int_{B(0,2R) \setminus \overline{D}} |\nabla \mathcal{E}(f)(y)| dy \end{aligned}$$

and $4\rho < 2^{k_0+m+1}\delta(x)$, we get

$$I_{32} \le c_{22}\delta(x)^\lambda \int_{B(0,2R)\setminus\overline{D}} |\nabla\mathcal{E}(f)(y)|dy.$$

Similarly we can estimate

$$I_4 \le c_{23}\left(\tilde{F}(x) + \delta(x)^\lambda \int_{B(0,2R)\setminus\overline{D}} |\nabla\mathcal{E}(f)(y)|dy\right).$$

Thus we get

$$I \le c_{24}\left(\tilde{F}(x) + \delta(x)^\lambda \int_{B(0,2R)\setminus\overline{D}} |\nabla\mathcal{E}(f)(y)|dy\right).$$

From this and Lemma 3.2 we obtain

$$\begin{aligned}
&\int_{D_1} |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \\
&= \int_{D_1} |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p\lambda}d\nu^+_{\alpha,p}(x) \\
&\le \sum_{Q\in\mathcal{V}(D),\, Q\cap D_1\neq\emptyset} \int_Q |\nabla_x\mathcal{E}(K_1f - \Phi f(x_0))(x)|^p\delta(x)^{p\lambda}d\nu^+_{\alpha,p}(x) \\
&\le c_{25}\int \tilde{F}(x)^p d\nu^+_{\alpha,p}(x) + c_{25}\int_D \delta(x)^{(p-1)\lambda}dx \left(\int_{B(0,2R)\setminus\overline{D}} |\nabla\mathcal{E}(f)(y)|dy\right)^p \\
&\le c_{26}\int F(x)^p d\nu^-_{\alpha,p}(x) + c_{26}\|f\|^p_{\mathcal{B}^-_{\alpha,p}} \left(\int_{B(0,2R)\setminus\overline{D}} \delta(y)^{-q(1-\alpha-(d-\beta)/p)}dy\right)^{p/q}.
\end{aligned}$$

Since $-q(1-\alpha-(d-\beta)/q) + d - \beta > 0$ and

$$\int F(y)^p d\nu^-_{\alpha,p}(y) = \int_{\mathbf{R}^d\setminus\overline{D}} |\nabla_x\mathcal{E}(f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx,$$

we obtain

$$\int_{D_1} |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \le c_{27}\|f\|^p_{\mathcal{B}^-_{\alpha,p}}.$$

Finally we shall show that

$$\int_{D\cap\{\delta(x)\ge t_0\rho\}} |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \le c_{28}\|f\|^p_{\mathcal{B}^-_{\alpha,p}}. \tag{4.3}$$

To do so, let k_1 be the greatest integer such that $Q\cap\{x\in D; \delta(x)\ge t_0\rho\}\neq\emptyset$ for some k_1-cube Q. Let Q be a k-cube satisfying $k\le k_1$ and put $2^{-k} = l$. Then

$$\begin{aligned}
&\int_Q |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \\
&\le c_{29}\int_Q \delta(x)^{-p(1+\beta)}\delta(x)^{p-p\alpha-d+\beta}dx \left(\int_{B(a,sl)} |K_1f(z)|d\mu(z)\right)^p \\
&\le c_{30}l^{-p\alpha}\|K_1f\|^p_p.
\end{aligned}$$

By [W1, Lemma] the number of k-cube included in D is at most $c_{31}2^{k\beta}$. Therefore we have

$$\sum_{Q\in\mathcal{V}_k(D),\, Q\cap D_1\neq\emptyset}\int_Q |\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx \leq c_{32}l^{-p\alpha-\beta}\|K_1f\|_p^p,$$

where $\mathcal{V}_k(D) = \{Q\in\mathcal{V}(D); Q \text{ is a } k\text{-cube}\}$. This and (4.1) imply

$$\begin{aligned}&\int_{D\cap\{\delta(x)\geq t_0\rho\}}|\nabla_x\mathcal{E}(K_1f)(x)|^p\delta(x)^{p-p\alpha-d+\beta}dx\\ &\leq c_{33}\sum_{k=-\infty}^{k_1}\left(2^{-k}\right)^{-p\alpha-\beta}\|K_1f\|_p^p \leq c_{34}\|f\|^p_{\mathcal{B}^-_{\alpha,p}},\end{aligned}$$

which gives (4.3).

Thus we see that K_1 is a bounded operator from $\mathcal{B}^-_{\alpha,p}$ to $\mathcal{B}^+_{\alpha,p}$.

(ii) With the aid of Remarks 2.1, 3.2 and Lemma 3.3 we can also prove by a similar method that K_2 is a bounded operator from $\mathcal{B}^+_{\alpha,p}$ to $\mathcal{B}^-_{\alpha,p}$.

□

References

[Ho] Hörmander, L., L^p estimates for (pluri-) subharmonic functions, *Math. Scand.* **20** (1967), 65-78.

[HN] Harrison, J. and Norton, A., Geometric integration on fractal curves in the plane, *Indiana Univ. Math. J.* 40 (1991), 567-594.

[Hu] Hutchinson, J. E., Fractals and selfsimilarity, *Indiana Univ. Math. J.* **30** (1981), 713-747.

[JW] Jonsson, A. and Wallin, H., A Whitney extension theorem in L_p and Besov spaces, *Ann. Inst. Fourier, Grenoble* 28, 1 (1978), 139–192.

[S] Stein, E. M., Singular integrals and differentiability properties of functions, Princeton University Press, Princeton- New Jersey, 1970.

[W1] Watanabe, H., The double layer potentials for a bounded domain with fractal boundary, *Potential Theory-ICPT94*, 463-471, Walter de Gruyter, Berlin-New York, 1996.

[W2] Watanabe, H., Layer potentials for a bounded domain with fractal boundary, *Sci. Bull. Josai Univ.* 1 (1997), 33-44.

[W3] Watanabe, H., Double layer potentials of functions in a Besov space for a bounded domain with fractal boundary, *Proceedings of the Fifth International Colloquium on Complex Analysis* (1997), 337-343.