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Chaotic bifurcation of one-parameter family

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Abstract

We give a defining equation of the boundary of the real cut of escape locus in the real moduli space for the real quadratic rational maps. And we discuss problems of non-monotone bifurcations for two cases of one-parameter families: quadratic rational maps and cubic polynomials. We present counter examples by computer experiments to the monotonicity conjecture and the antimonotonicity conjecture.

1 Introduction

In the paper [11], we obtaine a defining equation for a real cut of hyperbolic region, called escape locus, of real moduli space for the quadratic rational maps And, by using this moduli space and properties of escape locus, we develope arguments for bifurcation phenomenas. While in the paper [12], we discussed some one-parameter family with monotone or non-monotone bifurcations and presented two counter examples for conjectures described in [2] and [16].

Now combining these two papers [11] and [12], we presents some results in this paper. Therefor we give notice that this paper does not include new results.

This paper has two parts. First one is concerned with geometry of the space $\mathrm{Rat}_2(\mathbf{C})$ of the quadratic rational maps, namely its moduli space $\mathcal{M}_2(\mathbf{C})$, consisting of all holomorphic conjugacy classes of maps, which can be described as an orbifold whose underlying space is isomorphic to $\mathbf{C}^2$, and having a natural compactification, isomorphic to the projective plane $\mathbf{CP}^2$. Maps which are hyperbolic on their Julia set give rise to hyperbolic components in the moduli space. M. Rees shows in ([17]) that the hyperbolic components can be divided into four classes: type B,C,D and E. There is just one hyperbolic component of type E in $\mathcal{M}_2(\mathbf{C})$, so-called (hyperbolic) escape component, consisting of maps with totally disconnected Julia set. This component has a more complicated topology. If we work in the real space, there are just two escape components in the real moduli space $\mathcal{M}_2(\mathbf{R})$, namely two real slices. We call these loci upper escape locus and lower escape locus. For a map in the upper escape locus with two real critical points, its real dynamics is completely trivial: the compactified real line converges to the real fixed point under iterations. Milnor gives in ([8]) a defining equation of this boundary. On the other hand, a map in the lower escape locus has complicated real dynamics. We will give a defining equation of the boundary of this lower escape locus, as Theorem 1 in section 2.1.2.

The second one is concerned with some topics from the bifurcation problems for a one parameter real family of quadratic rational maps or of cubic polynomials.

System of iterated maps, viewed as real dynamical systems is considered as an important model for the chaotic behavior in certain parameterized systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes, often illustrated by the pitchforks oriented either one-way or both-ways. J. Milnor and W. Thurston ([9]) proved by using Teichmüller theory that the logistic family $\{\lambda x(1-x)\ ;\ \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure, namely the pitchforks oriented both-ways.

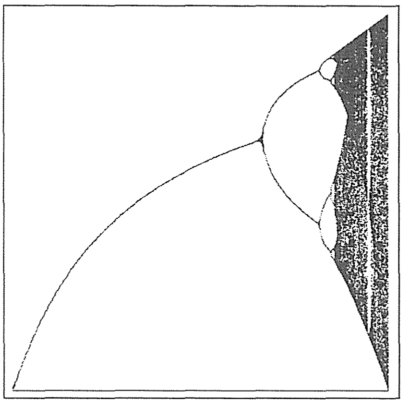

Figure 1: One-way pitchfork in the logistic family. $\{\lambda x(1-x)\}, (1 \leq \lambda \leq 4)$.

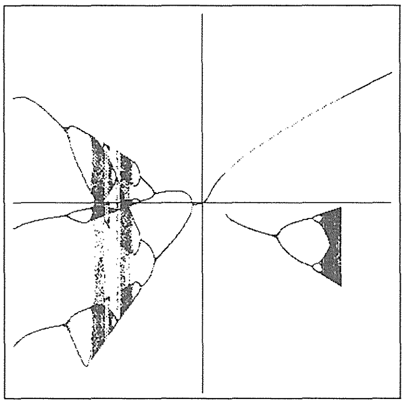

Figure 2: Both-ways pitchforks in the quadratic rational family. $\{m(\frac{1}{5} + \frac{x}{1+x^2})\}$, $(-20 \leq m \leq 20)$.

We discuss monotonicity conjecture (M) indicated in several papers, now reformulated in [16] as follows:

(M) *Let $f_m(x) = mf(x)$ be a one-parameter family of differential maps from closed interval I_m into itself which satisfies the following properties:* (1) *f_m is concave on I_m,* (2) *the set of periodic points of f_1 consists of two fixed points,* (3) *f_m has a negative schwarzian derivative. As the parameter m is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [5]).

Our method of approach to a bifurcation problem is to analyze an algebraic curve, defined by one-parameter family in the moduli space associated of a family, e.g., we examine "which hyperbolic locus does the curve lie in?" or "which dynamical curves does the curve intersect with?"

To the monotonicity conjecture, we will give a counter example using the defining equation of the lower escape locus, obtained in the section 2.1.2.

Next, we present a counter example to the antimonotonicity conjecture (A), enounced in the paper ([2]) with their heuristic argument and numerical evidence:

> (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*

Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition:(Anti). To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([14], [4]), in the moduli space of the cubic maps with the multiplier-coordinates system.

2 Moduli spaces

2.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [8]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$ (see [5]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. Since $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$, the set

$$\mathcal{S} = \{\langle f \rangle\, ;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is defined and called the symmetry locus.

For each $\mu \in \mathbf{C}$, let $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f having a fixed point with multiplier μ. Each of $\mathrm{Per}_1(\mu)$ forms a straight line as follows:

$$\mathrm{Per}_1(\mu) = \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\}$$

(Lemmas 3.4 and 3.6 in [8]).

2.1.1 Topological partition

For map $f \in \mathrm{Rat}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S^1))$ contains no (resp. one, two) critical points ([8]).

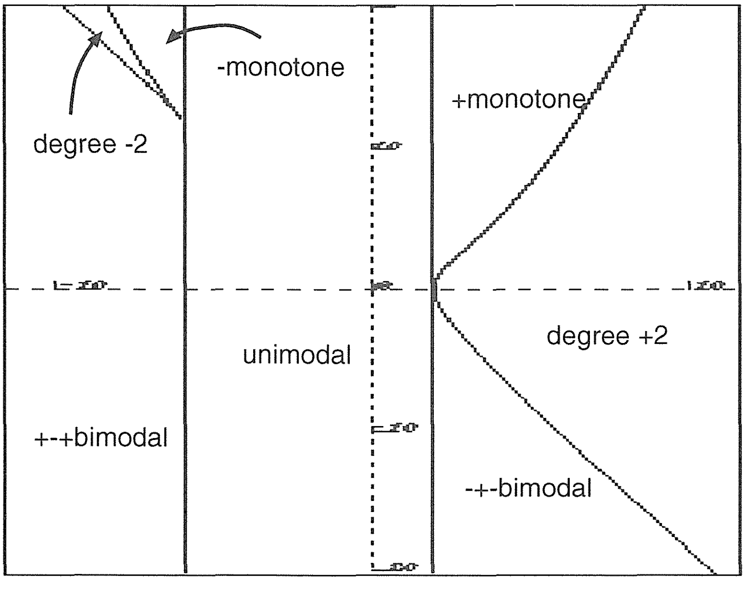

Figure 3: The topological partition of the moduli space of real quadratic rational maps. These regions are bounded by the real cut of the symmetry locus and two lines $\sigma_1 = 2, 6$.

2.1.2 Real slices of hyperbolic escape locus

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([17]) shows that the hyperbolic components can be divided into four classes as follows. The names are due to J. Milnor ([8]).

Type B: Bitransitive. Each of the two critical points belongs to the immediate basin of some attracting periodic point, where these two periodic points are distinct but belong to the same orbit. Evidently the period must be two or more.

Type C: Capture. Only one critical point belongs to the immediate basin on a periodic point, but the orbit of the other critical point eventually falls into this immediate basin. Again the period must be two or more.

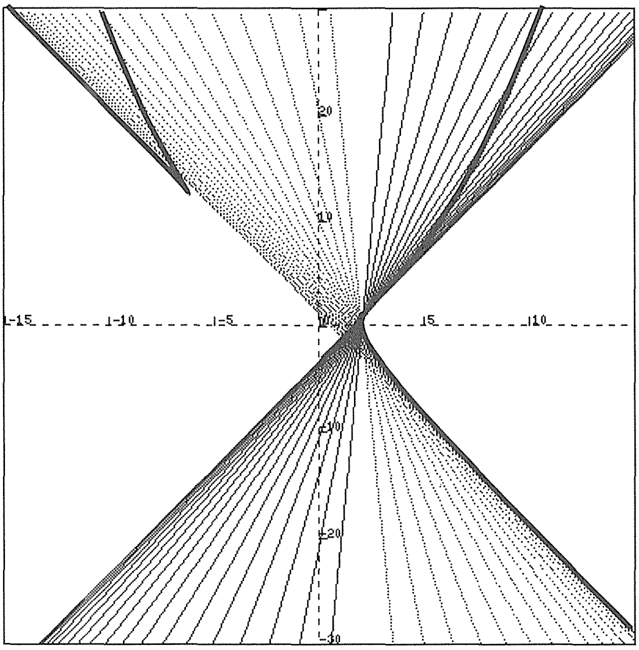

Figure 4: Lines $\mathrm{Per}_1(\mu)$: gray lines show $-1 < \mu < 0$, black lines show $0 < \mu < 1$ and thick curve shows the symmetry locus.

Type D: Disjoint attractors. The two critical points belong to the attracting basins for two disjoint attracting periodic orbits.

Type E: Escape. Both critical orbits converge to the same attracting fixed point. There is just one such hyperbolic component.

In the complex case the escape locus is connected. But the real cut of this component splits into two parts; the upper part and the lower part. The boundary curve of the upper part is given by Milnor (Caption of Figure 16 in [8]).

Now, we specify the lower boundary. This boundary curve will play a key role in our later discussions of section 3.

Theorem 1 *Escape loci on the real moduli space is the union of the following sets;*

$$\{\sigma_2 > -2\sigma_1 + 1, \quad \sigma_2 > 2\sigma_1 - 3\}, \ \{\sigma_2 < 2\sigma_1 - 3, \quad \sigma_1 < -1\},$$
$$\{\sigma_2 < \tfrac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, \quad \sigma_1 \geq -1\}.$$

Proof. Here, we use the following real two-parameter family of quadratic rational maps induced by M. Bier and T. C. Bountis ([1]) and rewritten by H. E. Nusse and J. A. Yorke ([16]);

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

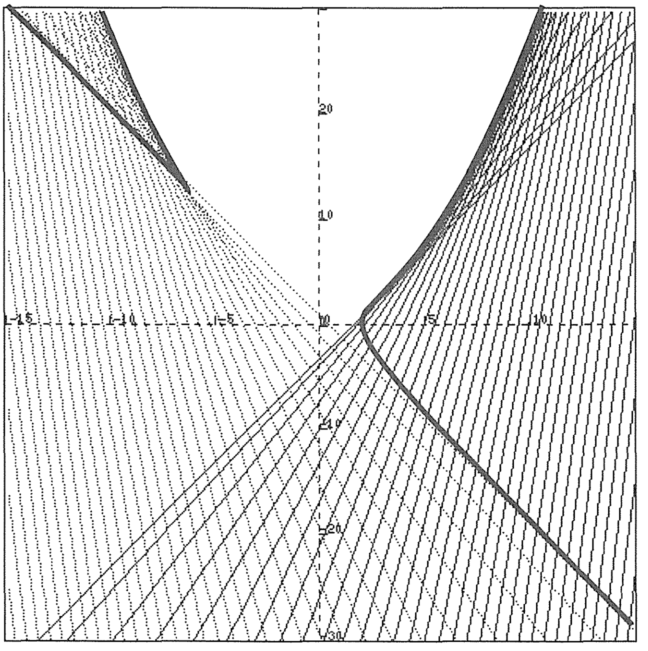

Figure 5: Lines $\mathrm{Per}_1(\mu)$: gray lines show $\mu < -1$, black lines show $\mu > 1$ and thick curve shows the symmetry locus.

This family covers the real moduli space $\{(\sigma_1, \sigma_2)\}$ expect for the degree ± 2 regions and the half line $\{(\sigma_1, \sigma_2)\,;\, \sigma_1 = 2, \sigma_2 < -1\}$ of the quadratic polynomial region (See [13]). Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$. We note that the algebraic curve defined by one-parameter family $\{f_{m,0}\}$ coincides with the boundary curve of degree ± 2 regions.

The fixed pints of $f_{m,r}$ are the three roots of equation,

$$x^3 - mrx^2 - (m-1)x - mr = 0.$$

Two critical points of this map are ± 1 and its critical values are $mr \pm \frac{m}{2}$. Therefore this family can not cover the degree ± 2 regions.

The denominator of map $f_{m,r}$ is always positive, and

$$\lim_{x \to \pm\infty} f_{m,r} = mr,$$

$$f_{m,r}(0) = mr.$$

Hence the horizontal line $y = mr$ is unique asymptotic line of this map. For the case of $m > 0$ (resp. $m < 0$), graph is $(-+-)$ (resp. $(+-+)$), and it is sufficient to consider the dynamics on the closed interval $[mr - \frac{m}{2}, mr + \frac{m}{2}]$ (resp. $[mr + \frac{m}{2}, mr - \frac{m}{2}]$).

From the graphical analysis it is clear that a map $f_{m,r}$ belongs to the escape if and only if $f_{m,r}$ satisfying one of the following condition.

1. In $\pm$ monotone- regions, $f_{m,r}$ has only one attracting fixed point.
2. In unimodal- region,
   - $f_{m,r}$ do not have real fixed point except for an attracting fixed point with multiplier $-1 < \mu < 0$, or
   - $f_{m,r}$ has an attracting fixed point with multiplier $0 < \mu < 1$ and two repelling fixed points, and its two minimal intervals containing each critical orbits has intersection.
3. In $(-+-)$-bimodal- region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points, and its two minimal intervals containing each critical orbits has non-trivial intersection.
4. In $(+-+)$-bimodal- region, $f_{m,r}$ has an attracting fixed point and two repelling fixed points.

In the parameter space $\{(r,m)\}$, after specifying "escape"- regions in each cases, we map these regions to the real moduli space by using transformation formula:

$$\begin{cases} ST1: 4m^2r^2 - m^2 + (\sigma_1+2)m - 4 = 0, \\ ST2: -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases}$$

For example, the escape region corresponding to the above condition 3 ($(-+-)$-bimodal case) is given as the condition $f^2_{m,r}(-1) < f^3_{m,r}(-1)$, translated into an inequality $N_{m,r}/D_{m,r} > 0$:

$$\begin{aligned} N_{m,r} =\ & m^2(-2mr+m-2)^3(4m^3r^3 - 4(m+1)m^2r^2 + (m^2+6m+4)mr - 2m^2 - 2m - 4) \\ D_{m,r} =\ & (m^2(2r-1)^2+4)(16m^6r^6 - 32m^6r^5 + 8(3m^2+4m+6)m^4r^4 \\ & -8(m^2+6m+8)m^4r^3 + (m^4+24m^3+48m^2+32m+48)m^2r^2 \\ & -4(m^3+6m^2+4m+8)m^2r + 5m^4 + 8m^2 + 16). \end{aligned}$$

From calculation we can see the factor $(-2mr+m-2)$ of $N_{m,r}$ and the second factor of $D_{m,r}$ are always positive under the condition 3. Therefore the condition 3 is finally reduced to an inequality:

$$4m^3r^3 - 4(m+1)m^2r^2 + (m^2+6m+4)mr - 2m^2 - 2m - 4 > 0.$$

Mapping this parameter region to the moduli space by transformation formula, we have

$$2\sigma_1^2 + (\sigma_2+7)\sigma_1 + 2\sigma_2 + 10 < 0, \quad \sigma_1 > 2.$$

For the other cases, we can get region of "escape" in the same way.

We conclude this proof by treating the region where the family $\{f_{m,r}\}$ do not cover.

For the case of degree ± 2 regions, a map belongs to the escape if and only if the map has unique attracting fixed point (See Caption of Figure 16 of [8]).

A quadratic polynomial family $\{x^2 + \sigma_2/4\}_{\sigma_2}$ coincides with the line $\{(2,\sigma_2)\}$ on the moduli space. Hence two half line $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 < -8\}$, $\{(\sigma_1,\sigma_2)\,;\,\sigma_1 = 2, \sigma_2 > 1\}$ belong to escape loci. ∎

2.2 Moduli space of cubic polynomials

Let $\mathrm{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from $\mathbf{C}$ to itself. The group $\mathrm{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbf{C})$, by conjugation, $g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbf{C})$ for $g \in \mathrm{Poly}_1(\mathbf{C})$, $p \in \mathrm{Poly}_3(\mathbf{C})$. Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbf{C})$ are holomorphically conjugate, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbf{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p. For each $p \in \mathrm{Poly}_3(\mathbf{C})$, let $z_1,\ z_2,\ z_3,\ z_4(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \leq i \leq 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$
\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}
$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that $3 - 2\sigma_1 + \sigma_2 = 0$. Now an affine structure is imposed on $\mathrm{M}_3(\mathbf{C})$ by this multipliers coordinate system (σ_1, σ_3). We remark that the singular part of this moduli space is given the following algebraic variety:

$$
S_3(\sigma_1, \sigma_3) = 4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0. \tag{1}
$$

A map in $\mathrm{Poly}_3(\mathbf{C})$ is always conjugate to a map of the normal form $z^3 + az + b$, and its parameters (a, b^2) is used as a coordinate system of $\mathrm{M}_3(\mathbf{C})$ which is isomorphic to $\mathbf{C}^2$ ([6]). These coordinates relate to (σ_1, σ_3) as follows:

$$
\begin{aligned}
\sigma_1 &= -3a + 6, \\
\sigma_3 &= 27b^2 + a(2a - 3)^2,
\end{aligned} \tag{2}
$$

Let $\mathrm{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. We simply define the real moduli space $\mathrm{M}_3(\mathbf{R})$ for $\mathrm{Poly}_3(\mathbf{R})$ as the real (σ_1, σ_3)-plane.

3 Bifurcations

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

3.1 Counter example to the Monotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of quadratic rational maps, to the monotonicity conjecture enounced in the paper [16].

3.1.1 Monotone and non-monotone bifurcations of quadratic rational families

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [1] and rewritten by H. E. Nusse and J. A. Yorke ([16]):

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [7]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m = 0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$. Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Theorem 2 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*

For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 \\ &= 0. \end{aligned}$$

For $r = \frac{1}{2}$ or $r = 0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper [5].

Example (Primary bubbling)
For $r = 0.58$, the one parameter family $\{f_{m,0.58}\}_m$ is non-monotone. More precisely, this bifurcation diagram is so-called primary bubbling ([5]).
Example (Antimonotone) Consider the one-parameter family defined on a suitable interval I_m,

$$F_m(x) = m\frac{x^2+ax+b}{1+x^2},$$

where constant a is the positive root of the following equation

$$49a^2 - 32 = 0,$$

and b is the unique positive root of the following equation

$$117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 + 1670655b^3 + 991301b^2 - 257125b = 0.$$

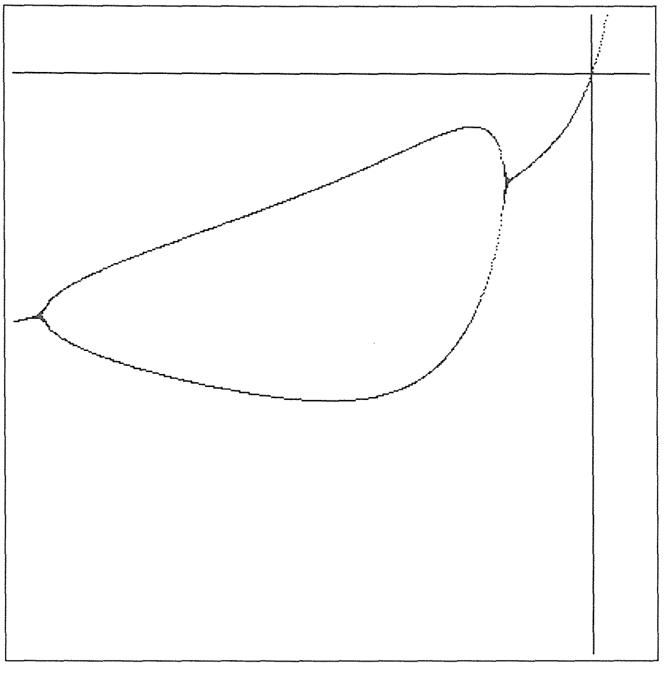

Figure 6: Primary bubbling bifurcation for $\{f_{m,0.58}(x)\}_m : -10 \leq m \leq 1,\ -2 \leq x \leq 0.2$.

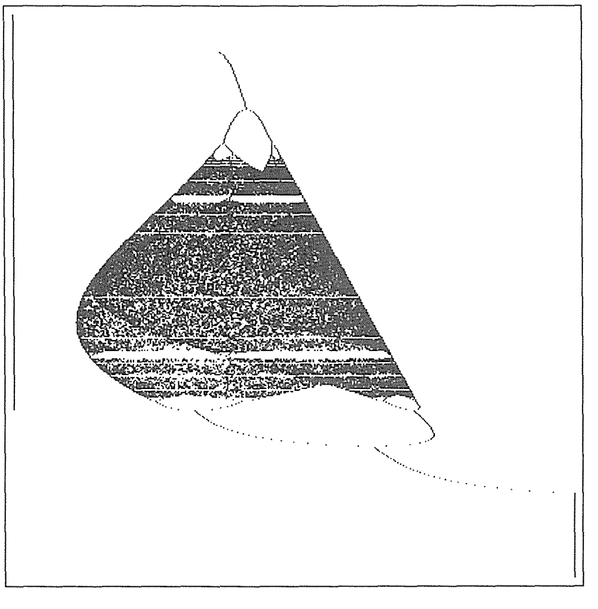

Figure 7: Bifurcation diagram of family $\{F_m(x)\}_m$: $-0.8 < x < 0.2$, $-25 < m < 5$

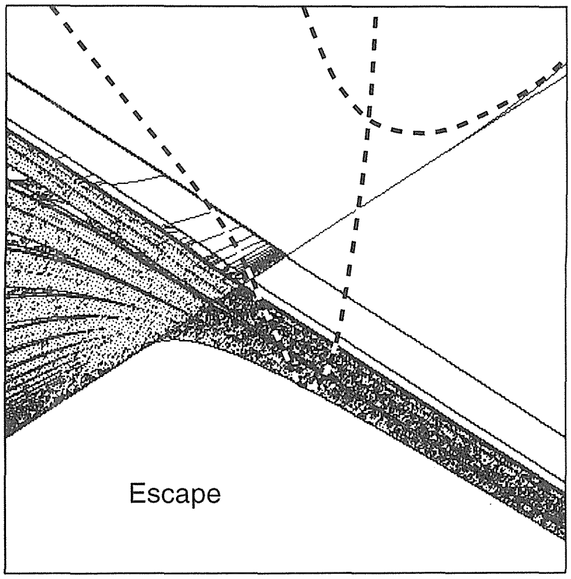

Figure 8: Real moduli space with an algebraic curve defined by $\{F_m\}_m$: $-3 < \sigma_1 < 5,\ -15 < \sigma_2 < 10$

It is clear that this family satisfies the conditions of monotonicity conjecture (M), namely, (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

In this moduli space, a defining equation of the algebraic curve defined by $\{F_m\}_m$ is given as follows;

$$\begin{aligned} S_{a,b} =&(2\sigma_1^3 + (\sigma_2 - 1)\sigma_1^2 + (-8\sigma_2 + 12)\sigma_1 - 4\sigma_2^2 + 12\sigma_2 - 36)a^6 + ((2\sigma_1^3 + (\sigma_2 + 24)\sigma_1^2 + \\ &(12\sigma_2 + 72)\sigma_1 + 36\sigma_2)b^2 + (-14\sigma_1^3 + (-6\sigma_2 - 20)\sigma_1^2 + (32\sigma_2 + 24)\sigma_1 + 16\sigma_2^2 + 24\sigma_2 + \\ &144)b + \sigma_1^4 + 4\sigma_1^3 + (-3\sigma_2 - 12)\sigma_1^2 - 12\sigma_2\sigma_1 + 36\sigma_2)a^4 + ((-10\sigma_1^3 + (-4\sigma_2 - 132)\sigma_1^2 + \\ &(-48\sigma_2 - 504)\sigma_1 - 144\sigma_2 - 432)b^3 + (2\sigma_1^4 + 46\sigma_1^3 + (4\sigma_2 + 188)\sigma_1^2 + (-16\sigma_2 - 216)\sigma_1 - \\ &240\sigma_2 - 720)b^2 + (-4\sigma_1^4 - 30\sigma_1^3 + (4\sigma_2 + 84)\sigma_1^2 + (48\sigma_2 + 152)\sigma_1 - 112\sigma_2 - 336)b + \\ &2\sigma_1^4 - 6\sigma_1^3 + (-4\sigma_2 - 12)\sigma_1^2 + (16\sigma_2 + 56)\sigma_1 - 16\sigma_2 - 48)a^2 + (\sigma_1^4 + 24\sigma_1^3 + 216\sigma_1^2 + \\ &864\sigma_1 + 1296)b^4 + (-4\sigma_1^4 - 64\sigma_1^3 - 288\sigma_1^2 + 1728)b^3 + (6\sigma_1^4 + 48\sigma_1^3 - 48\sigma_1^2 - 576\sigma_1 + \\ &864)b^2 + (-4\sigma_1^4 + 96\sigma_1^2 - 256\sigma_1 + 192)b + \sigma_1^4 - 8\sigma_1^3 + 24\sigma_1^2 - 32\sigma_1 + 16 = 0. \end{aligned}$$

We remark that this curve tangent to a boundary curve of the lower locus of escape. Then we see this family is antimonotone at this tangent point [3].

3.2 Example to the Antimonotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of cubic polynomials, to the antimonotonicity conjecture enounced in the paper [2].

3.2.1 One parameter cuvic polynomials with Monotone bifurcations

The one-parameter family $f_\lambda(x) = -x^3 + 1.2675x - \lambda$, defined in [2], is antimonotone under (Anti). It turns out that this family exactly on a half line $\sigma_1 = -3.8025$ in the moduli space.

On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of whose two critical points maps to another one (see [14], [15]). The set BC1 corresponds to the one parameter family:

$$BC1: \quad g_a(x) = -x^3 + ax + (1 + \frac{2}{3}a)\sqrt{\frac{a}{3}}.$$

We can show with computer experiments that this family is monotone (naturally not antimonotone) under (Anti).

Recently we know that J. Milnor and Ch. Tresser also treat of this problem and they said in [10] that

> *The analogue of the Antimonotonicity Conjecture for the stunted sawtooth families is certainly false, since by 5.8, it is very easy to find smooth curves along which there are only orbit creations. Thus, if the conjecture is true for the cubic family, then any complexity preserving correspondence between the stunted sawtooth and cubic parameter triangles must be very wild indeed.*

We remark that the entropy of the family $\{f_\lambda\}_\lambda$ is not monotone but one of our family $\{g_a\}_a$ is monotone.

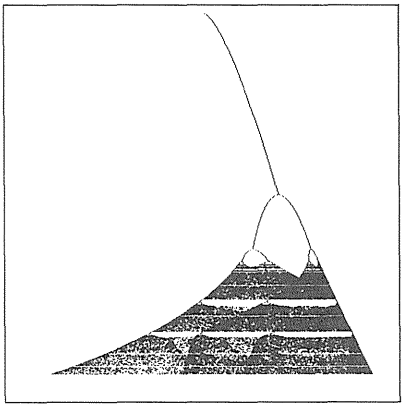

Figure 9: Bifurcation diagram of family $\{g_a(x)\}_a$: $-2 < x < 2$, $0 < a < 2$

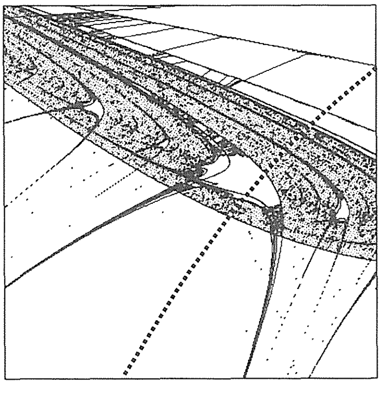

Figure 10: Real moduli space with a center curve BC1: $0 < \sigma_1 < 4$, $-60 < \sigma_3 < 0$

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