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Projective Moduli Space of the Polynomials: Cubic Case

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Abstract

The space of all cubic polynomials is a smooth complex four-manifold. On the other hand, the moduli space, consisting of all affine conjugacy classes of the maps, has a much simpler structure: the moduli space can be treated as an orbifold whose underlying space is isomorphic to $\mathbb{C}^2$. Therefore the moduli space has a natural compactification, isomorphic to the projective plane $\mathbb{CP}^2$. But for the polynomials of a higher degree $n(\geq 4)$, the moduli space parameterized by multipliers is no longer isomorphic to the space $\mathbb{C}^{n-1}$ (see [1]). It is interested to note that this parameterized space has a complicated structure with singular parts, attributed in most cases to the Fatou's index theorem of fixed points.

We define a projective moduli space, which corresponds the polynomials of degree $n(\geq 3)$ and less, and fully equip this space with multipliers-parameterization.

Introduction

The space of all cubic polynomials is a smooth complex four-manifold. On the other hand, the moduli space, consisting of all affine conjugacy classes of the maps, has a much simpler structure. We know that the moduli space can be treated as an orbifold whose underlying space is isomorphic to $\mathbb{C}^2$. Therefore the moduli space has a natural compactification, isomorphic to the projective plane $\mathbb{CP}^2$. Precisely saying, the moduli space is isomorphic to the space $\mathbb{C}^2$, under the form of an affine space with multipliers coordinate system, linked with three fixed points (see [4], [7]). The corresponding compactification is considered as a set of all affine conjugacy classes together with a two-sphere of ideal points at infinity. But what is a two-sphere of ideal points at infinity? Roughly, elements of this space can be regard as limits of cubic polynomials as they degenerate to a quadratic, linear or constant map.

For the polynomials of a higher degree $n(\geq 4)$, we show in [1] that the moduli space parameterized by multipliers is no longer isomorphic to the space $\mathbb{C}^{n-1}$. It is interested

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to note that this parameterized space has a complicated structure with singular parts, attributed in most cases to the Fatou's index theorem of fixed points.

We define a projective moduli space, which corresponds the polynomials of degree $n(\geq 3)$ and less, and fully equip this space with multipliers-parameterization

The reason why we adhere the multipliers-parameterization is that the affine structure imposed by taking these parameters as affine coordinates has special properties, for example, in analyzing some dynamical curves.

In this paper, we mainly describe several results in the case $n = 3$. One result is that this projective moduli space is nothing but the compactification of the moduli space of the cubic polynomials.

Remarkably different results in case $n = 4, 5$ are given in ([8]).

1 Definitions and preliminaries

In this section we will give definitions and preliminaries about the moduli space for the cubic polynomials, mainly obtained in ([1]).

1.1 Moduli space and its coordinate systems

Let $\mathrm{Poly}_3(\mathbb{C})$ be the space of all cubic polynomial maps of the form

$$\begin{aligned} &p : \mathbb{C} \to \mathbb{C}, \\ &p(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 \quad (a_3 \neq 0). \end{aligned}$$

The group $\mathrm{Poly}_1(\mathbb{C})$ of affine transformations acts on the space $\mathrm{Poly}_3(\mathbb{C})$ by conjugation,

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_3(\mathbb{C}) \quad \text{for} \quad g \in \mathrm{Poly}_1(\mathbb{C}),\ p \in \mathrm{Poly}_3(\mathbb{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_3(\mathbb{C})$ are *holomorphically conjugate*, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathrm{Poly}_1(\mathbb{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_3(\mathbb{C})$ under this action will be denoted by $\mathrm{M}_3(\mathbb{C})$, and called the *moduli space* of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p.

This action of the affine transformations group $\mathrm{Poly}_1(\mathbb{C})$ is not free. The group $G(2)$ of square roots of unity acts on the space $\mathrm{Poly}_3(\mathbb{C})$ by $p(z) \mapsto p(\eta z)/\eta$, where $\eta \in G(2)$ and $p \in \mathrm{Poly}_3(\mathbb{C})$.

Under the conjugacy of the action of $\mathrm{Poly}_1(\mathbb{C})$, it can be assumed that any cubic polynomial map is "monic" and "centered", i.e., $p(z) = z^3 + c_1 z + c_0$. However, the following two monic and centered polynomials belong to a same conjugacy class and transformed each other under the action of $G(2)$:

$$z^3 + c_1 z + c_0 \quad \text{and} \quad z^3 + c_1 z - c_0.$$

Hence, this $p = z^3 + c_1 z + c_0$ is uniquely determined up to the action of the group $G(2)$.

Let $\mathcal{P}_1(3)$ be the affine space of all monic and centered cubic polynomials with coordinates (c_1, c_0). Then we have a ramified covering map

$$\Phi_3 : \mathcal{P}_1(3) \to \mathrm{M}_3(\mathbb{C})$$

from $\mathcal{P}_1(3)$ onto $\mathrm{M}_3(\mathbb{C})$. Thus we can use $\mathcal{P}_1(3)$ as coordinate space for $\mathrm{M}_3(\mathbb{C})$ though there remains the ambiguity up to the group $G(2)$. This coordinate system is called *coefficients coordinate system.*

For a family of functions (which is not necessarily polynomials), many kind of coordinate systems are considered. For example, there is a technique introduced by J. Milnor for the quadratic rational maps ([5]). He use the derivatives at the fixed points as a coordinate system. Using his technique we intend to explore another coordinates in $\mathrm{M}_3(\mathbb{C})$ as follows; for each $p \in \mathrm{Poly}_3(\mathbb{C})$, let $z_1,\ z_2,\ z_3,\ z_4 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 3)$, and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned}
\sigma_{3,1} &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_{3,2} &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_{3,3} &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_{3,4} &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{aligned}$$

Note that these multipliers are invariant under the affine conjugacy.

By using Fatou index theorem (see [3]), we have the following relation among these multipliers:

$$3 - 2\sigma_{3,1} + \sigma_{3,2} = 0.$$

In view of above procedure: for any point $\langle p \rangle \in \mathrm{M}_3(\mathbb{C})$, solving the fixed points of p, calculating it's multipliers and these elementary symmetric functions, we have the natural projection to a couple $(\sigma_{3,1}, \sigma_{3,3}) \in \mathbb{C}^2$:

$$\Psi_3 : \mathrm{M}_3(\mathbb{C}) \to \mathbb{C}^2.$$

It is interesting to investigate whether this map is surjective or not. We call this problem *inverse problem*: for a given couple $(\tau_1, \tau_3) \in \mathbb{C}^2$ put $\tau_2 = -3 + \tau_3$ by Fatou index theorem, then solving a cubic polynomial with the condition that the elementary symmetric functions of its multipliers are τ_i $(i = 1, 2, 3)$. As we see later, the case $n = 3$ is nicely solved: Ψ_3 is bijective. (This fact is mentioned in [5] without any details.)

Therefore, we identified the moduli space $\mathrm{M}_3(\mathbb{C})$ with the affine space $\mathbb{C}^2$ with coordinates of two invariants $\sigma_{3,1}$ and $\sigma_{3,3}$. This coordinate system is called *multipliers coordinate system.*

In fact, for a map $z^3 + c_1 z + c_0$ in $\mathrm{Poly}_3(\mathbb{C})$ its parameters (c_1, c_0) is considered as a coordinate system of $\mathrm{M}_3(\mathbb{C})$ ([4]). Two systems, $\{(c_1, c_0)\}$ and $\{(\sigma_{3,1}, \sigma_{3,3})\}$, are linked together as follows:

$$\begin{aligned} \sigma_{3,1} &= -3c_1 + 6, \\ \sigma_{3,3} &= 27c_0^2 + c_1(2c_1 - 3)^2, \end{aligned} \tag{1}$$

or

$$\begin{aligned} c_1 &= (6 - \sigma_{3,1})/3, \\ c_0^2 &= (4\sigma_{3,1}^3 - 36\sigma_{3,1}^2 + 81\sigma_{3,1} + 27\sigma_{3,3} - 54)/729. \end{aligned} \tag{2}$$

We call these relations *transformation formula.*

1.2 Dynamical curves

On a space of polynomials, Ω, for each $\mu \in \mathbb{C}$ we shall mean by $\mathcal{P}\mathrm{er}_\Omega(m;\mu)$ the set of all holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p having a periodic point of period m and its multiplier μ. When $\Omega = \mathrm{Poly}_n(\mathbb{C})$, we write $\mathcal{P}\mathrm{er}_n(m;\mu)$ for brevity.

Now $\mathrm{M}_3(\mathbb{C})$ is an affine space with imposed structure by multipliers coordinate system. The loci $\mathcal{P}\mathrm{er}_3(i;\mu)$ $(i = 1, 2)$ have simple expressions ([1] Proposition 3): The locus $\mathcal{P}\mathrm{er}_3(1;\mu)$ is a straight line:

$$\mathcal{P}\mathrm{er}_3(1;\mu): \quad \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu. \tag{3}$$

The locus $\mathcal{P}\mathrm{er}_3(2;\mu)$ is an algebraic curve of degree three:

$$\begin{aligned} \mathcal{P}\mathrm{er}_3(2;\mu): \quad & \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 + (61\mu - 252)\sigma_1^2 \\ & -(4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 - 99\mu - 459 = 0. \end{aligned} \tag{4}$$

This curve is irreducible if and only if $\mu \neq 1$. In the case of $\mu = 1$,

$$\mathcal{P}\mathrm{er}_3(2;1) = \{(\sigma_1, \sigma_3); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0\} \bigcup \mathcal{P}\mathrm{er}_3(1;-1).$$

Let $\mathcal{S}_n$ $(\subset \mathrm{M}_n(\mathbb{C}))$ be the set consisting of all conjugacy classes $\langle p \rangle$ of polynomial maps admitting non-trivial automorphisms. We shall call $\mathcal{S}_n$ *symmetry locus*, following J. Milnor ([5]).

It is known that a cubic polynomial map $z^3 + c_1 z + c_0$ has non-trivial automorphisms if and only if $c_0 = 0$ ([1] Theorem 2). Therefore the family $\{z^3 + c_1 z\}$ forms an algebraic curve on the moduli space, and its defining equation given by ([1] Proposition 4):

$$S_3(\sigma_1, \sigma_3) = 4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0. \tag{5}$$

We have a result between lines $\mathcal{P}\mathrm{er}_3(1;\mu)$ and $\mathcal{S}_3$ ([1] Corollary 2): the envelope of $\{\mathcal{P}\mathrm{er}_3(1;\mu)\}_\mu$ coincides with the symmetry locus $\mathcal{S}_3$.

2 Extended moduli space

2.1 Projective space of multipliers coordinate space

In this section, we introduce the projective moduli space:

$$\widehat{\mathrm{M}}_3(\mathbb{C}) = \mathrm{M}_3(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_1$$

of the moduli space for the polynomials with degree three and less, and we consider a suitable parameterization for $\widehat{\mathrm{M}}_3(\mathbb{C})$. And we will show that there exists a map from $\widehat{\mathrm{M}}_3(\mathbb{C})$ to $\mathbb{CP}^2$ and its restriction on $\mathrm{M}_3(\mathbb{C})$ coincides with the isomorphism Ψ_3 from $\mathrm{M}_3(\mathbb{C})$ to $\mathbb{C}^2$. And we will show that this compactification of $\mathrm{M}_3(\mathbb{C})$ (project space $\mathbb{CP}^2$) is nothing more than the projective moduli space $\widehat{\mathrm{M}}_3(\mathbb{C})$.

Let $\{(s_0 : s_1 : s_3)\}$ be a set of homogeneous coordinates for $\mathbb{CP}^2$. Then the non-homogeneous for $\mathbb{CP}^2$ with respect to s_0 (i.e. affine space $\mathbb{C}^2$) can be treated as $\mathrm{M}_3(\mathbb{C})$. Let $L_\infty \cong \mathbb{CP}^1$ be the line at infinity. Now we will intend to investigate the correspondence between L_∞ and $\mathrm{M}_i(\mathbb{C})$ $(i = 1, 2)$.

The space $\cup_{i=1}^3 \mathrm{Poly}_i(\mathbb{C})$ is isomorphic to $\mathbb{C}^4$ by considering $\{(a_3, a_2, a_1, a_0)\,;\, a_3z^3 + a_2z^2 + a_1z + a_0\}$. Then we define the transformation formula from $\mathbb{C}^4$ to projective plane $\{(s_0 : s_1 : s_3)\} = \mathbb{CP}^2$.

Definition 1
The transformation formula between coefficients coordinates $\{(a_3, a_2, a_1, a_0)\,;\, a_3z^3 + a_2z^2 + a_1z + a_0\} \cong \mathbb{C}^4$ *and projective plane* $\{(s_0 : s_1 : s_3)\}$ *is given by*

$$\begin{aligned} PS_1 &: PS_1(s_0, s_1, s_3, a_3, a_2, a_1, a_0) = ((3a_1 - 6)a_3 - a_2^2)s_0 + s_1a_3 = 0 \\ PS_3 &: PS_3(s_0, s_1, s_3, a_3, a_2, a_1, a_0) = (-27a_0^2a_3 + (6a_0a_1 + 6a_0)a_2 - a_1^3 + 3a_1)s_0 \\ &\qquad\qquad -4s_1a_0a_2 + s_1a_1^2 - 2s_1a_1 + s_3 = 0. \end{aligned} \tag{6}$$

Let $\widehat{\Psi}_3$ be the map from $\widehat{\mathrm{M}}_3(\mathbb{C})$ to $\mathbb{CP}^2$ defined by this transformation formula. Now we investigate the map $\widehat{\Psi}_3$.

On affine 2-space, above formula coincides with transformation formula (1) by considering monic and centered normal form since $s_0 = 1$. Therefore, the map $\widehat{\Psi}_3$ restricted to the affine 2-space $\mathrm{M}_3(\mathbb{C}) \cong \mathbb{C}^2(\sigma_{3,1}, \sigma_{3,3})$ is the composition of the map Ψ_3 and $(\sigma_{3,1}, \sigma_{3,3}) \mapsto (1 : \sigma_{3,1} : \sigma_{3,3})$.

While on L_∞, substituting $s_0 = 0$ to (6) we get the following relations:

$$\begin{cases} s_1a_3 = 0 \\ -4s_1a_0a_2 + s_1a_1^2 - 2s_1a_1 + s_3 = 0. \end{cases} \tag{7}$$

If $s_1 = 0$ then s_3 must be 0 from (7); however, this contradicts with $(0 : 0 : 0) \notin \mathbb{CP}^2$. Therefore $s_1 \neq 0$ so a_3 must be 0.

Consequently the set

$$\{(0 : s_1 : s_3) \in \mathbb{CP}^1 = \hat{\mathbb{C}} \mid s_3 = s_1(4a_0a_2 - a_1^2 + 2a_1)\} \ (\cong \mathbb{C})$$

corresponds with a part of L_∞.

From this result and the fact that each quadratic polynomial $a_2z^2+a_1z+a_0$ with $a_2 \neq 0$, conjugates to monic and centered quadratic polynomial with form $z^2 + \frac{1}{4}(4a_0a_2 - a_1^2 + 2a_1)$, we have the following:

Lemma 2
For each point $(0 : s_1 : s_3)$ on the subset $L_\infty \setminus \{(0 : 0 : 1)\}$ there is a conjugacy class determined by transformation formula (6). That is, the map

$$\widehat{\Psi}_3|_{\mathrm{M}_2(\mathbb{C})} : \mathrm{M}_2(\mathbb{C}) \to L_\infty \setminus \{(0 : 0 : 1)\}$$

is surjective if we define $\langle z^2 + \alpha \rangle \mapsto (0 : 1 : \alpha)$.

In the last, we regard that the moduli space for the linear maps is embedded in the last remaining point $(0 : 0 : 1)$.

Theorem 3
The projection map $\widehat{\Psi}_3$ is defined as follows:

- *The restriction on $\mathrm{M}_3(\mathbb{C})$ is bijective, and coincides with the composition of Ψ_3 and $(\sigma_{3,1}, \sigma_{3,3}) \mapsto (1 : \sigma_{3,1} : \sigma_{3,3})$.*
- *The restriction on $\mathrm{M}_2(\mathbb{C})$ is bijective, and coincides with the map $\langle z^2 + \alpha \rangle \mapsto (0 : 1 : \alpha)$.*
- *The restriction on $\mathrm{M}_1(\mathbb{C})$ is ∞ to 1 map, and coincides with the map $\{\langle z + 1 \rangle, \langle m_0 z \rangle\} \mapsto (0 : 0 : 1)$.*

Remark 1
From the point of view of dynamical systems, the family of the linear maps can be regarded as the only one point. Hence the Theorem 3 implies $\widehat{\Psi}_3$ is bijective map from $\widehat{\mathrm{M}}_3(\mathbb{C})$ to $\mathbb{CP}^2$. Therefore via the map $\widehat{\Psi}_3$, the homogeneous coordinates decided from multipliers is as a coordinate system of projective moduli space $\widehat{\mathrm{M}}_3(\mathbb{C})$.

2.2 Projective dynamical curves

The space $\widehat{\mathrm{M}}_3(\mathbb{C})$ is isomorphic to $\mathbb{CP}^2$. Therefore we give expressions of dynamical curves in section 1.2 on projective space $\mathbb{CP}^2$. For curves on $\mathbb{CP}^2$, we use notation "over-hat" a s $\widehat{\mathcal{S}}_3$ or $\widehat{\mathcal{P}\mathrm{er}}_n(m;\mu)$.

In this section, we will show that for a given dynamical curve on the multipliers coordinate system its homogenized curve coincides with the dynamical curve on $\mathbb{CP}^2$ which is induced by $\widehat{\Psi}_3$.

Proposition 4

On the projective plane, the locus $\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu)$ *is a straight line:*

$$\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu): \quad (\mu^3-3\mu)s_0 - s_1\mu^2 + 2s_1\mu - s_3 = 0. \tag{8}$$

The locus $\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu)$ *is an algebraic curve of degree three:*

$$\begin{aligned}\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu): \quad & (-\mu^3+51\mu^2-99\mu-459)s_0^3 + (-4s_1\mu^2 - 246s_1\mu + \\ & 1134s_1 + 252s_3)s_0^2 + ((61s_1^2 - s_3s_1)\mu - 252s_1^2 - 57s_3s_1 + s_3^2)s_0 \\ & -4s_1^3\mu + 16s_1^3 + 4s_3s_1^2 = 0.\end{aligned} \tag{9}$$

This curve is irreducible if and only if $\mu \neq 1$*. In the case of* $\mu = 1$,

$$\widehat{\mathcal{P}\mathrm{er}_3}(2;1) = \{(s_0:s_1:s_3); 254s_0^2 + (-61s_1+s_3)s_0 + 4s_1^2\} \bigcup \widehat{\mathcal{P}\mathrm{er}_3}(1;-1).$$

Proof Clearly (8) and (9) coincide with (3) and (4) on the affine 2-space $\mathrm{M}_3(\mathbb{C})$, since $s_0 = 1$.

While on L_∞,

$$\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu): \quad -s_1\mu^2 + 2s_1\mu - s_3 = 0.$$

and

$$\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu): \quad 4s_1^2(-s_1\mu + 4s_1 + s_3) = 0$$

are obtained by substituting $s_0 = 0$ to (8) and (9). If $s_1 \neq 0$, a conjugacy class corresponding with $(0:s_1:s_2)$ has monic and centered representative $f(z) = z^2 + \frac{s_3}{4s_1}$. Then consider the following each conditions

$$\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu): \quad \begin{cases} f(z) = z \\ f'(z) = \mu \end{cases} \tag{10}$$

and

$$\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu): \quad \begin{cases} f(z) \neq z \\ f^2(z) = z \\ (f^2)'(z) = \mu. \end{cases} \tag{11}$$

Hence we have $s_1\mu^2 - 2s_1\mu + s_3 = 0$ from (10) and $-s_1\mu + 4s_1 + s_3$ from (11).

If $s_1 = 0$, the curve $\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu)$ does not pass through the point at infinity $(0:0:1)$ while $\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu)$ pass through the point at infinity $(0:0:1)$.

At last, we will show that $\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu)$ is irreducible curve if and only if $\mu \neq 1$. The degree of defining equation $\widehat{\mathcal{P}\mathrm{er}_3}(2;\mu)$ with respect to s_3 is two,

$$\begin{aligned} & s_3^2 + (-s_1\mu + 4s_1^2 - 57s_1 + 252)s_3 - \mu^3 + (-4s_1 + 51)\mu^2 \\ & \quad +(-4s_1^3 + 61s_1^2 - 246s_1 - 99)\mu + 16s_1^3 - 252s_1^2 + 1134s_1 - 459 = 0. \end{aligned} \tag{12}$$

And its discriminant is given by,

$$(\mu + 4s_1 - 33)^2(4\mu + s_1^2 - 16s_1 + 60).$$

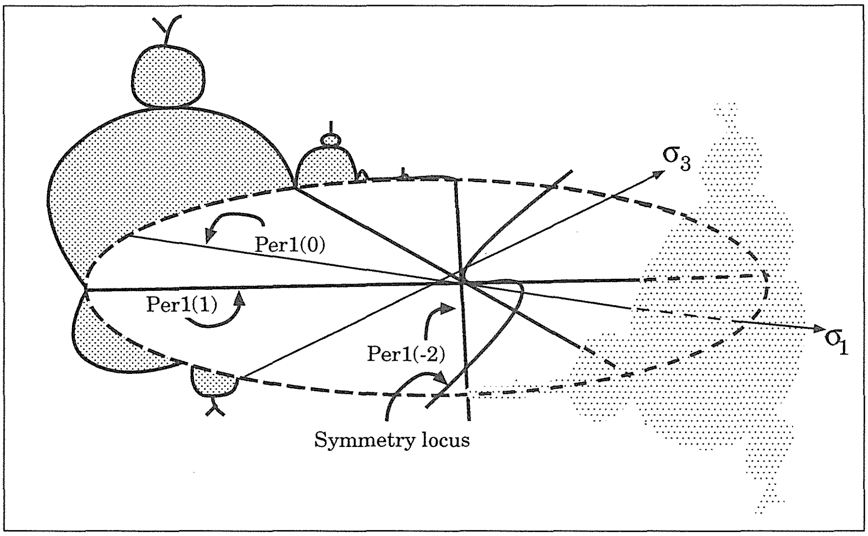

Figure 1: This figure shows the real cut of the lines $\mathcal{P}\mathrm{er}_3(1;\mu)$ stretched to the real part of the Mandelbrot set lies on the moduli space of quadratic polynomials on L_∞. Especially it seems that $\mathcal{P}\mathrm{er}_3(1;0)$ lands on the center point of the "main body", and $\mathcal{P}\mathrm{er}_3(1;-2)$ lands on the top of the "antenna".

Therefore (12) is reducible if and only if the equation $4\mu + s_1^2 - 16s_1 + 60 = 0$ has a double root, and this implies $\mu = 1$. ∎

Proposition 5

On the projective plane, the symmetry locus $\widehat{\mathcal{S}}_3$ is an algebraic curve and its defining equation $\widehat{S}_3$ is given by

$$\widehat{S}_3(s_0 : s_1 : s_3) = 4s_1^3 - 36s_0s_1^2 + 81s_0^2s_1 + 27s_0^2s_3 - 54s_0^3 = 0. \tag{13}$$

Proof It is clear that (13) coincides with (5) on $\mathrm{M}_3(\mathbb{C})$, since $s_0 = 1$. On L_∞ we have

$$\widehat{S}_3(0 : s_1 : s_3) = 4s_1^3 = 0.$$

This implies $\widehat{\mathcal{S}}_3$ passes through the point $(0 : 0 : 1)$. ∎

Remark 2

The symmetry locus for the moduli space of quadratic polynomials is empty. This fact does not contradict to Proposition 5.

Proposition 6

The envelope of $\{\widehat{\mathcal{P}\mathrm{er}_3}(1;\mu)\}_\mu$ coincides with the symmetry locus $\widehat{\mathcal{S}}_3$.

Proof Removing m from two equation $\widehat{S}_3 = 0$ and $\frac{\partial}{\partial\mu}\widehat{S}_3 = 0$, we obtain

$$(-s_1 + s_3 + 2s_0)(4s_1^3 - 36s_0s_1^2 + 81s_0^2s_1 + 27s_0^2s_3 - 54s_0^3) = 0.$$

The first factor is the set of singular points of this line family (stationary points). Hence we have the second factor as envelope. ∎

3 The case of higher degree

In this section we summarize results for the case of degree four and five. See [8], for details.

3.1 The case of polynomials of degree four

For the case of polynomials of degree four, its moduli space is no longer isomorphic to $\mathbb{C}^3$. The map Φ_4 is a three sheeted ramified covering map from $\mathcal{P}_1(4)$ onto $\mathrm{M}_4(\mathbb{C})$. While the projection Ψ_4 from $\mathrm{M}_4(\mathbb{C})$ into $\mathbb{C}^3$ has relatively complicated nature than cubic case. In fact, for a given triple $(\tau_1, \tau_2, \tau_4) \in \mathbb{C}^3$ and $\tau_3 = 4 - 3\tau_1 + 2\tau_2$ by Fatou index theorem, the inverse problem is solved as follows:

Main Results

The family of class with two multiple fixed points (hence there is not any other finite fixed point) is mapped on the point $(4, 6, 1)$, *hence on this point the map* Ψ_4 *is* ∞*-to-one. The symmetry locus* S_4 *is mapped on an algebraic curve* $S_4 \subset \mathbb{C}^3$, *and on this curve the map* Ψ_4 *is one-to-one. Any other class is mapped on* $\mathbb{C}^3 \setminus (\mathcal{E} \cup \{(4,6,1)\} \cup S_4)$ *and on this set the map* Ψ_4 *is two-to-one. The set* $\mathcal{E}$ *is a punctured curve on* $\mathbb{C}^3$ *defined by the quadratic polynomials.*

Therefore the moduli space has a natural compactification, the projective moduli space $\widehat{\mathrm{M}}_4(\mathbb{C})$, isomorphic to the projective space $\mathbb{CP}^3$.

Then by defining the transformation formula between the coefficients coordinates isomorphic to $\mathbb{C}^5$ and projective space $\mathbb{CP}^3$, we have the fact that the polynomials with degree four and less correspond with the plane at infinity $\mathbb{CP}^2$ and moreover the quadratic polynomials corresponds with $\widehat{\mathcal{E}}$ which is homogenized punctured curve of $\mathcal{E}$.

3.2 The case of polynomials of degree five

For the case of polynomials of degree five, there is a projection Ψ_5 from $\mathrm{M}_5(\mathbb{C})$ into $\mathbb{C}^4$. About the inverse problem, there are two distinctive sets : one is a one-point set corresponding to a family of infinite classes and another is a punctured curve where any class is not mapped on.

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