JOSAI MATHEMATICAL MONOGRAPHS
Vol. **2** *(2000) pp. 11 - 13*

A Remark on the Polynomial Determining Superstable Points in the Mandelbrot Set

Hiroshi SEKIGAWA*

NTT Communication Science Laboratories

Abstract

We give a detailed proof of the fact that all of the roots of the polynomial determining superstable points in the Mandelbrot set are simple. The proof is different from that in [1], which is a sketch.

1 Problem

Consider the following problem:

Problem 1
Define a sequence of polynomials $F_1(s)$, $F_2(s)$, ... as follows:

$$\begin{aligned} F_1(s) &= s, \\ F_2(s) &= s^2 + s, \\ F_3(s) &= (s^2+s)^2 + s, \\ &\vdots \\ F_n(s) &= F_{n-1}(s)^2 + s, \\ &\vdots \end{aligned}$$

Prove that all of the roots of $F_n(s) = 0$ are simple for every n.

The purpose of this short note is to give a detailed proof of the problem.

Proof We prove that the greatest common divisor of F_n and its derivative F_n' is constant.

Let $G \in \mathbb{Z}[s]$ be the primitive greatest common divisor of two integer-coefficient polynomials F_n and F_n'. By Gauss' Lemma, we can write $F_n' = GH$, where $H \in \mathbb{Z}[s]$.

*sekigawa@cslab.kecl.ntt.co.jp

We write $\overline{P}$ to signify a polynomial $P \in \mathbb{Z}[s]$ considered as a polynomial with coefficients modulo 2. Then, we have

$$\overline{F_n'} = \overline{G}\,\overline{H} = 1,$$

because $F_n' = 2F_{n-1}F_{n-1}' + 1$. Therefore, $\overline{G}$ is constant.

Since F_n is a monic polynomial, the leading coefficient of G is either 1 or -1. Therefore, G is constant. ∎

In July 1999, Professor Nishizawa of Josai University introduced the problem during a break at a workshop on computer algebra held at the Research Institute for Mathematical Sciences, Kyoto University. However, she also mentioned that there was no detailed proof for the problem.

Fortunately, the author has been able to obtain the above proof. After Professor Nishizawa was informed of this, she mentioned that there was a sketch of a proof in [1] (page 230). The two proofs are different; the proof in [1] is based on algebraic numbers, while the author's is based on integer-coefficient polynomials.

2 Background

In this section, we describe the origin of the problem. First, we define the Mandelbrot set.

Definition 1 (Mandelbrot set)
The Mandelbrot set M is defined to be the set of all complex parameter values s such that the orbit of $z = 0$ under the map

$$q_s : z \mapsto z^2 + s$$

is bounded.

We call $s \in M$ a superstable point when the point $z = 0$, which is the unique critical point of the map q_s, is a periodic point of q_s. Writing the n-fold composed map $q_s \circ \cdots \circ q_s$ as $q_s^{\circ n}$, we have the following equations:

$$\begin{aligned}
q_s(0) &= s, \\
q_s^{\circ 2}(0) &= s^2 + s, \\
q_s^{\circ 3}(0) &= (s^2 + s)^2 + s, \\
&\vdots \\
q_s^{\circ n}(0) &= (q_s^{\circ (n-1)}(0))^2 + s, \\
&\vdots
\end{aligned}$$

That is, $q_s^{\circ n}(0) = F_n(s)$. Therefore, $F_n(s) = 0$ if and only if s is a superstable point of period d that divides n.

3 Application

Let $p(m)$ be the number of superstable points of period m. We can compute $p(m)$ (page 230 in [1]). Since the degree of F_n is 2^{n-1} and all of the roots of $F_n(s) = 0$ are simple, we obtain

$$2^{n-1} = \sum_{d|n} p(d).$$

Applying the Möbius inversion formula, we have

$$p(n) = \sum_{d|n} 2^{n/d-1}\mu(d).$$

Here, $\mu(d)$ stands for the Möbius function:

$$\mu(d) = \begin{cases} 1, & \text{if } d = 1, \\ 0, & \text{if } d \text{ has a squared factor,} \\ (-1)^k, & \text{if } d \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

Acknowledgements

The author would like to thank Kiyoko Nishizawa for recommending that this note be written.

References

[1] J. Milnor, Self-Similarity and Hairiness in the Mandelbrot Set, *Computers in Geometry and Topology (M. C. Tangora, ed.), Lecture Notes in Pure and Applied Mathematics*, vol. 114, pp. 211–257, Marcel Dekker, 1989.