Josai Mathematical Monographs
Vol. **2** *(2000) pp. 15 - 34*

Spiral traveling wave solutions of some parabolic equations on annuli

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Abstract

This paper deals with spiral traveling wave solutions of some parabolic equations on annuli related to a model of the motion of screw dislocations. We prove the existence, stability and uniqueness of spiral traveling wave solutions. Next we consider a model equation for screw dislocations and study the properties of spiral solutions for the equation of interface motion which is formally derived in the singular limit of the model equation.

1 Introduction

In this paper we shall investigate a semilinear parabolic equation on a two-dimensional annulus:

$$\begin{cases} u_t = \Delta u + g(u-\theta), & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbf{R}^2 \mid a < |x| < b\}$, (r, θ) denotes the polar coordinates of $x \in \overline{\Omega}$ and g is the derivative of a multi-well potential.

Our motivation for studying problem (1) originates from crystallization processes in material sciences. Screw dislocations are observed on the surface of actual crystals such as silicon carbide, calcogen, paraffin and polyethylene ([19]). Frank [6], [3] originally proposed the following mechanism of the formation of screw dislocations: Crystals generally contain lattice defects. Once a lattice defect reaches the surface of a crystal, the defect creates a mononuclear layer (or a step) on it. Since the velocity of progress of the step is assumed to be the same at any point, the angular velocity near the corner of the defect is faster than that at the edge. Thus, the dislocation proceeds in a spiral shape.

Recently Kobayashi [10] has proposed the following reaction-diffusion equation as a model of the motion of screw dislocations:

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$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & x \in \Omega,\ t>0 \\ u_r = 0, & x \in \partial\Omega,\ t>0, \end{cases} \tag{2}$$

where the parameter $\varepsilon > 0$ is sufficiently small and $f(\cdot\,;\varepsilon)$ is the derivative of a multi-well potential for each ε. The unknown function $u(x,t)$ represents the normalized height of the crystal. Some numerical experiments imply that equation (2) has a rotating and growing solution with a spiral shape. The purpose of the present paper is to show the existence, uniqueness and stability of such a solution, which we call a *spiral traveling wave solution.* More precisely, a solution $\overline{u}(x,t)$ of (2) or (1) is called a spiral traveling wave solution with growth speed ω if it is written in the form

$$\overline{u}(x,t) = \varphi(r, \theta - \omega t) + \omega t, \quad x \in \Omega,\ t>0. \tag{3}$$

Since the reaction term is very large, equation (2) gives rise to sharp internal layers (or interfaces). As we will see later, the motion of such interfaces is driven by their curvature. To be more precise, each interface moves according to the equation

$$V = c - \kappa \tag{4}$$

in the singular limit as $\varepsilon \to 0$, where V and κ denote the normal velocity and the curvature of the interface respectively, and c is a positive constant determined by the nonlinearity f. Equation (4) also arises from the kinematic theory in excitable media as Belousov-Zhabotinskii reagent. For mathematical results in this area we refer to [9], [12] and references therein.

Our paper is organized as follows: In Section 2 we introduce basic notation and state our main results (Theorem A — on the existence — and Theorem B — on the uniqueness and the stability —). We prove Theorems A and B in Section 3. In Section 4 we present a formal derivation of the equation of interface motion corresponding to equation (2). In Section 5 we study spiral solutions with constant angular speed for the interface equation (Theorem C). In Appendix we recall monotonicity and convergence results in order-preserving dynamical systems in the presence of symmetry obtained by Ogiwara and Matano [16, Propositions B1 and B2]. These results play a crucial role in the proof of Theorems A and B.

By (3) a spiral traveling wave solution $\overline{u}$ with growth speed ω satisfies

$$\overline{u}(x, t+T_0) = \overline{u}(x,t) + 2\pi, \qquad x \in \overline{\Omega},\ t>0, \tag{5}$$

where $T_0 = 2\pi/\omega$. Solutions with property (5) have been studied for other equations such as systems of ordinary differential equations ([11], [7], [2]) and parabolic equations in the whole space $\mathbf{R}^N$ ([15]). The methods of these literatures are based on the theory of dynamical systems and are, in essence, same as that of [16]. For our problem (1), as we will see in Lemma 5, if a solution $\overline{u}$ satisfies (5) for some T_0 then it is a spiral traveling wave solution with growth speed $2\pi/T_0$.

The authors would like to express their gratitude to Professors Hiroshi Matano and Ryo Kobayashi for valuable advice and helpful comments.

2 Main results

Throughout this paper, we assume that the nonlinearity $g(v)$ satisfies the following:

(A1) g is a smooth, 2π-periodic function on $\mathbf{R}$;

(A2) g has three zeroes $0 < \zeta < 2\pi$ in the interval $[0, 2\pi]$;

(A3) $\displaystyle\int_0^{2\pi} g(v)dv > 0$.

It is known that, for any $u_0 \in C(\overline{\Omega})$, there exists a solution $u(x,t)$ of (1) with initial data $u(\cdot, 0) = u_0$ (see [13]). Here $C(\overline{\Omega})$ denotes Banach space of continuous functions on $\overline{\Omega}$ endowed with the norm $\|u_0\|_{C(\overline{\Omega})} = \sup\{|u_0(x)| \mid x \in \overline{\Omega}\}$. For u_1, $u_2 \in C(\overline{\Omega})$ we write

$$\begin{array}{lll} u_1 \le u_2 & \text{if} & u_1(x) \le u_2(x), \quad x \in \overline{\Omega}, \\ u_1 < u_2 & \text{if} & u_1(x) \le u_2(x) \text{ and } u_1(x) \not\equiv u_2(x), \quad x \in \overline{\Omega}, \\ u_1 \ll u_2 & \text{if} & u_1(x) < u_2(x), \quad x \in \overline{\Omega}. \end{array} \tag{6}$$

Let $\{\Phi_t\}_{t \in [0,\infty)}$ be the local semiflow on $C(\overline{\Omega})$ generated by (1). In other words, the map Φ_t on $C(\overline{\Omega})$ is defined by

$$\Phi_t(u_0) = u(\cdot, t) \qquad \text{for each } t \in [0, \infty),$$

where $u(x,t)$ is a solution of (1) with initial data $u(\cdot, 0) = u_0$. The strong maximum principle ([17]) shows that Φ_t is strongly order-preserving, that is, $u_1 < u_2$ implies $\Phi_t(u_1) \ll \Phi_t(u_2)$ for each $t > 0$. Further the standard parabolic estimate ([13]) shows that Φ_t is a compact map on $C(\overline{\Omega})$ for each $t > 0$.

Definition 1
A solution $\overline{u}(x,t)$ of (1) is called a spiral traveling wave solution if it is written in the form

$$\overline{u}(x,t) = \varphi(r, \theta - \omega t) + \omega t, \quad x \in \Omega, \, t > 0$$

for some function $\varphi(r, \xi)$ and some constant ω. We call the constant ω the growth speed of the spiral traveling wave solution $\overline{u}$.

Remark 1
Clearly, if $\overline{u}(x,t) = \overline{u}(r, \theta, t)$ is a spiral traveling wave solution of (1), then $\overline{u}(x, t + \tau)$ is also a spiral traveling wave solution for any constant τ. Further, $\overline{u}(r, \theta - \alpha, t) + \alpha$ is also a spiral traveling wave solution for any constant α.

It is easily seen that if $\varphi(r,\theta-\omega t)+\omega t$ is a spiral traveling wave solution of (1) then $\varphi(r,\xi)$ satisfies

$$-\omega\varphi_\xi+\omega=\Delta\varphi+g(\varphi-\xi). \tag{7}$$

Lemma 2
If a spiral traveling wave solution for (1) exists, then its growth speed is positive.

Proof Let $\varphi(r,\theta-\omega t)+\omega t$ be a spiral traveling wave solution. Then (7) is fulfilled. Multiplying both sides of (7) by $\varphi_\xi-1$ and integrating over Ω, we have

$$\begin{aligned}-\omega\int_\Omega(\varphi_\xi-1)^2\,dx &= \int_\Omega\{\Delta\varphi\cdot(\varphi_\xi-1)+g(\varphi-\xi)\cdot(\varphi_\xi-1)\}dx\\ &= -\frac{b^2-a^2}{2}\int_0^{2\pi}g(v)\,dv,\end{aligned}$$

and hence

$$\omega=\frac{(b^2-a^2)\displaystyle\int_0^{2\pi}g(v)\,dv}{2\displaystyle\int_\Omega(\varphi_\xi-1)^2\,dx}.$$

This proves the lemma. ∎

Definition 3
A spiral traveling wave solution $\overline{u}$ of (1) is called stable if for any $\varepsilon>0$ there exists some $\delta>0$ such that

$$\|u(\cdot,t)-\overline{u}(\cdot,t)\|_{C(\overline{\Omega})}<\varepsilon,\qquad t>0$$

holds for any solution u of (1) satisfying $\|u(\cdot,0)-\overline{u}(\cdot,0)\|_{C(\overline{\Omega})}<\delta$.

Concerning the existence, stability and uniqueness of spiral traveling wave solutions, we obtain the following:

Theorem A
For any $b>a>0$, (1) possesses a spiral traveling wave solution.

Theorem B

(i) *A spiral traveling wave solution $\overline{u}$ of (1) is stable and is monotone increasing in t, that is, $\overline{u}_t(x,t)>0$ for all $x\in\Omega$, $t>0$. Further it is unique up to translation to the t-direction, namely, if u is a spiral traveling wave solution of (1) then there exists some $\tau_0\in\mathbf{R}$ such that $u(\cdot,t)=\overline{u}(\cdot,t+\tau_0)$ for $t>0$.*

(ii) *For any solution u of (1), there exists some τ_0 such that*

$$\lim_{t\to\infty}\|u(\cdot,t)-\overline{u}(\cdot,t+\tau_0)\|_{C(\overline{\Omega})}=0. \tag{8}$$

Remark 2

From Theorem B, we see that a spiral traveling wave solution $\overline{u}$ of (1) is stable with asymptotic phase, namely, it is stable and, for any solution u of (1) with initial data sufficiently close to $\overline{u}$, there exists some τ_0 such that (8) holds.

3 Proof of Theorems A and B

In this section, we prove Theorems A and B. In what follows $x \in \overline{\Omega}$ will be often identified with (r, θ), the polar coordinates of x.

Lemma 4

Let $v(x,t)$ be a solution of (1) with initial data $v(\cdot, 0) \equiv 0$. Then there exists some constant $M > 0$ such that

$$\max\{v(x,t) \mid x \in \overline{\Omega}\} - \min\{v(x,t) \mid x \in \overline{\Omega}\} < M$$

for all $t > 0$.

Proof Differentiating (1) by θ, we see that the function $w(x,t) = v_\theta(x,t) - 1$ satisfies

$$\begin{cases} w_t = \Delta w + g'(v - \theta)w, & x \in \Omega,\ t > 0, \\ w_r = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{9}$$

Since $w(\cdot, 0) = -1 < 0$, from the strong maximum principle it follows that $w(\cdot, t) < 0$, namely

$$v_\theta(\cdot, t) < 1, \qquad t > 0.$$

Hence, using the fact that $v(r, 0, t) = v(r, 2\pi, t)$, we have

$$\theta - 2\pi < v(r,\theta,t) - v(r,0,t) < \theta, \qquad a \le r \le b,\ 0 \le \theta \le 2\pi,\ t > 0.$$

Thus

$$v(r,\theta,t) - v(a,\theta,t) - 2\pi < v(r,0,t) - v(a,0,t) < v(r,\theta,t) - v(a,\theta,t) + 2\pi \tag{10}$$

holds for $a \le r \le b$, $0 \le \theta \le 2\pi$, $t > 0$. Now fix $t_0 > 0$ arbitrarily and take a small constant $\delta > 0$ such that $(v_\theta - \delta v_t)(\cdot, t_0) < 1$ and $(v_\theta + \delta v_t)(\cdot, t_0) < 1$. Since $v_\theta \pm \delta v_t - 1$ are also solutions of (9), in the same way as above we get $(v_\theta \pm \delta v_t)(\cdot, t) < 1$ for $t > t_0$. This implies, for $t > t_0$,

$$-\frac{1 - v_\theta}{\delta} < v_t < \frac{1 - v_\theta}{\delta}. \tag{11}$$

Multiplying each side of (11) by $r \in (a, b)$ and integrating by θ from 0 to 2π, we have

$$-\frac{2\pi b}{\delta} < \int_0^{2\pi} r v_t \, d\theta < \frac{2\pi b}{\delta}.$$

Since v satisfies (1) for $t > t_0$, integration by parts yields

$$-2\pi b\,C < \int_0^{2\pi} (rv_r)_r\,d\theta < 2\pi b\,C$$

with $C = (1/\delta) + ||f||_{C(\overline{\Omega})}$. Integrating each side by r, dividing by r and integrating again, we get

$$-\frac{2\pi b(b-a)^2}{a}C < \int_0^{2\pi} \{v(r,\theta,t) - v(a,\theta,t)\}\,d\theta < \frac{2\pi b(b-a)^2}{a}C.$$

These inequalities and (10) yield

$$-\frac{b(b-a)^2}{a}C - 2\pi < v(r,0,t) - v(a,0,t) < \frac{b(b-a)^2}{a}C + 2\pi.$$

Therefore, again by (10), we obtain

$$-\frac{b(b-a)^2}{a}C - 4\pi < v(r,\theta,t) - v(a,0,t) < \frac{b(b-a)^2}{a}C + 4\pi.$$

Combining these inequalities and the fact that the set $\{v(\cdot,t) \mid 0 \le t \le t_0\}$ is a compact subset of $C(\overline{\Omega})$, we obtain the conclusion. ∎

Lemma 5
Let $\varphi(x) \in C(\overline{\Omega})$ satisfy $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for some $T_0 > 0$. Then $\varphi(r, \theta - \omega t) + \omega t$ is a solution of (1), where $\omega = 2\pi/T_0$.

We postpone the proof of Lemma 5 until the end of this section.

Proof of Theorem A Denote by $v(x,t)$ a solution of (1) with initial data $v(\cdot,0) \equiv 0$, in other words $v(\cdot,t) = \Phi_t(0)$. First we show that the orbit $\{v(\cdot,t) \mid t \ge 0\}$ is not bounded in $C(\overline{\Omega})$. Assuming that $\{v(\cdot,t) \mid t \ge 0\}$ is bounded in $C(\overline{\Omega})$, we will lead a contradiction. In this case, since a map Φ_t on $C(\overline{\Omega})$ is compact for each $t > 0$, the omega-limit set of 0 defined by

$$W(0) = \bigcap_{t>0} \overline{\{v(\cdot,s) \mid s > t\}} \subset C(\overline{\Omega})$$

is not empty. As is well-known, $W(0)$ is compact and Φ_t-invariant for each $t > 0$, namely $\Phi_t W(0) = W(0)$ (see for example [8]). Put

$$\alpha_0 = \inf\{\alpha > 0 \mid w_1 \le g_\alpha w_2 \text{ for any } w_1, w_2 \in W(0)\},$$

where $g_\alpha w(x) = g_\alpha w(r,\theta) = w(r, \theta - \alpha) + \alpha$ for $w(x) \in C(\overline{\Omega})$. Note that the map g_α on $C(\overline{\Omega})$ is commutative with Φ_t, namely, $g_\alpha \circ \Phi_t = \Phi_t \circ g_\alpha$.

Clearly $w_1 \le g_{\alpha_0} w_2$ holds for any $w_1, w_2 \in W(0)$. We show that $\alpha_0 = 0$. Assume that $\alpha_0 > 0$. If $w_1 < g_{\alpha_0} w_2$ for any $w_1, w_2 \in W(0)$ then $w_1 \ll g_{\alpha_0} w_2$ for any $w_1, w_2 \in W(0)$, since $W(0)$ is Φ_t-invariant and since Φ_t is strong order-preserving for any $t > 0$. In this case, compactness of $W(0)$ implies that if we choose $\delta > 0$ sufficiently small then $w_1 < g_{\alpha_0 - \delta} w_2$

for any $w_1, w_2 \in W(0)$, which contradicts the definition of α_0. Thus there exist some two elements $w_1, w_2 \in W(0)$ such that $w_1 = g_{\alpha_0} w_2$. Then it holds that

$$w \le w_1 \quad \text{for all } w \in W(0) \qquad \text{and} \qquad w_2 < w_1. \tag{12}$$

Since the latter inequality implies $\Phi_t(w_2) \ll \Phi_t(w_1)$ for any $t > 0$, by the definition of $W(0)$ there exist large t_1, $t_2 > 0$ satisfying

$$\Phi_{t_1}(0) \ll \Phi_{t_2}(0).$$

Therefore, if we choose $\varepsilon > 0$ sufficiently small then

$$\Phi_{t_1}(0) \ll g_{-\varepsilon}\Phi_{t_2}(0),$$

and hence

$$\Phi_{t_1+s}(0) \ll g_{-\varepsilon}\Phi_{t_2+s}(0)$$

for $s > 0$. Take a sequence $\{s_j\}_j$ such that $\Phi_{t_1+s_j}(0) \to w_1$ as $j \to \infty$. Replacing $\{\Phi_{t_2+s_j}(0)\}_j$ by its subsequence if necessary, we see that $\{\Phi_{t_2+s_j}(0)\}_j$ also converges to some $w_3 \in W(0)$. Then $w_1 \le g_{-\varepsilon} w_3$ holds. This and the former statement of (12) imply $w_3 \le g_{-\varepsilon} w_3$ and we are lead to a contradiction. Thus we obtain $\alpha_0 = 0$, from which for any w_1, $w_2 \in W(0)$ it follows that $w_1 \le w_2$ and $w_1 \ge w_2$, that is, $w_1 = w_2$. Hence $W(0)$ is a singleton. As is easily seen, if an omega-limit set is a singleton, then it consists of some equilibrium solution. This means that (1) possesses a spiral traveling wave solution with growth speed 0, which contradicts Lemma 2.

Thus we see that the orbit $\{v(\cdot,t) | t \ge 0\}$ is not bounded. Hence there exists some sequence $\{t_j\}_j$ such that $\|v(\cdot,t_j)\|_{C(\overline{\Omega})} \to \infty$ as $j \to \infty$. We discuss only the case where

$$\max\{v(x,t_j) \mid x \in \overline{\Omega}\} \to \infty, \qquad j \to \infty \tag{13}$$

and prove the existence of a spiral traveling wave solution with positive speed. The case where $\min\{v(x,t_j) \mid x \in \overline{\Omega}\} \to -\infty$ can be treated similarly. In the latter case there exists a spiral traveling wave solution with negative growth speed, which contradicts Lemma 2.

We show that there exists some $T_0 > 0$ such that $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ for some function $\varphi(x) \in C(\overline{\Omega})$. Then, by Lemma 5, we see that (1) possesses a spiral traveling wave solution with growth speed $2\pi/T_0$. As in Lemma 4, there exists some constant $M > 0$ such that

$$\max\{v(x,t) \mid x \in \overline{\Omega}\} - \min\{v(x,t) \mid x \in \overline{\Omega}\} < M, \quad t > 0. \tag{14}$$

We take $n(j) \in \mathbf{N}$ so that the function v_j defined by $v_j(x) = v(x,t_j) - 2\pi n(j)$ satisfies

$$v_j(x) \in [0, M + 2\pi], \quad x \in \overline{\Omega}.$$

Fix $s > 0$ arbitrarily. Then, replacing $\{\Phi_s(v_j)\}_j$ by its subsequence, we see that $\{\Phi_s(v_j)\}_j$ converges to some $\varphi \in C(\overline{\Omega})$.

Note that (13) and (14) imply $2\pi < v(\cdot, T)$ for some $T > 0$. Therefore $\Phi_{s+t}(2\pi) < \Phi_{s+t+T}(0)$ holds for all $t > 0$. Putting $t = t_j$ we have $\Phi_{s+t_j}(0) + 2\pi < \Phi_{s+t_j+T}(0)$ and hence $\Phi_{s+t_j}(0) - 2\pi n(j) + 2\pi < \Phi_{s+t_j+T}(0) - 2\pi n(j) = \Phi_T(\Phi_{s+t_j}(0) - 2\pi n(j))$, since $\Phi_t(u_0) + 2\pi m = \Phi_t(u_0 + 2\pi m)$ holds for any $t > 0$, $m \in \mathrm{N}$ and $u_0 \in C(\overline{\Omega})$. Letting $j \to \infty$, we get $\varphi + 2\pi \leq \Phi_T(\varphi)$. Now set

$$T_0 = \inf\{t \geq 0 \mid \varphi + 2\pi \leq \Phi_t(\varphi)\}.$$

Clearly $0 < T_0 \leq T$ and $\varphi + 2\pi \leq \Phi_{T_0}(\varphi)$. Suppose that $\varphi + 2\pi < \Phi_{T_0}(\varphi)$. Then, for any $\delta > 0$, $\Phi_\delta(\varphi + 2\pi) = \Phi_\delta(\varphi) + 2\pi \ll \Phi_{T_0+\delta}(\varphi)$. From this, for a sufficiently large j_0, it follows that

$$\Phi_{\delta+s}(v_{j_0}) + 2\pi \ll \Phi_{T_0+\delta+s}(v_{j_0}).$$

Therefore, there exists some $\varepsilon \in (0, T_0)$ such that

$$\Phi_{\delta+s}(v_{j_0}) + 2\pi \ll \Phi_{T_0-\varepsilon+\delta+s}(v_{j_0}),$$

and hence

$$\Phi_{\delta+s+t}(v_{j_0}) + 2\pi \ll \Phi_{T_0-\varepsilon+\delta+s+t}(v_{j_0}), \quad t > 0.$$

Adding $2\pi n(j_0) - 2\pi n(j)$ to both sides and putting $t = t_j - t_{j_0} - \delta$, we get

$$\Phi_s(v_j) + 2\pi \ll \Phi_{T_0-\varepsilon}(\Phi_s(v_j)).$$

Hence letting $j \to \infty$ implies

$$\varphi + 2\pi \leq \Phi_{T_0-\varepsilon}(\varphi),$$

which contradicts the definition of T_0. Therefore $\varphi + 2\pi = \Phi_{T_0}(\varphi)$ holds and the proof is completed. ∎

Lemma 6
Let $u_1, u_2 \in C(\overline{\Omega})$ satisfy $u_1 + 2\pi = \Phi_{T_1}(u_1)$ and $u_2 + 2\pi = \Phi_{T_2}(u_2)$ for some $T_1, T_2 > 0$. Then $T_1 = T_2$.

Proof Suppose that the conclusion of the lemma does not hold. Without loss of generality, we may assume that $T_1 < T_2$. Take $n_0 \in \mathrm{N}$ satisfying $u_1 - 2n_0\pi \leq u_2$. Then $\Phi_{nT_2}(u_1) - 2n_0\pi \leq u_2 + 2n\pi$ for all $n \in \mathrm{N}$, and hence $\Phi_{n(T_2-T_1)}(u_1) - 2n_0\pi \leq u_2$. This contradicts $\|\Phi_{l_nT_1+s_n}(u_1)\|_{C(\overline{\Omega})} = \|\Phi_{s_n}(u_1) + 2l_n\pi\|_{C(\overline{\Omega})} \to \infty$ as $n \to \infty$, where $n(T_2 - T_1) = l_nT_1 + s_n$ with $l_n \in \mathrm{N}$, $s_n \in [0, T_1)$. ∎

Proof of Theorem B (i) First, by applying Proposition B1 in [16] (which will be mentioned in Appendix of the present paper), we prove the uniqueness and monotonicity of a spiral traveling wave solution. Set an ordered metric space $X = C(\overline{\Omega})$ with order relation induced by (6) and put

$$X_1 = Y = \{u_0 \in C(\overline{\Omega}) \mid u_0 + 2\pi = \Phi_T(u_0) \text{ for some } T > 0\}.$$

Clearly each spiral traveling wave solution $\overline{u}$ of (1) satisfies $\overline{u}(\cdot,0) \in Y$. By Lemma 6

$$Y = \{u_0 \in C(\overline{\Omega}) \mid u_0 + 2\pi = \Phi_{T_0}(u_0)\}$$

holds for some $T_0 > 0$. The semiflow $\{\Phi_t\}_{t\in[0,\infty)}$ generated by (1) can be defined on Y for all $t \in \mathbf{R}$. Thus $\{\Phi_t\}_{t\in[0,\infty)}$ is extended to a one-parameter group acting on Y. Denote this group by G. Then condition (G2) in Appendix is satisfied. Further (G1) is fulfilled. Indeed the map Φ_t on Y is also order-preserving for $t < 0$. Fix a spiral traveling wave solution $\overline{u}$ arbitrarily. Then a pair Y and $\overline{\varphi} = \overline{u}(\cdot,0)$ satisfies (H1) and (H2). Further (H3) holds since by the strong maximum principle $\psi < h\overline{\varphi}$ implies $\psi \ll h\overline{\varphi}$ for any $\psi \in Y$, $h \in G$. Applying Proposition B1 in [16], we see that $Y = G\overline{\varphi}$ and that Y is homeomorphic and order-isomorphic to $\mathbf{R}$. By $Y = G\overline{\varphi}$ we obtain the uniqueness of a spiral traveling wave solution up to translation to the t-direction. Moreover Lemma 2 and monotonicity of $Y = G\overline{\varphi}$ yield that $\overline{u}_t(x,t) \geq 0$ and $\overline{u}_t(x,t) \not\equiv 0$ for $x \in \overline{\Omega}$, $t > 0$. Therefore, from the strong maximum principle it follows that $\overline{u}_t(x,t) > 0$ for $x \in \overline{\Omega}$, $t > 0$.

Next we show that a spiral traveling wave solution $\overline{u}$ is stable. By the positivity of $\overline{u}_t$ if $t_1 < t_2$ then $\overline{u}(\cdot,t_1) \ll \overline{u}(\cdot,t_2)$. Further by the maximum principle we have, for any $\delta_0 > 0$,

$$\overline{u}(\cdot,-\delta_0) \leq u(\cdot,0) \leq \overline{u}(\cdot,\delta_0) \quad \text{implies} \quad \overline{u}(\cdot,t-\delta_0) \leq u(\cdot,t) \leq \overline{u}(\cdot,t+\delta_0), \ \ t > 0.$$

This proves the stability of a spiral traveling wave solution. Indeed, for any $\varepsilon > 0$, take a $\delta_0 > 0$ satisfying $\|\overline{u}(\cdot,\delta_0) - \overline{u}(\cdot,-\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$ and set

$$\delta = \min\{\overline{u}(x,\delta_0) - \overline{u}(x,0) \mid x \in \overline{\Omega}\} = \min\{\overline{u}(x,0) - \overline{u}(x,-\delta_0) \mid x \in \overline{\Omega}\} > 0.$$

Then, for any solution u of (1) satisfying $\|u(\cdot,0) - \overline{u}(\cdot,0)\|_{C(\overline{\Omega})} < \delta$, we have

$$\overline{u}(\cdot,-\delta_0) < u(\cdot,0) < \overline{u}(\cdot,\delta_0).$$

Therefore, from the inequalities

$$\begin{aligned} \overline{u}(\cdot,t-\delta_0) &< u(\cdot,t) < \overline{u}(\cdot,t+\delta_0), \\ \overline{u}(\cdot,t-\delta_0) &< \overline{u}(\cdot,t) < \overline{u}(\cdot,t+\delta_0) \end{aligned}$$

it follows that

$$\|u(\cdot,t) - \overline{u}(\cdot,t)\|_{C(\overline{\Omega})} < \|\overline{u}(\cdot,t+\delta_0) - \overline{u}(\cdot,t-\delta_0)\|_{C(\overline{\Omega})} = \|\overline{u}(\cdot,\delta_0) - \overline{u}(\cdot,-\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$$

for all $t > 0$. ∎

Proof of Theorem B (ii) As we have shown above, (1) possesses a unique (up to translation to the t-direction) spiral traveling wave solution $\overline{u}$. We denote by ω the growth speed of $\overline{u}$.

Define a map F on $X = C(\overline{\Omega})$ by

$$F(u_0)(r,\theta) = \Phi_{T_0}(u_0)(r,\theta) - 2\pi,$$

where $T_0 = 2\pi/\omega$. Then, $\overline{\varphi} = \overline{u}(\cdot, 0)$ is a fixed point of F and further $\overline{\varphi} - 2m\pi$, $\overline{\varphi} + 2m\pi$ are also fixed points for all $m \in \mathbf{N}$. For any $u_0 \in X$ a sequence $\{F^n(u_0)\}_n$ is bounded in X, since $\overline{\varphi} - 2m\pi \le u_0 \le \overline{\varphi} + 2m\pi$ implies $\overline{\varphi} - 2m\pi \le F^n(u_0) \le \overline{\varphi} + 2m\pi$ for $m, n \in \mathbf{N}$. Hence the set $K(u_0) = \bigcap_{n \in \mathbf{N}} \overline{\{F^m(u_0) \mid m > n\}} \subset X$ is not empty. Set

$$Y = \{K(u_0) \mid u_0 \in X\}$$

and an acting group G being as in the proof of Theorem B (i). Clearly (G1) and (G2) in Appendix are fulfilled. A pair Y and $\{\overline{\varphi}\}$ satisfies (H4) and (H5). Further the strong maximum principle verifies (H6). Hence applying Proposition B2 in [16] (which will be mentioned in Appendix of the present paper), we see that for any $u_0 \in C(\overline{\Omega})$ there exists some τ_0 satisfying

$$\lim_{n \to \infty} \|F^n u_0 - \overline{u}(\cdot, \tau_0)\|_{C(\overline{\Omega})} = 0.$$

By the definition of F we obtain the conclusion. ▮

Proof of Lemma 5 As we have shown in the proof of Theorem B (i), a function satisfying

$$w(x) + 2\pi = \Phi_{T_0}(w)(x), \qquad x \in \overline{\Omega} \tag{15}$$

is unique up to action of one-parameter group $\{\Phi_t\}_{t \in \mathbf{R}}$. Since $\varphi(r, \theta - (2\pi/m)) + (2\pi/m)$ also satisfies (15) for any $m \in \mathbf{N}$, there exists some $s \in \mathbf{R}$ such that

$$\varphi\left(r, \theta - \frac{2\pi}{m}\right) + \frac{2\pi}{m} = \Phi_s(\varphi)(r, \theta), \qquad a \le r \le b,\ 0 \le \theta \le 2\pi.$$

It follows from this that

$$\varphi\left(r, \theta - \frac{2 \cdot 2\pi}{m}\right) + \frac{2 \cdot 2\pi}{m} = \Phi_s(\Phi_s(\varphi))(r, \theta) = \Phi_{2s}(\varphi)(r, \theta), \qquad a \le r \le b,\ 0 \le \theta \le 2\pi.$$

Repeating this calculation, we obtain $\Phi_{ms}(\varphi) = \varphi + 2\pi$. If $ms \neq T_0$ then $\{\Phi_t(\varphi) \mid t \ge |T_0 - ms|\}$ is a periodic orbit with period $|T_0 - ms|$, which contradicts

$$\|\Phi_{nT_0}(\varphi)\|_{C(\overline{\Omega})} = \|\varphi + 2\pi n\|_{C(\overline{\Omega})} \to \infty, \qquad n \to \infty.$$

Hence we get $ms = T_0$, namely $s = T_0/m$. Thus we have, for any $k \in \mathbf{N}$,

$$\varphi\left(r, \theta - \frac{k \cdot 2\pi}{m}\right) + \frac{k \cdot 2\pi}{m} = \Phi_{\frac{kT_0}{m}}(\varphi)(r, \theta), \qquad a \le r \le b,\ 0 \le \theta \le 2\pi$$

and further, for any rational number $p > 0$,

$$\varphi(r, \theta - 2\pi p) + 2\pi p = \Phi_{pT_0}(\varphi)(r, \theta), \qquad a \le r \le b,\ 0 \le \theta \le 2\pi.$$

Since the set of positive rational numbers is dense in $(0, \infty)$, if we set $\omega = 2\pi/T_0$ then

$$\varphi(r, \theta - \omega t) + \omega t = \Phi_t(\varphi)(r, \theta), \qquad a \le r \le b,\ 0 \le \theta \le 2\pi$$

holds for any $t > 0$. The proof is completed. ▮

4 A formal derivation of the interface equation

In this section, we consider equation (2):

$$\begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & x \in \Omega,\ t>0 \\ u_r = 0, & x \in \partial\Omega,\ t>0. \end{cases}$$

We assume that $f(v\,;\varepsilon) = -\frac{\partial W}{\partial v}(v\,;\varepsilon)$ is a smooth function derived from a multi-well potential $W(v\,;\varepsilon)$ whose local minima lie at $v = 2m\pi$ ($m \in \mathbf{Z}$) for all $\varepsilon \geq 0$. More precisely, we assume that $f(v\,;\varepsilon)$ satisfies the following conditions:

(F1) $f(v\,;\varepsilon)$ is 2π-periodic in v for each $\varepsilon \geq 0$,

(F2) $f(\cdot\,;\varepsilon)$ has exactly three zeroes $0 < \zeta(\varepsilon) < 2\pi$ in $[0, 2\pi]$ for each $\varepsilon \geq 0$,

(F3) $\dfrac{\partial f}{\partial v}(0\,;\varepsilon) < 0$ for each $\varepsilon \geq 0$,

(F4) $\displaystyle\int_0^{2\pi} f(v\,;0)dv = 0,\ \int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv > 0.$

By Theorems A and B, under the conditions (F1)–(F4), there exists a unique spiral traveling wave solution for each $\varepsilon > 0$. Roughly speaking, condition (F4) means that the difference of well-depth $W(2\pi\,;\varepsilon) - W(0\,;\varepsilon)$ is negative and that $W(2\pi\,;\varepsilon) - W(0\,;\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. It follows from (F1)–(F4) that there exists a unique solution $(\psi_\varepsilon(z), c(\varepsilon))$ of

$$\begin{cases} \psi_{zz} + \varepsilon c(\varepsilon)\psi_z + f(\psi\,;0) = 0, \quad z \in \mathbf{R}, \\ \psi(-\infty) = 2\pi,\ \ \psi(0) = \zeta(\varepsilon),\ \ \psi(+\infty) = 0, \end{cases} \tag{16}$$

for each $\varepsilon \geq 0$ ([5]). Note that $c(\varepsilon) > 0$ for $\varepsilon > 0$ and

$$c = \lim_{\varepsilon \to 0} c(\varepsilon) = \frac{\displaystyle\int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv}{\displaystyle\int_{\mathbf{R}} \{\psi_0'(z)\}^2 dz}. \tag{17}$$

Let u^ε be a solution of (2). Since the reaction term is very large and the potential W is multi-well type, u^ε approaches $\theta + 2m\pi$ for some $m \in \mathbf{Z}$ if $\theta + \zeta(\varepsilon) + 2(m-1)\pi < u^\varepsilon(x, 0) < \theta + \zeta(\varepsilon) + 2m\pi$. Accordingly, a sharp interface appears between the regions $\{u^\varepsilon \approx \theta + 2m\pi\}$ and $\{u^\varepsilon \approx \theta + 2(m+1)\pi\}$ for each $m \in \mathbf{Z}$. By virtue of (F1), $u^\varepsilon(x,t) = u^\varepsilon(r, \theta, t)$ can be extended to a function (also denoted by u^ε) defined for all $\theta \in \mathbf{R}$ satisfying the following equation:

$$\begin{cases} u_t = u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} + \dfrac{1}{\varepsilon^2} f(u-\theta\,;\varepsilon), & (r,\theta) \in (a,b)\times\mathbf{R},\ t>0, \\ u_r(a,\theta,t) = 0 = u_r(b,\theta,t), & \theta \in \mathbf{R},\ t>0. \end{cases}$$

We fix $T > 0$ and define

$$\widetilde{\Gamma}_t^{\varepsilon,m} = \{(r,\theta) \in (a,b) \times \mathbf{R} \mid u^\varepsilon(r,\theta,t) = \theta + \zeta(\varepsilon) + 2m\pi\}$$

for $t \in [0,T]$. Since u^ε is 2π-periodic in θ, we have $\widetilde{\Gamma}_t^{\varepsilon,m} = \sigma_{-2m\pi}\widetilde{\Gamma}_t^{\varepsilon,0}$ where σ_s is the translation $\sigma_s : (r,\theta) \mapsto (r, \theta+s)$. For simplicity, we assume that $\widetilde{\Gamma}_t^{\varepsilon,0}$ is a smooth embedded curve in $(a,b)\times\mathbf{R}$ with two boundary points on both $\{a\}\times\mathbf{R}$ and $\{b\}\times\mathbf{R}$ for each $t \in [0,T]$. It follows from the homogeneous Neumann boundary conditions that the closure of $\widetilde{\Gamma}_t^{\varepsilon,0}$ intersects with the lines $r = a$ and $r = b$ perpendicularly at the boundary points. We denote by $\widetilde{D}_t^m$ the domain in $(a,b) \times \mathbf{R}$ between the two curves $\widetilde{\Gamma}_t^{\varepsilon,m}$ and $\widetilde{\Gamma}_t^{\varepsilon,m-1}$. Let Π be the covering map from $(a,b) \times \mathbf{R}$ to Ω defined by $\Pi(r,\theta) = (r\cos\theta, r\sin\theta)$. We take a neighborhood $\widetilde{N}_t$ of $\widetilde{\Gamma}_t^{\varepsilon,0}$ in $(a,b) \times \mathbf{R}$ so that the map $\Pi|_{\widetilde{N}_t}$ is injective. We put $N_t = \Pi(\widetilde{N}_t)$, $D_t^j = \Pi(\widetilde{D}_t^j \cap \widetilde{N}_t)$ $(j = 0,1)$ and

$$N = \bigcup_{t\in[0,T]} (N_t \times \{t\}).$$

In what follows we write $\theta = \theta(x)$ for $x \in N_t$ if $x = \Pi(r,\theta)$.

We call the set

$$\Gamma^\varepsilon = \bigcup_{t\in[0,T]} (\Gamma_t^\varepsilon \times \{t\})$$

the interface, where

$$\Gamma_t^\varepsilon = \{\Pi(r,\theta) \in \Omega \mid (r,\theta) \in \widetilde{\Gamma}_t^{\varepsilon,0}\}.$$

We also call Γ_t^ε the interface at time t. We remark that if $x \in \Gamma_t^\varepsilon$ then $u^\varepsilon(x,t) = \theta(x) + \zeta(\varepsilon)$ and that $\overline{\Gamma_t^\varepsilon}$ perpendicularly intersects with $\partial\Omega$.

Let $d^\varepsilon(x,t)$ be the signed distance function to Γ^ε defined in N by

$$d^\varepsilon(x,t) = \begin{cases} \operatorname{dist}(x,\Gamma_t^\varepsilon), & \text{if } x \in D_t^0, \\ -\operatorname{dist}(x,\Gamma_t^\varepsilon), & \text{if } x \in D_t^1, \end{cases}$$

where $\operatorname{dist}(x,\Gamma_t^\varepsilon)$ is the distance from $x \in N_t$ to the curve Γ_t^ε in $\mathbf{R}^2$. We remark that $d^\varepsilon(x,t) = 0$ if $x \in \Gamma_t^\varepsilon$ and $|\nabla d^\varepsilon| = 1$. We assume that d^ε has the expansion

$$d^\varepsilon(x,t) = d_0(x,t) + \varepsilon d_1(x,t) + \varepsilon^2 d_2(x,t) + \cdots$$

and define

$$\begin{aligned} \Gamma_t &= \{x \in N_t \mid d_0(x,t) = 0\}, \\ \Omega_t^0 &= \{x \in N_t \mid d_0(x,t) > 0\}, \\ \Omega_t^1 &= \{x \in N_t \mid d_0(x,t) < 0\}, \\ \Gamma &= \bigcup_{t\in[0,T]} (\Gamma_t \times \{t\}), \\ Q_0 &= \bigcup_{t\in[0,T]} (\Omega_t^0 \times \{t\}), \\ Q_1 &= \bigcup_{t\in[0,T]} (\Omega_t^1 \times \{t\}). \end{aligned}$$

Roughly speaking, Γ_t represents the position of the interface at time t in the limit as $\varepsilon \to 0$, while d_0 represents the signed distance function to Γ. In what follows we derive the equation of motion of the interface Γ_t by using matched asymptotic expansions. See [1], [4], [14] and [18] for details.

We assume that the solution u^ε has the expansions

$$u^\varepsilon(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots \tag{18}$$

away from Γ^ε (the outer expansion) and

$$u^\varepsilon(x,t) = U_0(\xi,x,t) + \varepsilon U_1(\xi,x,t) + \varepsilon^2 U_2(\xi,x,t) + \cdots \tag{19}$$

near Γ^ε (the inner expansion), where $\xi = d^\varepsilon(x,t)/\varepsilon$. To make these expansions consistent, we require the matching conditions

$$\begin{array}{ll} U_k(+\infty,x,t) = u_k^1(x,t) & \text{if } x \in \Omega_t^1 \cup \Gamma_t \\ U_k(-\infty,x,t) = u_k^0(x,t) & \text{if } x \in \Omega_t^0 \cup \Gamma_t \end{array} \tag{20}$$

for all $(x,t) \in N$ and $k \geq 0$, where u_k^j $(j=0,1)$ denote the terms of the outer expansion (18) in the region Q_j $(j=0,1)$. Since $u^\varepsilon(x,t) = \theta(x) + \zeta(\varepsilon)$ on Γ^ε, we also require the normalization conditions $U_0(0,x,t) = \theta(x) + \zeta_0$, $U_k(0,x,t) = \zeta_k$ $(k \geq 1)$, where ζ_j denote the terms of the expansion $\zeta(\varepsilon) = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots$.

Substituting the outer expansion (18) into (2) and the collecting the ε^{-2} and ε^{-1} terms respectively, we have

$$\begin{aligned} &f(u_0(x,t) - \theta(x)\,;0) = 0, \\ &\frac{\partial f}{\partial v}(u_0(x,t) - \theta(x)\,;0)u_1(x,t) + \frac{\partial f}{\partial \varepsilon}(u_0(x,t) - \theta(x)\,;0) = 0, \end{aligned}$$

in $Q_0 \cup Q_1$. The first equation implies that

$$u_0(x,t) = \begin{cases} \theta(x) & \text{in } Q_0, \\ \theta(x) + 2\pi & \text{in } Q_1. \end{cases}$$

Hence from the second equation, we get $u_1(x,t) = 0$ in $Q_0 \cup Q_1$.

Next, substituting the inner expansion (19) into (2) and the collecting the ε^{-2} and ε^{-1} terms, we have

$$U_{0\xi\xi} + f(U_0 - \theta(x)\,;0) = 0, \tag{21}$$

$$\begin{aligned} U_{1\xi\xi} + \frac{\partial f}{\partial v}(U_0 - \theta(x)\,;0)U_1 = {} & U_{0\xi}(d_{0t} - \Delta d_0) - 2\nabla(U_{0\xi}) \cdot \nabla d_0 \\ & - \frac{\partial f}{\partial \varepsilon}(U_0 - \theta(x)\,;0). \end{aligned} \tag{22}$$

In both equations we regard x and t as parameters. From (21) together with the matching conditions and the normalization conditions, we obtain

$$U_0(\xi,x,t) = \psi_0(\xi) + \theta(x) \tag{23}$$

where ψ_0 is the unique solution to (16) for $\varepsilon = 0$.

Substituting (23) into (22) and recalling the normalization conditions, we get

$$\begin{cases} U_{1\xi\xi} + \dfrac{\partial f}{\partial v}(\psi_0(\xi)\,;0)U_1 = (d_{0t} - \Delta d_0)\psi_0'(\xi) - \dfrac{\partial f}{\partial \varepsilon}(\psi_0(\xi)\,;0), \\ U_1(0,x,t) = \zeta_1. \end{cases} \tag{24}$$

By Lemma 4.1 in [1], (24) has a bounded solution if and only if

$$(d_{0t} - \Delta d_0)\int_{\mathbf{R}} \{\psi_0'(\xi)\}^2 d\xi - \int_{\mathbf{R}} \frac{\partial f}{\partial \varepsilon}(\psi_0(\xi)\,;0)\psi_0'(\xi) d\xi = 0. \tag{25}$$

Under the solvability condition (25), the solution U_1 of (24) incidentally satisfies the matching conditions (20), since the right-hand side of the first equation of (24) tends to 0 exponentially as $\xi \to \pm\infty$. By (25), we get

$$d_{0t} = \Delta d_0 - c, \tag{26}$$

where c is the positive constant defined in (17). It is known that $-d_{0t} = V$ and $\Delta d_0 = \kappa$, where V and κ are the normal velocity and the curvature of the interface Γ_t, respectively. Thus (26) is equivalent to (4):

$$V = c - \kappa \quad \text{on } \Gamma_t.$$

Moreover $\overline{\Gamma_t}$ intersects with $\partial\Omega$ perpendicularly.

5 Existence of a spiral for the interface equation

In this section we consider the interface equation

$$\begin{cases} V = c - \kappa & \text{on } \Gamma_t, \\ \langle \nu(x), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \cap \overline{\Gamma_t}, \end{cases} \tag{27}$$

where $\mathbf{n} = \mathbf{n}(x,t)$ and $\nu(x)$ is the outward unit normal at each point of Γ_t and $\partial\Omega$, respectively. We seek for a solution of (27) which is written in the form

$$\overline{\Gamma}(t) = \{(r\cos(\theta(r) + \omega t), r\sin(\theta(r) + \omega t) \mid a \le r \le b,\ t \ge 0\}$$

for some function $\theta(r)$ and some constant ω. We call such $\overline{\Gamma}(t)$ a spiral with angular speed ω. One can easily see that $\overline{\Gamma}(t)$ is a solution of (27) if and only if $q(r) = r\theta'(r)$ satisfies

$$\begin{cases} \dfrac{dq}{dr} = h(r,q\,;\omega), & r > a, \\ q(a) = q(b) = 0, \end{cases}$$

where $h(r,q\,;\omega) = (1+q^2)\left(-c\sqrt{1+q^2} - \frac{q}{r} + \omega r\right)$.

Theorem C

Fix $a > 0$ arbitrarily.

(i) *For any $b > a$, there exists a spiral with angular speed $\omega(b) > 0$. In addition, the spiral is unique up to rotation.*

(ii) *The angular speed $\omega(b)$ is strictly monotone decreasing in b and there exists $\omega_\infty > 0$ such that $\lim_{b\to\infty} \omega(b) = \omega_\infty$.*

(iii) *In the case where $\Omega = \{x \in \mathbf{R} \mid |x| > a\}$, there exists a spiral with speed ω_∞ such that $\lim_{r\to\infty} \theta'(r) = -\dfrac{\omega_\infty}{c}$.*

Remark 3

The statement (iii) of Theorem C shows that the shape of the spiral for (27) looks like Archimedean spiral as $r \to \infty$ in the case where $b = +\infty$.

In what follows we denote by $q(r\,;\omega)$ the solution of the initial value problem

$$\begin{cases} \dfrac{dq}{dr} = h(r, q\,;\omega), \qquad r > a, \\ q(a) = 0, \end{cases} \tag{28}$$

and let (a, R_ω) be the maximal interval of the existence of $q(r\,;\omega)$.

Lemma 7

(i) *If $\omega_1 < \omega_2$ then $q(r\,;\omega_1) < q(r\,;\omega_2)$ for $a < r < \min\{R_{\omega_1}, R_{\omega_2}\}$.*

(ii) *If $\omega > c/a$ then $R_\omega = +\infty$ and $q(r\,;\omega) > 0$ for $r > a$.*

(iii) *R_ω is nondecreasing in $\omega \in \mathbf{R}$.*

(iv) *If ω_n converges to ω_0 then $\liminf_{n\to\infty} R_{\omega_n} \geq R_{\omega_0}$. If, in addition, $\omega_n \leq \omega_0$ for large n then $\lim_{n\to\infty} R_{\omega_n} = R_{\omega_0}$.*

Proof (i) The statement immediately follows from the fact that $h(r, q\,;\omega)$ is strictly increasing in ω for $r > a$.
(ii) If $\omega > c/a$ then $h(r, 0\,;\omega) = -c + \omega r > 0$ for $r > a$. Therefore $q(r\,;\omega) > 0$ for $a < r < R_\omega$. Since $h(r, q\,;\omega) < 0$ if $q \geq \omega r^2$, we have $0 < q(r\,;\omega) \leq \omega r^2$ for any $r \in (a, R_\omega)$. This implies $R_\omega = +\infty$.
(iii) If $R_\omega < +\infty$ then $\lim_{r\nearrow R_\omega} q(r\,;\omega) = -\infty$, since $h(r, q\,;\omega) < 0$ for $q \geq \max\{\omega r^2, 0\}$. Therefore by virtue of (i), R_ω is nondecreasing in ω.
(iv) Put $p_n(r) = q(r\,;\omega_n) - q(r\,;\omega_0)$. Then p_n satisfies

$$\begin{cases} \dfrac{dp_n}{dr} = H_n(r, p_n), \qquad r > a, \\ p_n(a) = 0, \end{cases} \tag{29}$$

where $H_n(r, p) = h(r, q_0(r) + p\,;\omega_n) - h(r, q_0(r)\,;\omega_0)$ and $q_0(r) = q(r\,;\omega_0)$. For any $R < R_{\omega_0}$ and $\delta > 0$ there exists $L > 0$ such that

$$|H_0(r, p) - H_0(r, \widetilde{p})| \leq L|p - \widetilde{p}|, \qquad |p|, |\widetilde{p}| \leq \delta,\ a \leq r \leq R$$

and that

$$\gamma_n = \sup_{\substack{|p|\le\delta \\ a\le r\le R}} |H_n(r,p) - H_0(r,p)| \to 0, \qquad n\to\infty.$$

We define $R_n = \sup\{a < r < R \mid |p_n(r)| \le \delta\}$. Then by (29) we have

$$|p_n(r)| \le \gamma_n(R-a) + L\int_a^r |p_n(s)|ds$$

for $a \le r \le R_n$. Therefore by Gronwall's inequality, we have

$$|p_n(r)| \le \gamma_n(R-a)e^{L(r-a)} \le \gamma_n(R-a)e^{L(R-a)}$$

for $a \le r \le R_n$. This implies $R_n = R$ for sufficiently large n. Thus we get $R_{\omega_n} > R$ for large n, hence

$$\liminf_{n\to\infty} R_{\omega_n} \ge R_{\omega_0}. \tag{30}$$

Combining (ii) and (30), we obtain $\lim_{n\to\infty} R_{\omega_n} = R_{\omega_0}$ if $\omega_n \le \omega_0$ for large n. ∎

Lemma 8
There exists $\widetilde{\omega} \le c/a$ such that $R_{\widetilde{\omega}} > b$ and $q(b;\widetilde{\omega}) \le 0$.

Proof Suppose that the statement of the lemma does not hold. Then for any $\omega \le c/a$, either of the following holds:

$$\text{(a) } R_\omega \le b, \qquad \text{(b) } R_\omega > b \text{ and } q(b;\omega) > 0.$$

By Lemma 7 (ii), the statement (b) holds for $\omega > c/a$. We define $\omega_0 = \sup\{\omega \in \mathbf{R} \mid R_\omega \le b\}$. Then we have $\omega_0 \ge c/b$, since $h(r,0;\omega) < 0$ for $a \le r \le b$ if $\omega < c/b$. Clearly $\omega \le c/a$. By virtue of Lemma 7 (iii), we obtain $R_{\omega_0} \le b$, hence

$$\lim_{r\nearrow R_{\omega_0}} q(r;\omega_0) = -\infty.$$

On the other hand, $R_\omega > b$ and $q(b;\omega) > 0$ for any $\omega > \omega_0$. Let $r_0 \in (a, R_{\omega_0})$ be such that $q(r_0;\omega_0) < -(\omega_0+1)b^2$ and that $h(r_0, q(r_0;\omega_0);\omega_0) < 0$. Then $q(r_0;\omega_1) < -(\omega_0+1)b^2$ and $h(r_0, q(r_0;\omega_1);\omega_1) < 0$ for some ω_1 sufficiently close to ω_0. Since $h(r_1,q;\omega) > h(r_2,q;\omega)$ for $a \le r_1 < r_2 \le b$, $q \le -(\omega_0+1)b^2$ and $\omega_0 \le \omega \le \omega_0+1$, we have

$$\frac{dq(r;\omega_1)}{dr} = h(r, q(r;\omega_1);\omega_1) < h(r_0, q(r_0;\omega_1);\omega_1) < 0$$

for all $r \ge r_0$ satisfying $q(r;\omega_1) = q(r_0;\omega_1)$. Hence $q(r;\omega_1) < q(r_0;\omega_1)$ for $r > r_0$, contradicting the fact that $q(b;\omega_1) > 0$. ∎

Proof of Theorem C (i) Let $\widetilde{\omega} \in [c/b, c/a]$ be such that $R_{\widetilde{\omega}} > b$ and $q(b;\widetilde{\omega}) < 0$. Then $q(b;\omega)$ is well-defined for $\omega \ge \widetilde{\omega}$ and is continuous in $\omega \ge \widetilde{\omega}$. Since $q(b;\omega) > 0$ for

$\omega > c/a$, there exists $\omega(b) \in (\widetilde{\omega}, c/a]$ satisfying $q(b\,;\omega(b)) = 0$. The uniqueness of $\omega(b)$ is an immediate consequence of Lemma 7 (i). ∎

Proof of Theorem C (ii) By Lemma 7 (i), the rotation speed $\omega(b)$ is strictly monotone decreasing in b. Therefore $\omega(b)$ converges to some $\omega_\infty \geq 0$, since $\omega(b) \geq c/b$. Note that $q(r\,;0) < 0$ for $r \in (a, R_0)$ and hence $q(r\,;0)$ satisfies

$$\frac{dq}{dr} \leq \frac{c}{2} q(1+q^2), \qquad r \geq \frac{2}{c}.$$

This implies $R_0 < +\infty$, from which and the following lemma we obtain $\omega_\infty > 0$. ∎

Lemma 9
$R_{\omega_\infty} = +\infty$.

Proof Suppose that R_{ω_∞} is finite. We fix $b_0 > a$, $k > \omega(b_0)/c$ and take

$$\widetilde{R} = \max\left\{\frac{2k}{ck - \omega(b_0)}, R_{\omega_\infty}\right\}.$$

Then, if $\omega \leq \omega(b_0)$, $-kr$ is a supersolution of (28) for $r \geq \widetilde{R}$ since $h(r, -kr\,;\omega) \leq -k$. We take $\widetilde{q} = \min\{-k\widetilde{R}, -\omega(b_0)\widetilde{R}^2\}$. Then we get $h(r_1, q\,;\omega) \geq h(r_2, q\,;\omega)$ for $a \leq r_1 < r_2 \leq \widetilde{R}$, $q \leq \widetilde{q}$ and $\omega \leq \omega(b_0)$. By the similar argument in the proof of Lemma 8, there exists $b > b_0$ and $r_0 < R_{\omega_\infty}$ such that $q(r_0\,;\omega(b)) < \widetilde{q}$ and $h(r_0, q(r_0\,;\omega(b))\,;\omega(b)) < 0$. Again by the argument in the proof of Lemma 8, we obtain $q(r\,;\omega(b)) < q(r_0\,;\omega(b)) < \widetilde{q}$ for $r_0 < r < \widetilde{R}$ and $q(r\,;\omega(b)) < -kr$ for $r \geq \widetilde{R}$, contradicting the fact that $q(b\,;\omega(b)) = 0$. This contradiction proves the lemma. ∎

Proof of Theorem C (iii) By Lemma 9, $q(r\,;\omega_\infty)$ exists for all $r > a$. Furthermore $q(r\,;\omega_\infty)$ is negative since $q(r\,;\omega(b)) < 0$ for $a < r < b$. This corresponds to a spiral with angular speed ω_∞ for $\Omega = \{x \in \mathbf{R} \mid |x| > a\}$. To complete the proof, we show that

$$\lim_{r \to +\infty} \frac{q(r\,;\omega_\infty)}{r} = -\frac{\omega_\infty}{c}.$$

Fix $k > \omega_\infty/c$. We take b_0 so that $k > \omega(b_0)/c$ and put $r_0 = 2k/(ck - \omega(b_0))$. Since $-kr$ is a supersolution of (28) for $r \geq r_0$ if $\omega \leq \omega(b_0)$, we have $q(r\,;\omega(b)) > -kr$ for $r \geq r_0$ and $b \geq b_0$. This implies $q(r\,;\omega_\infty) \geq -kr$ for $r \geq r_0$, since $q(r\,;\omega(b))$ uniformly converges to $q(r\,;\omega_\infty)$ on any compact subset of $(a, +\infty)$. Hence we have

$$\liminf_{r \to +\infty} \frac{q(r\,;\omega_\infty)}{r} \geq -\frac{\omega_\infty}{c}.$$

We define

$$K_\infty = \left\{(r, q) \,\middle|\, r > \frac{c}{\omega_\infty}\sqrt{1+q^2},\ q < 0\right\}.$$

Let

$$l = \limsup_{n\to\infty} \frac{q(r\,;\omega_\infty)}{r}$$

and suppose that $l > -\omega_\infty/c$. Then there exists $r_0 > a$ such that $(r_0, q(r_0\,;\omega_\infty)) \in K_\infty$. Since $h(r,q\,;\omega_\infty) > 0$ for $(r,q) \in K_\infty$, we have $(r, q(r\,;\omega_\infty)) \in K_\infty$ for all $r \geq r_0$. Therefore by (28) we obtain

$$\frac{d}{dr}\left(\frac{q(r\,;\omega_\infty)}{r}\right) > \frac{-c\sqrt{1+q(r\,;\omega_\infty)^2}+\omega_\infty r}{r} > 0 \tag{31}$$

for $r \geq r_0$, hence

$$-\frac{\omega_\infty}{c} < l = \lim_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} \leq 0.$$

On the other hand, by (31) we have

$$\liminf_{r\to+\infty} \frac{d}{dr}\left(\frac{q(r\,;\omega_\infty)}{r}\right) \geq cl + \omega_\infty > 0.$$

This contradiction proves that

$$\limsup_{r\to+\infty} \frac{q(r\,;\omega_\infty)}{r} \leq -\frac{\omega_\infty}{c}.$$

The theorem is proved. ∎

Appendix

In this appendix we present two propositions in [16]. Proposition B1 is concerned with the structure of a subset of an ordered metric space under a group action. Proposition B2 is, in a sense, a set-valued version of the former half of Proposition B1.

Let X be an ordered metric space. In other words, X is a metric space on which a closed partial order relation is defined. We will denote by $\leq$ the order relation in X. Here, we say that a partial order relation in X is *closed* if $\varphi_n \leq \psi_n$ $(n = 1,2,3,\cdots)$ implies $\lim_{n\to\infty} \varphi_n \leq \lim_{n\to\infty} \psi_n$ provided that both limits exist. We write $\varphi < \psi$ if $\varphi \leq \psi$ and $\varphi \neq \psi$. For a subset $V \subset X$, the expression $\varphi \leq V$, $V \leq \varphi$ means $\varphi \leq \psi$, $\psi \leq \varphi$ for all points $\psi \in V$, respectively.

Let G be a metrizable topological group acting on some subset X_1 of X. We say G acts on X_1 if there exists a continuous mapping $\gamma: G \times X_1 \to X_1$ such that $g \mapsto \gamma(g,\cdot)$ is a group homomorphism of G into $Hom(X_1)$, the group of homeomorphisms of X_1 onto itself. For brevity, we write $\gamma(g,\varphi) = g\varphi$ and identify the element $g \in G$ with its action $\gamma(g,\cdot)$. We assume that

(G1) γ is order-preserving (that is, $\varphi \preceq \psi$ implies $g\varphi \preceq g\psi$ for any $g \in G$);

(G2) G is connected.

Let Y be a subset of X and $\overline{\varphi}$ be an element of $Y \cap X_1$ such that

(H1) $g\overline{\varphi} \in Y$ for any $g \in G$;

(H2) for any $\psi \in Y$, there exist some $g_1, g_2 \in G$ satisfying $g_1\overline{\varphi} < \psi < g_2\overline{\varphi}$;

(H3) for any $\psi \in Y$ with $\psi < h\overline{\varphi}$ for some $h \in G$, there exists some neighborhood B of the unit element of G such that $\psi < gh\overline{\varphi}$ for any $g \in B$.

Proposition ([16, Proposition B1])
Let G satisfy (G1), (G2) and Y, $\overline{\varphi}$ satisfy (H1), (H2), (H3). Then Y is a totally-ordered connected set and $Y = G\overline{\varphi}$. Furthermore, if Y is locally precompact, then Y is homeomorphic and order-isomorphic to $\mathbf{R}$.

A similar result holds for the case where the set Y consists of subsets of X. To be more precise, let Y be a set of subsets of X containing $\{\overline{\varphi}\}$ such that

(H4) $\{g\overline{\varphi}\} \in Y$ for any $g \in G$;

(H5) for any $V \in Y$, there exist some $g_1, g_2 \in G$ satisfying $g_1\overline{\varphi} \leq V \leq g_2\overline{\varphi}$ and $V \neq \{g_1\overline{\varphi}\}$, $\{g_2\overline{\varphi}\}$;

(H6) for any $V \in Y$ with $V \leq h\overline{\varphi}$ and $V \neq \{h\overline{\varphi}\}$ for some $h \in G$, there exists some neighborhood B of the unit element of G such that $V \leq gh\overline{\varphi}$ and $V \neq \{gh\overline{\varphi}\}$ for any $g \in B$.

Proposition ([16, Proposition B2])
Let G satisfy (G1), (G2) and Y, $\{\overline{\varphi}\}$ satisfy (H4), (H5), (H6). Then $Y = G\{\overline{\varphi}\} = \{\{g\overline{\varphi}\} \mid g \in G\}$.

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