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Symbolic Computation in Particle Physics and Astrophysics

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Abstract

We review some of symbolic computation methods used in particle physics and astrophysics. We first introduce a widely used mathematical package *the FeynArts* programmed on *the Mathematica.* We show a new model for the FeynArts, where one can compute the Feynman amplitudes in a symbolic manner for the hadron system. We then apply the model to the hadron physics. The pion photoproduction reaction on the nucleon $\gamma N \to \pi N$ has been studied with the model code. The Feynman diagrams and the Feynman amplitudes have been calculated in a symbolic manner. Next we show an application of the Mathematica to astrophysics. An analytic expression has been derived for the Sunyaev-Zel'dovich effect for the cluster of galaxies.

1 Introduction

In high-energy particle physics, symbolic computations on computers have a long history since *the REDUCE*[1] was invented in 1968. The code has been extensively used for the trace calculation of the Feynman amplitudes in Quantum Electrodynamics (QED). Recently more advanced and sophisticated tools have become available for symbolic manipulations of mathematical formulas on computers. They are, for example, *the Mathematica*[2] and *the Maple V*[3]. They have been widely used not only in sciences and technologies but also in other fields such as economics, business and education.

In high-energy particle physics, in particular, a mathematical package called *the FeynArts* [4, 5] has been developed. With the package, one can automatically generate the Feynman diagrams for a fixed reaction process. With the code, one can also obtain the Feynman amplitudes which correspond to the diagrams in a symbolic manner. The package is executable on *the Mathematica. The FeynArts* has been used for drawing the Feynman diagrams and for calculating the Feynman amplitudes for QED and Quantum Chromodynamics (QCD). So far the package is available only for renormalizable field theories such

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as QED and QCD, where one can compute higher order loop diagrams with any desired accuracy.

In the present paper, we show a model program of calculating the Feynman amplitudes for hadronic systems such as photons (γ), mesons (π, ρ, ω) and nucleons (N). However the following remark should be noted in applying the present model to the hadronic systems. The strong interaction of the hadronic system is NOT renormalizable. Therefore the use of the code should be restricted to tree diagrams (Not to loop diagrams). We have applied the model code to particle physics problems.

A brief introduction of *the FeynArts* is given in section 2. We then construct a model program for the hadronic system. The application of the code to particle physics is demonstrated. We will show two examples: the pion photoproduction reaction on the nucleon $\gamma N \to \pi N$, and its radiative reaction process $\gamma N \to \gamma\pi N$. We show an example of the symbolic computation in astrophysics in section 3. We summarize the present results in section 4.

2 Application in Particle Physics

2.1 FeynArts and Model Construction

The *FeynArts* is one of the mathematical packages on *the Mathematica* which calculates the Feynman amplitudes in a symbolic manner for QED, QCD and other interaction models. The information in for *the FeynArts* is given in detail in Ref. 2. In the present paper, we restrict ourselves for a model construction for the hadronic system. The model is essential in applying *the FeynArts* to the hadronic system. Again the following remarks are in order. Unlike QED and QCD the model for the hadron system is unrenormalizable. Therefore the use of the model is restricted to the tree level calculation of the Feynman diagrams.

Staring with the standard pseudovector Lagrangians[6], we find the following propagator and vertex expressions for photons, pions and nucleons.

Nucleon propagator:

$$\frac{i(\gamma_\mu p^\mu + m_N)}{p^2 - m_N^2}, \tag{1}$$

where m_N and p are the mass and the momentum of the nucleon, respectively. The Greek letter γ_μ denotes the Dirac gamma matrix.

Pion propagator:

$$\frac{i}{k^2 - m_\pi^2}, \tag{2}$$

where m_π and k are the mass and the momentum of the pion.

πNN vertex:

$$-i\,\frac{f_{\pi NN}}{m_\pi}\gamma_5\gamma_\mu\,\tau_i\,p^\mu, \tag{3}$$

where $f_{\pi NN}$ is the πNN coupling constant, and τ_i is the isospin Pauli matrix.

γNN vertex:

$$-e\left(\frac{1+\tau_3}{2}\right)\gamma_\mu\,, \tag{4}$$

where e is the electric charge of the proton.

$\gamma\pi\pi$ vertex:

$$-ie\,(k_\mu + k'_\mu)\,\epsilon_{3ik}\,. \tag{5}$$

$\gamma\pi NN$ vertex:

$$-e\,\frac{f_{\pi NN}}{m_\pi}\gamma_5\gamma_\mu\tau_i\,\epsilon_{3ik}\,. \tag{6}$$

$\gamma\gamma\pi\pi$ vertex:

$$2\,e^2\,g_{\mu\nu}(\delta_{ij} - \delta_{i3}\,\delta_{j3})\,. \tag{7}$$

Using eqs. (1) - (7), it is now straightforward to construct a model of for the hadronic system. We show the list of the obtained model program "*model.PIN*" as under. In the list, propagators, line specifications and coupling forms are defined. F, V and S denote the nucleon (fermion), photon (vector particle) and pion, respectively.

```
(*  ************************   model.PIN   *************************  *)
            (** setting particle property specifications **)
PropList [PIN] =
{
  Prop[in] [ F[-13], F[13], mom] ==
                         PV[ NonCommnutative[ DiracSlash[mom] + MN]
                         I PropagatorDenominator[mom, MN] ] ,
  Prop[ex] [ F[13], mom]  == PV[NonCommutative[ LeptonSpinor[ mom, MN] ]] ,
  Prop[ex] [ F[-13], mom] == PV[NonCommutative[ LeptonSpinor[-mom, MN] ]] ,
  Prop[in] [ V[4, li1], V[4, li2], mom] ==
            PV[ I PropagatorDenominator[mom, MLA]
                 ( -MetricTensor[li1,li2] + (1 - 1/GaugeXi[A])
                   FourVector[mom,li1]  FourVector[mom,li2]
                   PropagatorDenominator[mom, MLA/Sqrt[GaugeXi[A]] ] ) ] ,
  Prop[ex] [ V[4,li2], mom] == PV[ Polarizationvector[mom,li2] ] ,
  Prop[in] [ S[4], S[4], mom] == PV[ I PropagatorDenominator[mom,MPI] ] ,
  Prop[ex] [ S[4], mom] == PV[1]
}
                    (**  setting line specifications **)
  LineSpec[ F[13] ]  = { straight, forward, "p" } ;
  LineSpec[ F[-13] ] = { straight, backward, "p" } ;
  LineSpec[ V[4] ]   = { wavy, none, Greek["g"] } ;
  LineSpec[ S[4] ]   = { dashed, none, Greek["p"] }
```

```
                      (**  setting coupling specifications **)
CoupList [ PIN ] =
{
  Coup[ { V[4,li1], mom1}, { F[-13], mom2}, { F[13], mom3} ] ==
      PV [ NonCommutative [DiracMatrix [li1] ] EL ,
  Coup[ { V[4,li1], mom1}, { S[4], mom2}, { S[4], mom3} ] ==
      PV[ (FourVector[mom2, li1]  -  FourVector[mom3, li1] ) EL ] ] ,
  Coup[ { S[4],mom1}, { F[-13], mom2}, { F[13], mom3} ] ==
      PV[ NonCommununutativer DiracSlash[mom1] . DiracMatrix[5]  FPI ] ,
  Coup[ { V[4,li1], mom1}, { S[4], mom2}, { F[-13], mom3}, { F[13], mom4} ]
   == PV[ NonCommutative [DiracMatrix [li1,5] I EL*FPI]
}
(*  *********************   end of model.PIN   *********************  *)
```

2.2 Application to $\gamma N \to \pi N$

Let us apply the constructed model to the pion photoproduction on the nucleon: $\gamma N \to \pi N$. Running the *FeynArts* with the model program "*model.PIN*", one obtains four Feynman diagrams and their amplitudes. We write the obtained Feynman amplitudes as follows.

amplitude # 1

$$e\, f_{\pi NN}\, \overline{u}_N(k_2)\, \gamma \cdot \epsilon(p_1) \frac{\{\gamma \cdot (p_2 - k_1) + m_N\}}{(p_2 - k_1)^2 - m_N^2}\, \gamma \cdot k_1 \gamma_5\, u_N(p_2)\,, \tag{8}$$

where $\overline{u}(k_2)$ and $u_N(p_2)$ are the nucleon spinors. The Greek letter $\epsilon(p_1)$ denotes the photon polarization vector.

amplitude # 2

$$e\, f_{\pi NN}\, \overline{u}_N(k_2)\, \frac{\gamma \cdot (p_1 - k_1)\gamma_5}{(p_1 - k_1)^2 - m_\pi^2}\, \gamma \cdot k_1 \gamma_5\, k_1 \cdot \epsilon(p_1)\, u_N(p_2)\,, \tag{9}$$

amplitude # 3

$$e\, f_{\pi NN}\, \overline{u}_N(k_2)\, \gamma \cdot k_1 \gamma_5 \frac{\{\gamma \cdot (p_1 + p_2) + m_N\}}{(p_1 + p_2)^2 - m_N^2}\, \gamma \cdot \epsilon(p_1)\, u_N(p_2)\,. \tag{10}$$

amplitude # 4

$$i\, e\, f_{\pi NN}\, \overline{u}_N(k_2)\, \gamma \cdot \epsilon(p_1)\gamma_5\, u_N(p_2)\,. \tag{11}$$

Equations (8) to (11) are the Feynman amplitudes for the reaction. All physics observables such as cross sections and polarizations are calculable with these amplitudes. The physics results have been shown in Ref. [6].

We have also applied the present model to the radiative reaction $\gamma N \to \gamma\pi N$. With the model code, we obtain sixteen topologically distinct diagrams and their Feynman amplitudes. We skip to show their explicit forms in the present paper since their expressions are rather lengthy. Instead we only refer the recent articles by C. Wolfe et al.[7]. In Ref [7], the physics implications have been explored in great detail.

3 Application in Astrophysics

We now show the second example of the symbolic computation in physics. It is well known that our universe is filled by the uniform Cosmic Microwave Background (CMB) radiation as predicted by the big bang universe scenario. The CMB photon has extremely uniform Planck distribution whose temperature is 2.7K.

On the other hand, it is also well known that the clusters of galaxies have extremely high–temperature electron plasmas whose temperature exceed 10keV. If the CMB photons pass through the electron plasmas in the cluster of galaxies, the photon distribution function is distorted by the Compton scattering of the photon by the electron gas. This effect is called as the Sunyaev-Zel'dovich effect[8]. The SZ effect together with the X-ray spectrum measurement of the cluster of galaxies, one can determine the Hubble constant, which is one of the most fundamental constants in astrophysics. Therefore it is extremely important to study the SZ effect theoretically. We have investigated the SZ effect. The details of the physics implications have been explored in great detail in Itoh et al.[9] and Nozawa et al.[10]. Therefore, we summarize some of main results in the present paper. The main calculation has been performed in a symbolic manner with Mathematica.

The time evolution of the photon distribution function $n(\omega)$ is written as

$$\begin{aligned} \frac{\partial n(\omega)}{\partial t} &= -2 \int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' \, W \\ & \quad \{n(\omega)[1+n(\omega')]f(E) - n(\omega')[1+n(\omega)]f(E')\} \,, \end{aligned} \tag{12}$$

$$W = \frac{(e^2/4\pi)^2}{2\omega\omega' E E'} \, \delta^4(p+k-p'-k') \, \overline{X} \,, \tag{13}$$

$$\overline{X} = -\left(\frac{\kappa}{\kappa'} + \frac{\kappa'}{\kappa}\right) + 4m^4 \left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right)^2 - 4m^2 \left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right) , \tag{14}$$

$$\kappa = -2\omega E \left(1 - \frac{|\vec{p}\,|}{E} \cos\alpha\right) , \tag{15}$$

$$\kappa' = 2\omega' E \left(1 - \frac{|\vec{p}\,|}{E} \cos\alpha'\right) . \tag{16}$$

In the above W is the transition probability corresponding to the Compton scattering. The four-momenta of the initial electron and photon are $p = (E, \vec{p})$ and $k = (\omega, \vec{k})$, respectively. The four-momenta of the final electron and photon are $p' = (E', \vec{p}')$ and $k' = (\omega', \vec{k}')$, respectively. The angles α and α' are the angles between $\vec{p}$ and $\vec{k}$, and between $\vec{p}$ and $\vec{k}'$, respectively. Throughout this paper, we use the natural unit $\hbar = c = 1$ unit, unless otherwise stated explicitly.

We expand eq. (12) in powers of Δx by assuming $\Delta x \ll 1$. We obtain the Fokker-Planck expansion

$$\begin{aligned} \frac{\partial n(\omega)}{\partial t} &= 2\left[\frac{\partial n}{\partial x} + n(1+n)\right] I_1 \\ &+ 2\left[\frac{\partial^2 n}{\partial x^2} + 2(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] I_2 \\ &+ 2\left[\frac{\partial^3 n}{\partial x^3} + 3(1+n)\frac{\partial^2 n}{\partial x^2} + 3(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] I_3 \\ &+ \cdots , \end{aligned} \tag{17}$$

where

$$I_k \equiv \frac{1}{k!}\int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' \, W \, f(E) \, (\Delta x)^k . \tag{18}$$

Analytic integration of eq. (18) is not possible except for doing power series expansions of the integrand. The calculation of I_k was performed with a symbolic manipulation computer algebra package *Mathematica.*

Finally we show the result of the Sunyaev-Zel'dovich effect of the cluster of galaxies. We have obtained the following expression for the fractional distortion of the photon spectrum:

$$\frac{\Delta n(X)}{n_0(X)} = \frac{y\,\theta_e X e^X}{e^X - 1}\left[Y_0 + \theta_e Y_1 + \theta_e^2 Y_2 + \theta_e^3 Y_3\right] , \tag{19}$$

$$Y_0 = -4 + \tilde{X} , \tag{20}$$

$$Y_1 = -10 + \frac{47}{2}\tilde{X} - \frac{42}{5}\tilde{X}^2 + \frac{7}{10}\tilde{X}^3 + \tilde{S}^2\left(-\frac{21}{5} + \frac{7}{5}\tilde{X}\right) , \tag{21}$$

$$\begin{aligned} Y_2 &= -\frac{15}{2} + \frac{1023}{8}\tilde{X} - \frac{868}{5}\tilde{X}^2 + \frac{329}{5}\tilde{X}^3 - \frac{44}{5}\tilde{X}^4 + \frac{11}{30}\tilde{X}^5 \\ &+ \tilde{S}^2\left(-\frac{434}{5} + \frac{658}{5}\tilde{X} - \frac{242}{5}\tilde{X}^2 + \frac{143}{30}\tilde{X}^3\right) \\ &+ \tilde{S}^4\left(-\frac{44}{5} + \frac{187}{60}\tilde{X}\right) , \end{aligned} \tag{22}$$

$$\begin{aligned} Y_3 &= \frac{15}{2} + \frac{2505}{8}\tilde{X} - \frac{7098}{5}\tilde{X}^2 + \frac{14253}{10}\tilde{X}^3 - \frac{18594}{35}\tilde{X}^4 \\ &+ \frac{12059}{140}\tilde{X}^5 - \frac{128}{21}\tilde{X}^6 + \frac{16}{105}\tilde{X}^7 \\ &+ \tilde{S}^2\left(-\frac{7098}{10} + \frac{14253}{5}\tilde{X} - \frac{102267}{35}\tilde{X}^2 + \frac{156767}{140}\tilde{X}^3 - \frac{1216}{7}\tilde{X}^4 + \frac{64}{7}\tilde{X}^5\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \tilde{S}^4\left(-\frac{18594}{35}+\frac{205003}{280}\tilde{X}-\frac{1920}{7}\tilde{X}^2+\frac{1024}{35}\tilde{X}^3\right)\\
&+ \quad \tilde{S}^6\left(-\frac{544}{21}+\frac{992}{105}\tilde{X}\right). \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
X &\equiv \frac{\omega}{k_B T_0}, & (24)\\
y &\equiv \sigma_T \int d\ell N_e, & (25)\\
\tilde{X} &\equiv X \coth\left(\frac{X}{2}\right), & (26)\\
\tilde{S} &\equiv \frac{X}{\sinh\left(\frac{X}{2}\right)}, & (27)
\end{aligned}
$$

where σ_T is the Thomson scattering cross section and N_e is the electron number density. The expansion parameter θ_e is defined by

$$\theta_e \equiv \frac{k_B T_e}{mc^2}. \qquad (28)$$

4 Summary

In the present paper, we have reviewed the results of the symbolic calculation in physics. First we have constructed a model code *model.PIN* for *the FeynArts*, which is a package software for *the Mathematica.* We have applied the code to the pion photoproduction reaction. With the code, we have calculated the Feynman amplitudes and the cross sections for the reaction.

As the second example, we have demonstrated the symbolic calculation in astrophysics. The Sunyaev-Zel'dovich effect has been calculated with the power series expansion method. The calculation has been performed with *the Mathematica.* The analytic expressions for the SZ effect have been obtain up to $O(\theta_e)$.

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