JOSAI MATHEMATICAL MONOGRAPHS
Vol. **2** *(2000) pp. 43 - 68*

Calculation of logarithmic function with Continued Fraction Expansions

Isao MAKINO* Takeshi AOYAMA†

Kogakuin Univ. Kogakuin Univ.

1 Introduction

In the study of engineering, we usually need calculations of elementary transcendental functions like exponential or logarithmic functions. We aim to get values of elementary transcendental functions quickly with arbitrary precision. In general, we know some calculation methods. For example, Taylor expansion, asymptotic expansion and continued fraction expansions, show us how to calculate a functions value. Recently, it is said that computer CPUs are good[skillful] at integer operation. In the above calculation method, continued fraction expansions is good for calculating mainly integer operations because it has good properties. So we have thought whether or not it is a good idea for us to make fast calculations on personal computer with continued fraction.

The purpose of this study is to investigate how quickly we can calculate logarithm using continued fraction. We have used natural logarithmic function as an example. In the following section, logarithmic function we say means natural logarithmic function.

We thank Prof. Hirayama and Hilano, for their suggestions and advice.

2 Continued Fraction

First, to sum up the major characteristics of continued fraction and thier properties we need some definitions. The following formula is called continued fraction:

$$q_0 + \cfrac{p_1}{q_1 + \cfrac{p_2}{q_2 + \cfrac{p_3}{\ddots \, \cfrac{}{q_{n-1} + \cfrac{p_n}{q_n + \ddots}}}}}, \tag{1}$$

*makino@sin.cc.kogakuin.ac.jp
†aotake@trex.cc.kogakuin.ac.jp

where, $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n$ are integers. It is called a finite continued fraction when n is finite. Other wise, it is called an infinite continued fraction. It has become customary to write continued fraction in a typographically more convenient form like the following:

$$q_0 + \frac{p_1|}{|q_1} + \frac{p_2|}{|q_2} + \cdots + \frac{p_n|}{|q_n} + \cdots. \tag{2}$$

2.1 General property of continued fraction

If the continued fraction (2) is converted to a fraction $\frac{P_n}{Q_n}$, then following theorem is obtained (See [2, 3]):

Theorem 1
Let $P_{-1} = 1, Q_{-1} = 0$.

$$\begin{cases} P_n = q_n P_{n-1} + p_n P_{n-2} \\ Q_n = q_n Q_{n-1} + p_n Q_{n-2} \end{cases} \quad (\text{for } n = 1, 2, 3, \cdots) \tag{3}$$

Theorem 2
Let $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n > 0$. *If*

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{n} p_i}{Q_n Q_{n-1}} = 0,$$

then P_n/Q_n *is convergent. Put the convergence* α, *then*

$$\left| \frac{P_n}{Q_n} - \alpha \right| < \frac{\prod_{i=1}^{n} p_i}{Q_n Q_{n-1}} \quad (p_i > 0) \tag{4}$$

By Theorem 1 and Theorem 2, the operations which we use in our calculations are addition, subtraction and multiplication except division operating at last one times. The error evaluation formula is derived from (4). However little is known about concrete error evaluation formulas for most functions. Thus we need to derive it when we calculate the value of functions using continued fraction.

3 Calculation of logarithmic function

As formula of continued fraction expansions for logarithmic function, the following continued fraction formulas are known for logarithmic functions:

$$\log \frac{1+z}{1-z} = \frac{2z|}{|1} + \frac{-z^2|}{|3} + \frac{-4z^2|}{|5} + \cdots + \frac{-n^2 z^2|}{|2n+1} + \cdots. \tag{5}$$

$$\log(1+z) = \frac{z|}{|1} + \frac{z|}{|2} + \frac{z|}{|3} + \frac{2z|}{|5} + \cdots + \frac{nz|}{|2} + \frac{nz|}{|2n+1} + \cdots.$$

We would like to show some calculation methods of logarithmic function with continued fraction expansions.

If an error evaluation formula derived from (4) is a "good" formula, we are able to reduce the calculation time. But if an error evaluation formula derived from (4) is a "bad" formula, the calculation time is long. In this paper, a "good" formula means a formula which is small in "the number of continued fraction expansion terms it uses to get the value of functions with necessary precision". Let us call the number the "loop numbers".

3.1 Error evaluation formula

To calculate with mainly integer operation, in equation (5) let $z = \frac{p}{q}$ (p, q are integers). Then following formula is derived:

$$\log\frac{1+z}{1-z} = \frac{2p|}{|q} + \frac{-p^2|}{|3q} + \frac{-4p^2|}{|5q} + \cdots + \frac{-n^2p^2|}{|(2n+1)q} + \cdots. \tag{6}$$

By equation (6) and equation (4), the following equation is obtained:

$$\begin{cases} P_n = (2n-1)qP_{n-1} - (n-1)^2p^2P_{n-2}, \\ Q_n = (2n-1)qQ_{n-1} - (n-1)^2p^2Q_{n-2}. \end{cases} \quad (n = 1, 2, 3, \cdots) \tag{7}$$

More over we get

Theorem 3

$$nqQ_{n-1} - n^2p^2Q_{n-2} > 0 \quad (0 < p < q)$$

Next theorem is derived from Theorem 3.

Theorem 4

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < z^{2n+1}\frac{1}{n+1} \quad (0 < z < 1) \tag{8}$$

[Simple proof] By Theorem 3, Q_n in reccusive formula (7) is changed like the following: $Q_n = (n-1)Q_{n-1} + nQ_{n-1} - (n-1)^2p^2Q_{n-2}$. So, we apply to Theorem 3 for the proceeding equation. Then we get the following inequality: $Q_n > (n-1)Q_{n-1}$. We apply to this inequality recursive. Next, we make the term Q_nQ_{n-1} and $\prod p_i$. By substituting these to the equation (4) we are shown the theorem.

Convergence is not so good especially when if z is near to 1 in Theorem 4. So we must improve the error evaluation formula.

3.2 Conditional error evaluation formula

Under the condition $z < \frac{1}{2}$, then the following theorem is derived:

Theorem 5

$$\frac{nq}{7}Q_{n-1} - (n-1)^2p^2Q_{n-2} > 0 \quad (n \geq 2, 2p < q).$$

By Theorem 5, the following theorem is derived like Theorem 4:

Theorem 6

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < \frac{676}{49(2n-1)}\left(\frac{7}{13}\right)^{2n-1} z^{2n-1} \quad (n \geq 2, z < \frac{1}{2}) \tag{9}$$

3.3 General error evaluation formula

We derived the above theorems with trial and error. We want to derive them in a more general way. We get following theorem.

Theorem 7
For $p, q \in \mathbf{Z}, 1 < \alpha \in \mathbf{R}, 0 < k \in \mathbf{R}, \frac{p}{q} < \frac{1}{\alpha}$ and $k = 2\alpha^2 + 1$, the following formula holds.

$$\frac{nq}{k}Q_{n-1} - (n-1)^2p^2Q_{n-2} > 0 \quad \text{for } n = 2, 3, \cdots.$$

Moreover,

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < \frac{(1-z)^2}{2n}\left(\frac{2\alpha^2-1}{4\alpha^2-3}\right)^{2n-1} z^{2n-1}.$$

Overview of proof We prove the theorem by induction. Let $k > 0$ and let $\alpha > 1$ and $\frac{p}{q} < \frac{1}{\alpha}$. When $n = 1$, obviously. When $n = 2$,

$$\begin{aligned}\frac{2q}{k}Q_1 - p^2Q_0 &= \frac{2q}{k}(3q^2 - p^2) - p^2q \\ &> \frac{1}{k}(6\alpha^2 - (2\alpha^2+1))p^2q \\ &= \frac{(4\alpha^2-1)p^2q}{k} > 0.\end{aligned}$$

We assume

$$\frac{nq}{k}Q_{n-1} - (n-1)^2p^2Q_{n-2} > 0 \tag{10}$$

Let n change $n+1$. Then we get

$$\begin{aligned}\frac{(n+1)q}{k}Q_n - n^2p^2Q_{n-1} &> \frac{(n-1)^2p^2}{kqn}\left[\left\{(2k-1)\alpha^2 - k^2\right\}n^2 + \right. \\ &\quad \left. +(k-1)\alpha^2 n - k\alpha^2\right]p^2Q_{n-2}\end{aligned}$$

by equation (10). Let the content of the bracket [ ] be $f(k)$, then the coefficient part of n^2 in $f(k)$ is positive where $n = 2, 3, 4, \cdots$ because if $\alpha^2 - \alpha\sqrt{\alpha^2(\alpha+1)(\alpha-1)} < k < \alpha^2 + \alpha\sqrt{\alpha^2(\alpha+1)(\alpha-1)}$, then let determination equation of $f(k)$ be D, $D = (k-1)^2\alpha^4 +$

$4\left\{(2k-1)\alpha^2 - k^2\right\} k\alpha^2 > 0$, and let solution of $f(k)$ be $\alpha, \beta (\alpha < \beta)$, then $\beta < 2$. So, the right hand side of above formula is positive. Under the condition

$$0 < k < \alpha^2 + \alpha\sqrt{\alpha^2(\alpha+1)(\alpha-1)}, \tag{11}$$

We obtain the following formula by (4).

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < \frac{2}{n\alpha^2}\left(\frac{k}{2k-1}\right)^{2n-1} z^{2n-1}.$$

Moreover, We apply to arithmetic - geometric mean for the right hand side of (11), then $2\alpha^2 - 1 < \alpha^2 + \alpha\sqrt{\alpha^2(\alpha+1)(\alpha-1)}$. Let $k = 2\alpha^2 - 1$. Then we can get the following formula:

$$\left|\frac{P_n}{Q_n} - \log\frac{1+z}{1-z}\right| < \frac{(1-z)^2}{2n}\left(\frac{2\alpha^2-1}{4\alpha^2-3}\right)^{2n-1} z^{2n-1}.$$

3.4 Divided calculation with matrix

Let $P_{m,n}/Q_{m,n}$ define

$$\frac{P_{m,n}}{Q_{m,n}} = \cfrac{p_m}{q_m + \cfrac{p_{m+1}}{q_{m+1} + \cfrac{p_{m+2}}{\ddots \cfrac{}{q_{n-1} + \cfrac{p_n}{q_n}}}}}. \tag{12}$$

Then recurrence equation of Theorem 1 is expressed as following [2]:

$$\begin{cases} P_n = Q_{m,n}P_{m-1} + P_{m,n}P_{m-2} \\ Q_n = Q_{m,n}Q_{m-1} + P_{m,n}Q_{m-2} \end{cases} \quad (n = 1, 2, 3, \cdots) \tag{13}$$

Namely, we don't calculate P_k, Q_k in order from $k = 1$ to $k = n$. We calculate respectively P_k, Q_k in $k = 1, 2, \cdots, m$ and $k = m+1, m+2, \cdots, n$. Finally, we calculate both fractions by using (13). In addition, the method of calculation is the same to the multiplication of matrices. The recurrence equation of theorem 1 is expressed as the following:

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} P_{n-2} & P_{n-1} \\ Q_{n-2} & Q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & p_n \\ 1 & q_n \end{pmatrix} \tag{14}$$

Let

$$M_k = \begin{pmatrix} 0 & p_k \\ 1 & q_k \end{pmatrix},$$
$$M_{1,k} = \begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix}.$$

Then

$$M_{1,n} = M_1 M_2 \cdots M_n = M_{1,n-1} M_n.$$

We call the calculation method of above equation "divided calculation".

4 Experiment

Before explaining the procedure of the experiment, we sum up the symbols we use.

Ef_1	$Ef_1 = z^{2n+1}\dfrac{1}{n+1} \quad (0 < z < 1)$
Ef_2	$Ef_1 = \dfrac{676}{49(2n-1)}\left(\dfrac{7}{13}\right)^{2n-1} z^{2n-1} \quad (n \geq 2, z < \dfrac{1}{2})$
Ef	Ef_1 or Ef_2.
N	The number of the significant figures.
l_m	$l_{min} = \min\left\{n \in \mathbf{Z} \,\middle\|\, \left\|\dfrac{P_n}{Q_n} - \log\dfrac{1+z}{1-z}\right\| < 10^{-N}\right\}$
l_1	$l_1 = \min\left\{n \in \mathbf{Z} \,\middle\|\, z^{2n+1}\dfrac{1}{n+1} < 10^{-N}\right\}$ (Theorem 4)
l_2	$l_2 = \min\left\{n \in \mathbf{Z} \,\middle\|\, \dfrac{676}{49(2n-1)}\left(\dfrac{7}{13}\right)^{2n-1} z^{2n-1} < 10^{-N}\right\}$ (Theorem 6)
l	The loop number, l_1 or l_2.
T_m	The calculation time used l_m.
T_1	The calculation time used l_1.
T_2	The calculation time used l_2.
T	T_m, T_1 or T_2
PARI	The calculation time by PARI.
Total	T_m, T_1 or T_2 + the time of garbage collection.
PARI-Total	*PARI* + the time of garbage collection.
`WIDTH`	The base number of term in "divided calculation".

4.1 loop number

We call this the loop number to error evaluation formula.We examine as the following procedure.

1. Calculate minimum of loop numbers l_m, and make table of l_m.
2. Calculate loop number l_1.
3. Calculate loop number l_2.
4. Compare these numbers.

Here we determine the minimum of loop numbers l_m according to the following procedure. Assume that we want to get a function value with a high precision of more than N in the calculation, and let $X_n = \frac{P_n}{Q_n}$. Then,

1. Calculate X_n, X_{n+1}.

2. Calculate $E_n = X_{n+1} - X_n$

3. The number which firstly satisfies $\frac{1}{10^N} - E_n > 0$ is minimum of loop numbers l_m.

4. If $\frac{1}{10^N} - E_n \leq 0$, then let n change $n+1$, and return step1.

4.2 Measure of calculation time

We want to obtain the T of $\log(a)$, with continued fraction expansion. We check the effectiveness of the calculation in comparison with software PARI-GP. We examine the following procedures:

1. Measure calculation time $PARI$ of $\log(a)$ by PARI-GP.

2. Measure calculation time T_1 of $\log(a)$ with l_1. We examine following two cases.
   (a) Use recursive formula (3).
   (b) Use equation (14), i.e. "divided calculation".

3. Measure T of $\log(a)$ with l_2. This calculation is used matrix.

When we calculate matrix, we use the following procedure. First, let

$$f(i,j) = M_i M_{i+1} \cdots M_j$$

1. Get loop number l by Ef for given N.

2. Determine `WIDTH`.

3. Calculate $f(1,l)$. In calculation $f(L,H)$,
   (a) if $H - L >$ `WIDTH` then
      - Calculate $f(L, M-1) \times f(M,H)$, where $m = [(L+H)/2]$.
   (b) if $H - L \leq$ `WIDTH` then
      - Calculate $M_L M_{L+1} \cdots M_H$ directory.

Here, the symbol [ ] is Gauss's symbol. Thus $[a] = \max\{n \in \mathbf{Z} | n \leq a\}$.

5 Result of experiment and consideration

By the experimentation in the loop number, we get the following: [1)]

1. N in the calculation and loop number are in proportion to each other.

[1)] Question:On the experimentation in the loop numbers,...

2. The larger the significant figures in the calculation is, the larger l_1 is than l_m.

3. $\frac{l_1}{l_m}$ is almost constant.

4. In calculation $\log 2$, l_2 is about sixty percent smaller than l_1.

5. l_2 is closer to l_m than l_1.

By the experiment in T, we get following things:

1. T becomes long if N is large.

2. T by Ef_1 with recursive equation (7) is longer than $PARI$.

3. T by Ef_1 with "divided calculation" is shorter than $PARI$.

4. T is long when the number of significant figures of z in $\log \frac{1+z}{1-z}$ is large.

5. T is longer than $PARI$ when the number of significant figures of z in $\log \frac{1+z}{1-z}$ is large.

6. We can calculate faster than PARI-GP by using the calculation method "divided calculation" at N of z is smaller than about 20.

By the results of the experiment of the loop number, we found the following things: The value of z is close 1 when the value $a \left(= \dfrac{1+\frac{p}{q}}{1-\frac{p}{q}} \right)$ to substitute for $\log(a)$ is large. Thus for N the loop number is large when the value a to substitute for $\log(a)$ is large. On the other hand, Ef_2 has the term $\left(\frac{7}{13}\right)^{2n-1}$, so Ef_2 is greater than Ef_1.

By the results of the experiment of calculation time, we found the following things: If the substituting value a for $\log(a)$ is large, then T is long even under reccusive equation (7) with l_m. As a result we cannot calculate quickly if we use the reccusive equation without "divided calculation". But "divided calculation" is a "good" method. If we use this method, then we can calculate faster than PARI-GP. The reason for this is the bit length of data in memory is shorter than the bit length in calculation used in the reccusive equation. It is the same reason that T is short when we calculate with "divided calculation".

6 Summary

In this paper, we tried to derive the error evaluation formula for $\log \frac{1+z}{1-z}$, and show that we were able to calculate faster than PARI when we used "divided calculation". Following our work, we list following thing:

- How to calculate faster than PARI $\log(a)$ where for $a \geq 3$

Thus we use the error evaluation formula in theorem 7 and find what value α is better in fast calculation. This work includes using the "divided calculation" method. Now, we consider to apply the following formula:

$$\log(a) = \log\left(\frac{a}{2^n}\right) + n \cdot \log(2) \quad \text{where} \quad a \geq 3.$$

Moreover, we want to install to Risa/Asir the command `log`.

A Experiments of Loop Numbers

A.1 Minimum of Loop Numbers

We show minimum of major loop numbers l_m at Table 1 and the ratio of N to log(2), log(3) and log(5) at Table 2. Each row of these tables is shown as the following:

N The required number of significant figures.

log(2) l_m for log(2).

log(3) l_m for log(3).

log(5) l_m for log(5).

$\frac{\log(2)}{N}$ The ratio of l_m to N for log(2).

$\frac{\log(3)}{N}$ The ratio of l_m to N for log(3).

$\frac{\log(5)}{N}$ The ratio of l_m to N for log(5).

The machine environment of these experiments is shown in the following table:

CPU	Cyrix 6x86L RP200+ 150MHz
Memory	SIMM 64M Byte
OS	FreeBSD 2.2.5
Asir	Version 950831.

Table 1: minimum of loop numbers l_{min}

N	log(2)	log(3)	log(5)
1000	656	876	1200
2000	1309	1752	2398
3000	1964	2628	3597
4000	2619	3504	4795
5000	3273	4379	5993
6000	3920	5247	7180
7000	4574	6124	8379
8000	5229	7000	9577
9000	5883	7874	10775
10000	6538	8750	11976

Table 2: The ratio of N to l_m

N	$\frac{\log(2)}{N}$	$\frac{\log(3)}{N}$	$\frac{\log(5)}{N}$
1000	0.7	0.9	1.2
2000	0.7	0.9	1.2
3000	0.7	0.9	1.2
4000	0.7	0.9	1.2
5000	0.7	0.9	1.2
6000	0.7	0.9	1.2
7000	0.7	0.9	1.2
8000	0.7	0.9	1.2
9000	0.7	0.9	1.2
10000	0.7	0.9	1.2

A.2 Loop Number 1

We show loop number l_1 at Table 3 ,the ratio of l_1 to l_m at Table 4 and the ratio of l_1 to N at Table 5. The items in each row have similar meaning to the Table of Minimum of Loop Numbers.

The items of each row in Table 3 have the following meanings:

N	The required number of significant figures.
log(2)	l_1 for log(2).
log(3)	l_1 for log(3).
log(5)	l_1 for log(5).

The items in each row in Table 4 have the following meanings:

N	The required number of significant figures.
log(2)	The ratio of l_1 to l_m, i.e. $\frac{l_1}{l_m}$, for log(2).
log(3)	The ratio of l_1 to l_m, i.e. $\frac{l_1}{l_m}$, for log(3).
log(5)	The ratio of l_1 to l_m, i.e. $\frac{l_1}{l_m}$, for log(5).

The items in each row in Table 5 have the following meanings:

N	The required number of significant figures.
$\frac{\log(2)}{N}$	The ratio of l_1 to N, i.e. $\frac{l_1}{N}$, for log(2).
$\frac{\log(3)}{N}$	The ratio of l_1 to N, i.e. $\frac{l_1}{N}$, for log(3).
$\frac{\log(5)}{N}$	The ratio of l_1 to N, i.e. $\frac{l_1}{N}$, for log(5).

The machine environment of these experiments is shown as following table:

CPU	MMX-Pentium 200MHz (233MHz)
Memory	SIMM 64M Byte
OS	PlamoLinux 1.3 + kernel 2.2.3
Asir	Version 950831.

Table 3: loop number l_1

N	log(2)	log(3)	log(5)
1000	1043	1655	2831
2000	2092	3317	5663
3000	3139	4976	8509
4000	4185	6634	11345
5000	5231	8293	14182
6000	6278	9952	17018
7000	7324	11610	19854
8000	8371	13269	22691
9000	9417	14928	25527
10000	10463	16616	28364

Table 4: The ratio of l_1 to l_m

N	log(2)	log(3)	log(5)
1000	1.6	1.9	1.4
2000	1.6	1.9	1.4
3000	1.6	1.9	1.4
4000	1.6	1.9	1.4
5000	1.6	1.9	1.4
6000	1.6	1.9	1.4
7000	1.6	1.9	1.4
8000	1.6	1.9	1.4
9000	1.6	1.9	1.4
10000	1.6	1.9	1.4

Table 5: The ratio of l_1 to N

N	$\frac{\log(2)}{N}$	$\frac{\log(3)}{N}$	$\frac{\log(5)}{N}$
1000	1.0	1.7	2.8
2000	1.0	1.7	2.8
3000	1.0	1.7	2.8
4000	1.0	1.7	2.8
5000	1.0	1.7	2.8
6000	1.0	1.7	2.8
7000	1.0	1.7	2.8
8000	1.0	1.7	2.8
9000	1.0	1.7	2.8
10000	1.0	1.7	2.8

A.3 Loop Number 2

We show loop number l_2 of log(2) at Table 6.

The item of each row in Table 6 mean as following:

- N The required number of significant figures.
- l_1 l_1 for log(2).
- l_2 l_2 for log(2).
- $\frac{l_2}{N}$ The ratio of l_2 to N of log(2).
- $\frac{l_2}{l_1}$ The ratio of l_2 to l_1 of log(2).

The machine environment of these experiments is shown as following table:

CPU	Celeron 300A Dual (463MHz)
Memory	SDRAM 128M Byte
OS	PlamoLinux 1.3 + kernel 2.2.3
Asir	Version 981001.

Table 6: loop number l_2 of log(2)

N	l_1	l_2	$\frac{l_2}{N}$	$\frac{l_2}{l_1}$
10000	10439	6704	0.6	0.7
20000	20986	13427	0.6	0.7
30000	31357	20107	0.6	0.7
40000	41904	26787	0.6	0.7
50000	52451	33466	0.6	0.7

B Experiment of Calculation Time

B.1 The calculation time by recursive equation with l_m

Tables from Table 7 to Table 9 show T_{calc} by reccusive equation with l_{min} for each N. Each row of the table shows the following:

N	The required number of significant figures.
T-$r.eq.$ with l_m	The calculation time by reccusive equation with l_m.
$PARI$	The calculation time by PARI/GP.

Item T-$r.eq.$ or $PARI$ in these tables describe "calculation time + garbage collection time (total time)" each other. The unit of these items is "second".

The machine environment of these experiments is the same environment in the experiment of minimum of loop numbers.

Table 7: T by r.eq. with l_m and $PARI$ for log(2)

N	T-$r.eq.$ with l_m	$PARI$
1000	0.160+ 0.050(0.210)	0.080+ 0.292(0.372)
2000	0.560+ 0.280(0.840)	0.430+ 1.008(1.438)
3000	1.250+ 0.800(2.050)	1.220+ 2.912(4.132)
4000	2.270+ 1.310(3.580)	2.200+ 4.501(6.701)
5000	3.410+ 2.630(6.040)	4.610+ 9.247(13.857)
6000	5.110+ 3.630(8.740)	6.530+13.040(19.570)
7000	7.040+ 4.780(11.820)	8.940+17.970(26.910)
8000	9.090+ 6.520(15.610)	11.980+23.820(35.800)
9000	11.770+ 8.440(20.210)	15.490+31.020(46.510)
10000	14.570+ 8.700(23.270)	25.290+50.670(75.960)

Table 8: T by r.eq. with l_m and $PARI$ for log(3)

N	T-$r.eq.$ with l_m	$PARI$
1000	0.250+ 0.130(0.380)	0.080+ 0.088(0.168)
2000	0.990+ 0.680(1.670)	0.430+ 0.842(1.272)
3000	2.210+ 1.530(3.740)	1.240+ 2.505(3.745)
4000	3.990+ 3.190(7.180)	2.250+ 4.193(6.443)
5000	6.470+ 4.710(11.180)	4.670+ 9.288(13.958)
6000	9.130+ 6.470(15.600)	6.610+13.100(19.710)
7000	12.740+ 7.990(20.730)	8.970+18.000(26.970)
8000	16.690+ 9.670(26.360)	12.120+24.200(36.320)
9000	21.180+11.910(33.090)	15.650+31.380(47.030)
10000	26.240+16.560(42.800)	25.160+50.670(75.830)

Table 9: T by r.eq. with l_m and $PARI$ for log(5)

N	T-$r.eq.$ with l_m	$PARI$
1000	0.470+ 0.270(0.740)	0.090+ 0.087(0.177)
2000	1.880+ 1.310(3.190)	0.430+ 0.672(1.103)
3000	4.330+ 3.500(7.830)	1.230+ 2.275(3.505)
4000	7.780+ 5.810(13.590)	2.280+ 4.643(6.923)
5000	12.420+ 8.070(20.490)	4.670+ 9.269(13.939)
6000	17.900+10.650(28.550)	6.590+13.100(19.690)
7000	24.830+13.620(38.450)	9.090+18.170(27.260)
8000	32.390+17.590(49.980)	12.050+24.000(36.050)
9000	41.430+21.610(63.040)	15.600+31.150(46.750)
10000	51.370+29.620(80.990)	25.490+50.850(76.340)

Table 10 shows the the ratio of T by reccusive equation with l_m to $PARI$ when let $T = 1$. Note that a letter T means the "calculation time" in the description "calculation time + garbage collection(total time)".

Table 10: The ratio of T by r.eq. with l_m to $PARI$

N	log(2)	log(3)	log(5)
1000	0.5	0.3	0.2
2000	0.8	0.4	0.2
3000	1.0	0.6	0.3
4000	1.0	0.6	0.3
5000	1.4	0.7	0.4
6000	1.3	0.7	0.4
7000	1.3	0.7	0.4
8000	1.3	0.7	0.4
9000	1.3	0.7	0.4
10000	1.7	1.0	0.5

B.2 T by reccusive equation with l_1

Each symbol in the tables means the following as long as we don't point at:

N	The required number of significant figures.
$Total$	Calculation time + Garbage collection time.
T-$r.eq.$ with l_1	The calculation time by reccusive equation with l_1.
$Total$-$r.eq.$ with l_1	The time of $Total$ by reccusive equation with l_1.
$PARI$	The calculation time by PARI/GP.

The machine environment of these experiments is the same environment in the experiment of minimum of loop numbers.

Table 11: T by r.eq. with l_1 and $PARI$ for log(2)

N	T-$r.eq.$ with l_1	$PARI$
1000	0.310+0.210(0.520)	0.080+0.292(0.372)
2000	1.340+0.960(2.300)	0.430+1.008(1.438)
3000	3.070+2.410(5.480)	1.220+2.912(4.132)
4000	5.630+4.370(10.000)	2.200+4.501(6.701)
5000	8.770+6.920(15.690)	4.610+9.247(13.857)
6000	13.030+9.930(22.960)	6.530+13.040(19.570)
7000	17.670+13.170(30.840)	8.940+17.970(26.910)
8000	23.310+15.910(39.220)	11.980+23.820(35.800)
9000	29.670+19.070(48.740)	15.490+31.020(46.510)
10000	36.970+19.860(56.830)	25.290+50.670(75.960)

Table 12: T by r.eq. with l_1 and $PARI$ for log(3)

N	T-$r.eq.$ with l_1	$PARI$
1000	0.850+0.610(1.460)	0.080+0.088(0.168)
2000	3.680+2.710(6.390)	0.430+0.842(1.272)
3000	8.170+6.240(14.410)	1.240+2.505(3.745)
4000	14.780+9.220(24.000)	2.250+4.193(6.443)
5000	23.260+14.510(37.770)	4.670+9.288(13.958)
6000	33.750+18.440(52.190)	6.610+13.100(19.710)
7000	46.380+25.150(71.530)	8.970+18.000(26.970)
8000	61.180+32.640(93.820)	12.120+24.200(36.320)
9000	77.680+42.070(119.750)	15.650+31.380(47.030)
10000	97.380+48.520(145.900)	25.160+50.670(75.830)

Table 13: T by r.eq. with l_1 and $PARI$ for log(5)

N	T-$r.eq.$ with l_1	$PARI$
1000	2.660+1.980(4.640)	0.090+0.087(0.177)
2000	10.920+8.830(19.750)	0.430+0.672(1.103)
3000	25.090+14.160(39.250)	1.230+2.275(3.505)
4000	45.160+25.880(71.040)	2.280+4.643(6.923)
5000	71.870+39.000(110.870)	4.670+9.269(13.939)
6000	106.600+48.230(154.830)	6.590+13.100(19.690)
7000	151.400+74.560(225.960)	9.090+18.170(27.260)
8000	205.400+93.730(299.130)	12.050+24.000(36.050)
9000	262.400+116.500(378.900)	15.600+31.150(46.750)
10000	328.500+123.900(452.400)	25.490+50.850(76.340)

Table 14: The ratio of T by r.eq. with l_m to $PARI$

N	log(2)	log(3)	log(5)
1000	3.9	10.6	29.6
2000	3.1	8.6	25.4
3000	2.5	6.6	20.4
4000	2.6	6.6	19.8
5000	1.9	5.0	15.4
6000	2.0	5.1	16.2
7000	2.0	5.2	16.7
8000	1.9	5.0	17.0
9000	1.9	5.0	16.8
10000	1.5	3.9	12.9

Table 15: $Total$ by r.eq. with l_1 and $PARI$ for log(2)

N	$Total$-$r.eq.$ with l_1	$PARI$
1000	0.310+0.210(0.520)	0.080+0.292(0.372)
2000	1.340+0.960(2.300)	0.430+1.008(1.438)
3000	3.070+2.410(5.480)	1.220+2.912(4.132)
4000	5.630+4.370(10.000)	2.200+4.501(6.701)
5000	8.770+6.920(15.690)	4.610+9.247(13.857)
6000	13.030+9.930(22.960)	6.530+13.040(19.570)
7000	17.670+13.170(30.840)	8.940+17.970(26.910)
8000	23.310+15.910(39.220)	11.980+23.820(35.800)
9000	29.670+19.070(48.740)	15.490+31.020(46.510)
10000	36.970+19.860(56.830)	25.290+50.670(75.960)

Table 16: $Total$ by r.eq. with l_1 and $PARI$ for log(3)

N	T-$r.eq.$ with l_1	$PARI$
1000	0.850+0.610(1.460)	0.080+0.088(0.168)
2000	3.680+2.710(6.390)	0.430+0.842(1.272)
3000	8.170+6.240(14.410)	1.240+2.505(3.745)
4000	14.780+9.220(24.000)	2.250+4.193(6.443)
5000	23.260+14.510(37.770)	4.670+9.288(13.958)
6000	33.750+18.440(52.190)	6.610+13.100(19.710)
7000	46.380+25.150(71.530)	8.970+18.000(26.970)
8000	61.180+32.640(93.820)	12.120+24.200(36.320)
9000	77.680+42.070(119.750)	15.650+31.380(47.030)
10000	97.380+48.520(145.900)	25.160+50.670(75.830)

Table 17: T by r.eq. with l_1 and $PARI$ for log(5)

N	T-$r.eq.$ with l_1	$PARI$
1000	2.660+1.980(4.640)	0.090+0.087(0.177)
2000	10.920+8.830(19.750)	0.430+0.672(1.103)
3000	25.090+14.160(39.250)	1.230+2.275(3.505)
4000	45.160+25.880(71.040)	2.280+4.643(6.923)
5000	71.870+39.000(110.870)	4.670+9.269(13.939)
6000	106.600+48.230(154.830)	6.590+13.100(19.690)
7000	151.400+74.560(225.960)	9.090+18.170(27.260)
8000	205.400+93.730(299.130)	12.050+24.000(36.050)
9000	262.400+116.500(378.900)	15.600+31.150(46.750)
10000	328.500+123.900(452.400)	25.490+50.850(76.340)

Table 18: The ratio of $Total$ by r.eq. with l_1 to $PARI$

N	log(2)	log(3)	log(5)
1000	3.9	10.6	29.6
2000	3.1	8.6	25.4
3000	2.5	6.6	20.4
4000	2.6	6.6	19.8
5000	1.9	5.0	15.4
6000	2.0	5.1	16.2
7000	2.0	5.2	16.7
8000	1.9	5.0	17.0
9000	1.9	5.0	16.8
10000	1.5	3.9	12.9

B.3 The calculation time by `divided calculation` with l_1

CPU	Celeron 300A Dual (450MHz)
Memory	SDRAM 128M Byte
OS	Plamo Linux 1.3 + kernel 2.2.3
Asir	Version 981001.

Table 19: *PARI* and T - $N = 10000$

PARI		PARI-total)
12.84sec		12.86sec.
WIDTH	T	gc(total)
1000	2.31sec	1.36sec(3.669sec)
900	2.24sec	0.91sec(3.151sec)
800	2.26sec	0.88sec(3.157sec)
700	2.2sec	0.97sec(3.176sec)
600	1.92sec	0.73sec(2.654sec)
500	1.85sec	0.82sec(2.671sec)
400	1.88sec	0.79sec(2.669sec)
300	1.71sec	0.71sec(2.424sec)
200	1.68sec	0.76sec(2.435sec)
100	1.64sec	0.68sec(2.321sec)
90	1.56sec	0.69sec(2.25sec)
80	1.56sec	0.62sec(2.185sec)
70	1.55sec	0.46sec(2.011sec)
60	1.53sec	0.49sec(2.015sec)
50	1.55sec	0.47sec(2.018sec)
40	1.53sec	0.49sec(2.027sec)
30	**1.49sec**	**0.53sec(2.011sec)**
20	1.56sec	0.45sec(2.018sec)
10	1.55sec	0.49sec(2.037sec)
9	1.53sec	0.52sec(2.044sec)
8	1.59sec	0.49sec(2.081sec)
7	1.54sec	0.54sec(2.082sec)
6	1.52sec	0.57sec(2.089sec)
5	1.64sec	0.45sec(2.089sec)

B.4 T by `divided calculation` with l_2

The machine environment of these experiments is the same environment in the experiment of T by `divided calculation` with l_1.

Table 20: T by `divided calculation` with l_2 for $\log 2$: $N = 10000$

PARI		
PARI		gc ($PARI - total$)
12.99sec		12.99sec
Risa/Asir : lpnm=10439		
WIDTH	T	gc ($Ttotal$)
No Div	12.54sec	9.51sec(22.05sec)
100	1.57sec	0.78sec(2.353sec)
80	1.54sec	0.58sec(2.12sec)
50	1.54sec	0.56sec(2.104sec)
10	1.5sec	0.6sec(2.107sec)
5	1.58sec	0.57sec(2.144sec)

C New Table(Use PARI-GP ver 2.0.16)

The machine environment of the following experiment is:

CPU	Celeron 300A Dual (463MHz)
Memory	128MByte
OS	Vine Linux 1.0 + kernel 2.2.11

C.1 Minimum of loop numbers

The table 21 shows minimum of loop numbers for given N.

Table 21: Minimum of Loop Numbers

N	log(2)	log(3)	log(5)	log(7)	log(11)	log(13)
1000	654	875	1197	1449	1851	2022
2000	1307	1749	2393	2896	3701	4044
3000	1960	2623	3590	4344	5551	6065
4000	2613	3497	4786	5791	7401	8086
5000	3266	4272	5982	7239	9251	10107
6000	3919	5246	7178	8686	11100	12128
7000	4752	6120	8375	10134	12950	14149
8000	5226	6994	9571	11581	14800	16171
9000	5879	7869	10767	13029	16650	18192
10000	6532	8743	11963	14476	18500	20213

C.2 The calculation time with l_1 and recursive equation

Table 22: log(2)

N	*Asir*[sec]			*PARI*[m-sec]
1000	0.11sec	+gc: 0.11sec	(0.2178sec)	20
2000	0.4sec	+gc: 0.61sec	(1.006sec)	100
3000	0.9sec	+gc: 1.03sec	(1.933sec)	270
4000	1.56sec	+gc: 1.88sec	(3.438sec)	490
5000	2.71sec	+gc: 2.62sec	(5.326sec)	1000
6000	3.95sec	+gc: 3.89sec	(7.845sec)	1440
7000	5.35sec	+gc: 5.39sec	(10.74sec)	1940
8000	7.22sec	+gc: 7.04sec	(14.26sec)	2630
9000	9.22sec	+gc: 9 sec	(18.22sec)	3410
10000	11.26sec	+gc: 8.45 sec	(19.71sec)	5440

Table 23: log(3)

N	*Asir*[sec]			*PARI*[m-sec]
1000	0.25sec	+gc: 0.32sec	(0.5717sec)	20
2000	1.14sec	+gc: 1.06sec	(2.205sec)	90
3000	2.55sec	+gc: 2.31sec	(4.881sec)	280
4000	4.55sec	+gc: 4.33sec	(8.879sec)	490
5000	7.21sec	+gc: 7.18sec	(14.39sec)	1010
6000	10.48sec	+gc: 9.98sec	(10.49sec)	1450
7000	14.7sec	+gc:13.87sec	(28.59sec)	1970
8000	19.88sec	+gc:17.34sec	(37.23sec)	2650
9000	25.92sec	+gc:21.16sec	(47.08sec)	3440
10000	31.49sec	+gc:22.9sec	(54.39sec)	5440

Table 24: log(5)

N	*Asir*[sec]			*PARI*[m-sec]
1000	0.8sec	+gc: 0.89sec	(1.69sec)	20
2000	3.36sec	+gc: 3.45sec	(6.812sec)	90
3000	9.02sec	+gc: 7.37sec	(15.39sec)	280
4000	14.76sec	+gc: 13.72sec	(28.48sec)	500
5000	23.57sec	+gc: 18.89sec	(42.46sec)	1010
6000	35.74sec	+gc: 24.94sec	(60.68sec)	1440
7000	53.95sec	+gc: 34.19sec	(88.15sec)	1990
8000	70.64sec	+gc: 42.3sec	(113sec)	2650
9000	91.03sec	+gc: 48.57sec	(139.6sec)	3420
10000	115.6sec	+gc: 61.71sec	(177.3sec)	5440

Table 25: log(7)

N	*Asir*[sec]			*PARI*[m-sec]
1000	1.72sec	+gc: 1.75sec	(3.473sec)	20
2000	7.36sec	+gc: 7.1sec	(14.46sec)	90
3000	17.87sec	+gc: 15.21sec	(33.08sec)	290
4000	34.31sec	+gc: 26.03sec	(60.34sec)	500
5000	52.42sec	+gc: 33.36sec	(85.78sec)	1010
6000	82.7sec	+gc: 47.13sec	(129.8sec)	1440
7000	116.9sec	+gc: 63.85sec	(180.7sec)	1990
8000	156.2sec	+gc: 78.48sec	(234.7sec)	2640
9000	202.8sec	+gc: 91.41sec	(294.3sec)	3420
10000	256.4sec	+gc:103.5sec	(359.9sec)	5440

Table 26: divided calculation : $N = 1000$

WIDTH	$Asir$[sec]		
1000	0.06sec	+ gc : 0.05sec	(0.1158sec)
500	0.04sec	+ gc : 0.03sec	(0.07804sec)
100	0.04sec	+ gc : 0.02sec	(0.05612sec)
50	0.03sec	+ gc : 0.01sec	(0.04408sec)
10	0.03sec	+ gc : 0.02sec	(0.05465sec)
5	0.04sec	+ gc : 0.02sec	(0.05817sec)

C.3 The calculation time with "divided calculation" and l_1

C.3.1 log(2)

Table 27: divided calculation : $N = 2000$

WIDTH	$Asir$[sec]		
1000	0.17sec	+ gc : 0.13sec	(0.2997sec)
500	0.1sec	+ gc : 0.1sec	(0.2026sec)
100	0.09sec	+ gc : 0.06sec	(0.1539sec)
50	0.09sec	+ gc : 0.05sec	(0.1414sec)
10	0.1sec	+ gc : 0.05sec	(0.1429sec)
5	0.11sec	+ gc : 0.04sec	(0.149sec)

Table 28: divided calculation : $N = 3000$

WIDTH	$Asir$[sec]		
1000	0.34sec	+ gc : 0.27sec	(0.6074sec)
500	0.25sec	+ gc : 0.17sec	(0.4211sec)
100	0.17sec	+ gc : 0.11sec	(0.2847sec)
50	0.17sec	+ gc : 0.09sec	(0.2641sec)
10	0.18sec	+ gc : 0.08sec	(0.2594sec)
5	0.19sec	+ gc : 0.1sec	(0.2824sec)

Table 29: divided calculation : $N = 4000$

WIDTH	$Asir$[sec]		
1000	0.41sec	+ gc : 0.35sec	(0.7639sec)
500	0.32sec	+ gc : 0.27sec	(0.5827sec)
100	0.28sec	+ gc : 0.17sec	(0.4507sec)
50	0.29sec	+ gc : 0.1sec	(0.3923sec)
10	0.26sec	+ gc : 0.13sec	(0.3867sec)
5	0.29sec	+ gc : 0.12sec	(0.4093sec)

Table 31: `divided calculation` : $N = 6000$

`WIDTH`	$Asir$[sec]		
1000	0.87sec	+ gc : 0.73sec	(1.606sec)
500	0.7sec	+ gc : 0.37sec	(1.068sec)
100	0.56sec	+ gc : 0.23sec	(0.7848sec)
50	0.54sec	+ gc : 0.21sec	(0.7497sec)
10	0.49sec	+ gc : 0.26sec	(0.7465sec)
5	0.55sec	+ gc : 0.18sec	(0.7334sec)

Table 30: `divided calculation` : $N = 5000$

`WIDTH`	$Asir$[sec]		
1000	0.65sec	+ gc : 0.49sec	(1.143sec)
500	0.49sec	+ gc : 0.27sec	(0.7657sec)
100	0.38sec	+ gc : 0.2sec	(0.5822sec)
50	0.39sec	+ gc : 0.17sec	(0.5591sec)
10	0.42sec	+ gc : 0.15sec	(0.561sec)
5	0.4sec	+ gc : 0.18sec	(0.5809sec)

Table 32: `divided calculation` : $N = 7000$

`WIDTH`	$Asir$[sec]		
1000	1.32sec	+ gc : 0.92sec	(2.244sec)
500	0.93sec	+ gc : 0.54sec	(1.478sec)
100	0.68sec	+ gc : 0.36sec	(1.04sec)
50	0.67sec	+ gc : 0.36sec	(1.033sec)
10	0.73sec	+ gc : 0.32sec	(1.053sec)
5	0.69sec	+ gc : 0.29sec	(0.9856sec)

Table 33: `divided calculation` : $N = 8000$

`WIDTH`	$Asir$[sec]		
1000	1.15sec	+ gc : 0.76sec	(1.915sec)
500	0.94sec	+ gc : 0.56sec	(1.504sec)
100	0.86sec	+ gc : 0.4sec	(1.264sec)
50	0.86sec	+ gc : 0.27sec	(1.125sec)
10	0.88sec	+ gc : 0.25sec	(1.13sec)
5	0.88sec	+ gc : 0.3sec	(1.174sec)

Table 34: `divided calculation` : $N = 9000$

WIDTH	*Asir*[sec]		
1000	1.46sec	+ gc : 1.06sec	(2.52sec)
500	1.18sec	+ gc : 0.55sec	(1.724sec)
100	1.02sec	+ gc : 0.44sec	(1.454sec)
50	1.04sec	+ gc : 0.39sec	(1.429sec)
10	1.03sec	+ gc : 0.46sec	(1.491sec)
5	1.06sec	+ gc : 0.55sec	(1.606sec)

Table 35: `divided calculation` : $N = 10000$

WIDTH	*Asir*[sec]		
1000	1.73sec	+ gc : 1.04sec	(2.769sec)
500	1.4sec	+ gc : 0.65sec	(2.051sec)
100	1.18sec	+ gc : 0.38sec	(1.567sec)
50	1.17sec	+ gc : 0.37sec	(1.535sec)
10	1.17sec	+ gc : 0.38sec	(1.546sec)
5	1.24sec	+ gc : 0.34sec	(1.581sec)

C.3.2 log(3)

Table 36: `divided calculation` : $N = 1000$

WIDTH	*Asir*[sec]		
1000	0.18sec	+ gc : 0.1sec	(0.2831sec)
500	0.1sec	+ gc : 0.08sec	(0.178sec)
100	0.07sec	+ gc : 0.03sec	(0.09934sec)
50	0.07sec	+ gc : 0.02sec	(0.0951sec)
10	0.06sec	+ gc : 0.03sec	(0.09003sec)
5	0.07sec	+ gc : 0.04sec	(0.1083sec)

Table 37: `divided calculation` : $N = 5000$

WIDTH	*Asir*[sec]		
1000	1.18sec	+ gc : 0.75sec	(1.923sec)
500	0.98sec	+ gc : 0.52sec	(1.503sec)
100	0.83sec	+ gc : 0.44sec	(1.287sec)
50	0.79sec	+ gc : 0.44sec	(1.258sec)
10	0.8sec	+ gc : 0.33sec	(1.199sec)
5	0.88sec	+ gc : 0.3sec	(1.179sec)

Table 38: `divided calculation` : $N = 10000$

WIDTH	Asir[sec]		
1000	3.38sec	+ gc : 1.74sec	(5.124sec)
500	2.92sec	+ gc : 0.86sec	(3.779sec)
100	2.61sec	+ gc : 0.78sec	(3.394sec)
50	2.65sec	+ gc : 0.54sec	(3.186sec)
10	2.64sec	+ gc : 0.57sec	(3.214sec)
5	2.67sec	+ gc : 0.63sec	(3.301sec)

C.4 The calculation time with `divided calculation` and l_2

Table 39: `divided calculation` : $N = 1000$

WIDTH	Asir[sec]		
1000	0.05sec	+ gc : 0.02sec	(0.07282sec)
500	0.03sec	+ gc : 0.03sec	(0.05948sec)
100	0.02sec	+ gc : 0.01sec	(0.03096sec)
50	0.02sec	+ gc : 0.01sec	(0.03073sec)
10	0.02sec	+ gc : 0.01sec	(0.03001sec)
5	0.03sec		(0.03205sec)

Table 40: `divided calculation` : $N = 2000$

WIDTH	Asir[sec]		
1000	0.11sec	+ gc : 0.09sec	(0.1928sec)
500	0.07sec	+ gc : 0.05sec	(0.1241sec)
100	0.05sec	+ gc : 0.03sec	(0.08205sec)
50	0.04sec	+ gc : 0.04sec	(0.07904sec)
10	0.05sec	+ gc : 0.02sec	(0.07075sec)
5	0.06sec	+ gc : 0.02sec	(0.07418sec)

Table 41: `divided calculation` : $N = 3000$

WIDTH	Asir[sec]		
1000	0.12sec	+ gc : 0.14sec	(0.2681sec)
500	0.12sec	+ gc : 0.09sec	(0.202sec)
100	0.09sec	+ gc : 0.06sec	(0.1484sec)
50	0.1sec	+ gc : 0.03sec	(0.1288sec)
10	0.08sec	+ gc : 0.04sec	(0.121sec)
5	0.09sec	+ gc : 0.04sec	(0.1279sec)

Table 42: divided calculation : $N = 4000$

WIDTH	Asir[sec]
1000	0.29sec + gc : 0.13sec (0.4173sec)
500	0.18sec + gc : 0.12sec (0.3003sec)
100	0.14sec + gc : 0.06sec (0.2004sec)
50	0.13sec + gc : 0.06sec (0.1942sec)
10	0.11sec + gc : 0.08sec (0.1979sec)
5	0.14sec + gc : 0.07sec (0.203sec)

Table 43: divided calculation : $N = 5000$

WIDTH	Asir[sec]
1000	0.4sec + gc : 0.25sec (0.6514sec)
500	0.3sec + gc : 0.15sec (0.4528sec)
100	0.17sec + gc : 0.11sec (0.2805sec)
50	0.2sec + gc : 0.07sec (0.2669sec)
10	0.19sec + gc : 0.09sec (0.2816sec)
5	0.24sec + gc : 0.05sec (0.2877sec)

Table 44: divided calculation : $N = 6000$

WIDTH	Asir[sec]
1000	0.39sec + gc : 0.2sec (0.5905sec)
500	0.32sec + gc : 0.13sec (0.4559sec)
100	0.27sec + gc : 0.09sec (0.3579sec)
50	0.25sec + gc : 0.11sec (0.3517sec)
10	0.26sec + gc : 0.1sec (0.3569sec)
5	0.28sec + gc : 0.09sec (0.3706sec)

Table 45: divided calculation : $N = 7000$

WIDTH	Asir[sec]
1000	0.54sec + gc : 0.25sec (0.7901sec)
500	0.43sec + gc : 0.18sec (0.6026sec)
100	0.37sec + gc : 0.1sec (0.4703sec)
50	0.33sec + gc : 0.12sec (0.4515sec)
10	0.34sec + gc : 0.11sec (0.4515sec)
5	0.32sec + gc : 0.15sec (0.4747sec)

Table 46: divided calculation : $N = 8000$

WIDTH	Asir[sec]
1000	0.67sec + gc : 0.34sec (1.009sec)
500	0.56sec + gc : 0.2sec (0.7644sec)
100	0.44sec + gc : 0.14sec (0.5826sec)
50	0.42sec + gc : 0.14sec (0.5581sec)
10	0.4sec + gc : 0.17sec (0.5724sec)
5	0.41sec + gc : 0.18sec (0.5855sec)

Table 47: `divided calculation` : $N = 9000$

`WIDTH`	$Asir$[sec]		
1000	0.91sec	+ gc : 0.47sec	(1.382sec)
500	0.72sec	+ gc : 0.31sec	(1.031sec)
100	0.52sec	+ gc : 0.25sec	(0.7687sec)
50	0.53sec	+ gc : 0.21sec	(0.7381sec)
10	0.53sec	+ gc : 0.22sec	(0.7451sec)
5	0.54sec	+ gc : 0.24sec	(0.7835sec)

Table 48: `divided calculation` : $N = 10000$

`WIDTH`	$Asir$[sec]		
1000	1.03sec	+ gc : 0.42sec	(1.452sec)
500	0.78sec	+ gc : 0.29sec	(1.068sec)
100	0.58sec	+ gc : 0.16sec	(0.7457sec)
50	0.59sec	+ gc : 0.15sec	(0.7354sec)
10	0.6sec	+ gc : 0.15sec	(0.7576sec)
5	0.6sec	+ gc : 0.2sec	(0.8044sec)

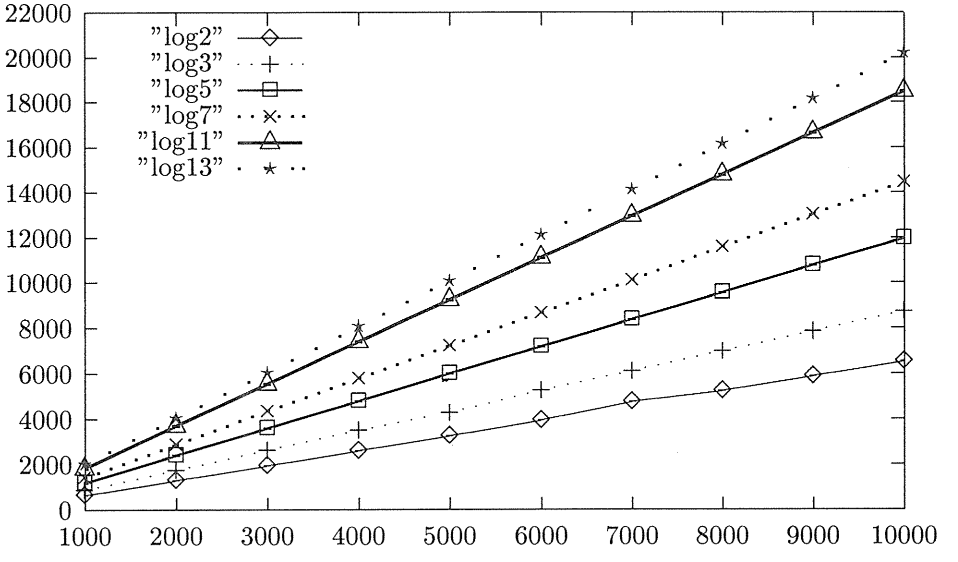

Figure 1: Minimum of loop numbers

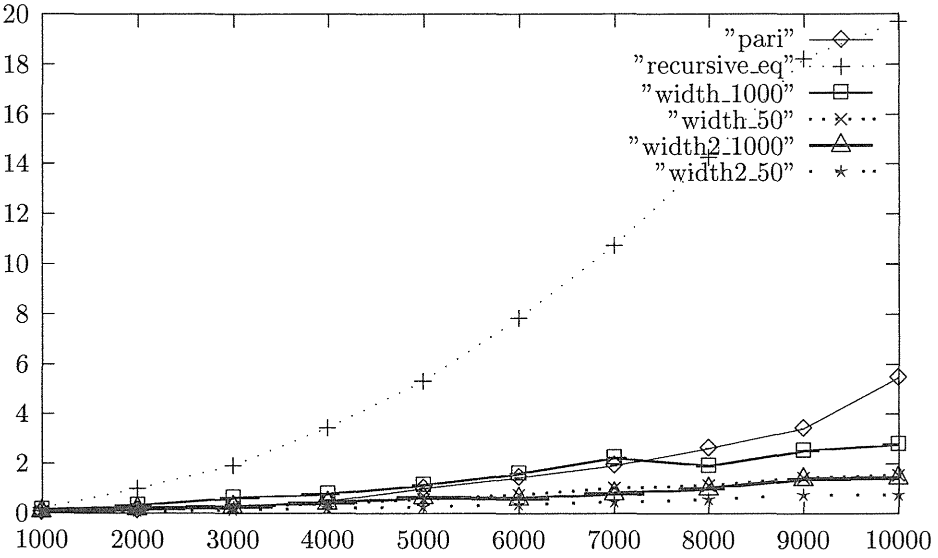

Figure 2: The calculation time T_{11}

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