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Quantifier Elimination in Control Theory

– Real Quantifier Elimination in Practice –

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Abstract

In this paper we focus on the applications of Quantifier Eliminations (QE) to Control Theory and we aim at actual applicability of QE methods to industrial size problems. This is also regarded as a typical case study about how we can resolve the unsolved important engineering problems.

1 Introduction

Quantifier elimination approach covers wide range of many mathematical and industrial problems as follows;

- Real implicitization of parametric algebraic surfaces.
- Automatic theorem proving and finding in real geometry.
- Geometric reasoning about three-dimensional objects, including parallel and central projections of objects, the reconstruction of objects from projections, lighting and shading, equidistance surfaces.
- Rounding, blending and boundary representation of solids.
- Collision and motion planing in robotics.
- The Birkhoff interpolation problem.
- Sign behavior of univariate polynomials.
- Implementation of guarded expressions for coping with degenerate cases in the evaluation of algebraic expressions.
- Stability analysis for ODE's and PDE's.
- Control theory.
- Simulation and error diagnosis of technical networks.
- Non-convex parametric linear, quadratic and hyperbolic optimization problems.
- Parametric scheduling.

(See [3], [40].) In this article, we focus on control theory and first we briefly explain the historical outline about applications of QE to control theory. Then we, in particular, give attention to "robust control problems" which is one of main concerns of control community.

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2 Quantifier Elimination

Many mathematical and industrial problems can be translated to formulas consisting of polynomial equations, inequalities, quantifiers ($\forall, \exists$) and Boolean operators ($\wedge, \vee, \neg, \rightarrow$, *etc*). Such formulas construct sentences in the so-called *first-order theory* of real closed fields and are called *first-order formulas.*

Let $f_i(X, U) \in \mathbf{Q}[X, U]$, $i = 1, 2, \cdots, t$, where $\mathbf{Q}$ is the fields of rational numbers, $X = (x_1, \cdots, x_n) \in \mathbf{R}^n$ a vector of quantified variables, and $U = (u_1, \cdots, u_m) \in \mathbf{R}^m$ a vector of unquantified parameter variables. Let $F_i = f_i(X, U) \ \#_i \ 0$, where $\#_i \in \{=, \geq, >, \neq\}$, for $i = 1, \cdots, s$, $Q_j \in \{\forall, \exists\}$, and X_j a block of q_j quantified variables for $j = 1, \cdots, s$. In general, quantified formula φ is given

$$\varphi = ({}^{Q_1}X_1 \cdots {}^{Q_s}X_s) \ G(F_1, \cdots, F_t) \tag{1}$$

where $G(F_1, \cdots, F_t)$ is a quantifier-free (qf) Boolean formula.

QE procedure is an algorithm to compute equivalent qf formula for a given first-order formula. If all variables are quantified, *i.e.* $m = 0$, QE procedure decides whether the given formula (1) is *true* or *false.* This problem is called *decision problem.* When there are some unquantified variables U, QE procedure find a qf formula $\varphi(U)$ describing the range of possible U where $\varphi(U)$ is true. If there is no such range QE outputs false. This problem is called *general quantifier elimination problem.*

The history of the algorithms for QE begins with *Tarski-Seidenberg decision procedure* in 1950's [36], [9]. But this is very intricate and far from feasible. In 1975, Collins presented a more efficient general purpose QE algorithm based on Cylindrical Algebraic Decomposition (CAD) [12]. The algorithm has improved by Collins and Hong [13] and was implemented on SACLIB as "QEPCAD" by Hong. Weispfenning has presented other QE algorithm by using Comprehensive Gröbner basis and the real root counting for multivariate polynomial systems [41].

Weispfenning presented a more efficient QE algorithm based on test terms [38],[29],[39]. Though there is some degree restriction of a quantified variable in input formulas for test terms approach, this approach seems very practical. Implementation of the method was done on Reduce as "REDLOG" and *Risa/Asir* [1] by Sturm [34], [35]. Moreover, L.González-Vega *et.al.* also presented a special QE algorithm based on Sturm-Habicht sequence for particular inputs some "sign definite" conditions [19]. We can say that the relevance of these special QE algorithms consists in its applicability to the actual important problems.

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[1] *Risa/Asir* is a computer algebra system [32] developed at Fujitsu Labs Ltd. FTP:*endeavor.fujitsu.co.jp:/pub/isis/asir*

3 Application of QE in Control Theory

Roughly speaking, control systems consists of a plant and a controller (compensator) and control problems are usually described as follows : *"Design the controller so that the controlled systems satisfy the desired properties (specifications) for a given plant."* If we consider all admissible noise, disturbance, and model uncertainties within the plants, the problems are called "Robust control problem". Usually, a plant and a controller are given by rational functions in s (s : Laplace variable), say $P(s), C(s)$ respectively, and $C(s)$ has some control parameters, say $p_1, \cdots, p_t$. And specifications are given by using functions Φ_i in $P(s)$,$C(s)$ and specific value γ_i: $\Phi_i(P(s), C(s)) < \gamma_i$. Then, control problem is described by

$$\exists c \in C(s; p_1, \cdots, p_t) \ s.t. \ \Phi_i(P(s), C(s)) < \gamma_i \ for\ a\ fixed\ plant\ P(c)$$

and robust control problem is

$$\exists c \in C(s; p_1, \cdots, p_t) \ s.t. \ \Phi_i(P(s), C(s)) < \gamma_i \ for\ all\ plants\ p \in \mathcal{P}.$$

where P is some family of plants.

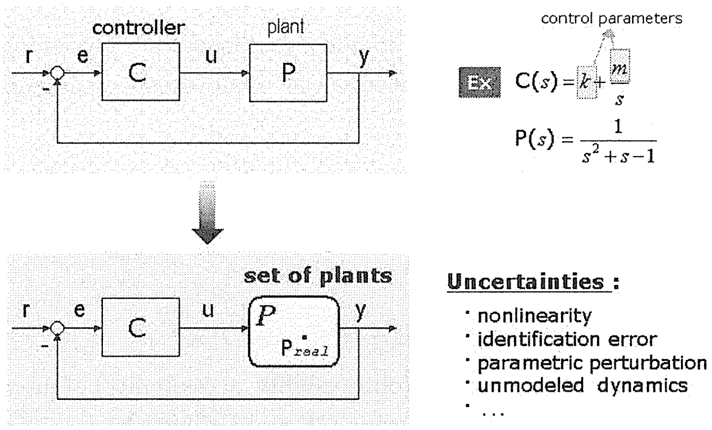

Figure 1: Robust Control Problems

These are surely constraint solving problems and usually solved by numerical methods. QE is regarded as one of powerful methods of "constraint solving" and enables us to

(a) obtain not only one feasible solution but also the feasible (possible) range of solutions,

(b) deal with non-convex optimization and

(c) examine decision problems exactly.

These features (advantages) of QE is useful to resolve many unsolved problems in engineering and industrial problems if we utilize numerical methods only.

Many interesting control system design and analysis problems can be reduced to quantifier elimination problems as shown in the followings (see Fig.2);

1. In 1975, Anderson *et.al.* [8]

   Application of Tarski-Seidenberg decision theory ([36],[9]) to the solution of the static output feedback stabilization problem,

2. In 1995, Dorato *et.al.*[15], in 1996 Abdallah *et.al.* [1] and in 1997 Doraot *et.al.*[16]

   Application of QE theory to a robust multi-objective design for linear systems (stability, robust stability, robust performance),

3. In 1996, Jirstrand [25]

   Application of QE theory to linear systems (stabilization, feedback design) and nonlinear systems (computation of stationary points and curve following in the state space).

4. In 1997, Neubacher [31]

   Application of QE theory to various stability problems and developing a specialized (more efficient) method which solves them either symbolically or numerically.

5. In 1998, Anai [4]

   Solving *Semidefinite Programming (SDP)* problems which are one of the generic *Linear Matrix Inequality (LMI)* problems by QE, in particular, when we consider the real parametric uncertainties.

6. In 1998, Nešić [30], in 1999 Anai *et.al* [7]

   Checking the fundamental properties (observability, accessibility) of discrete-time polynomial systems in finite time step by using QE and Gröbner basis.

7. In 1998, Yovine [28]

   Checking the observability of an important class of Hybrid Systems finite time step by using QE.

8. In 1999, Anai & Hara [5]

   Efficient robust control analysis and synthesis method by a special QE using a Sturm-Habicht sequence.

The first attempt to reduce some control problems to QE problems by Anderson *et al.*[8] was made in 1970's. But at that time the algorithm of QE was very intricate and no appropriate software was available. However, recently some improved algorithms have been

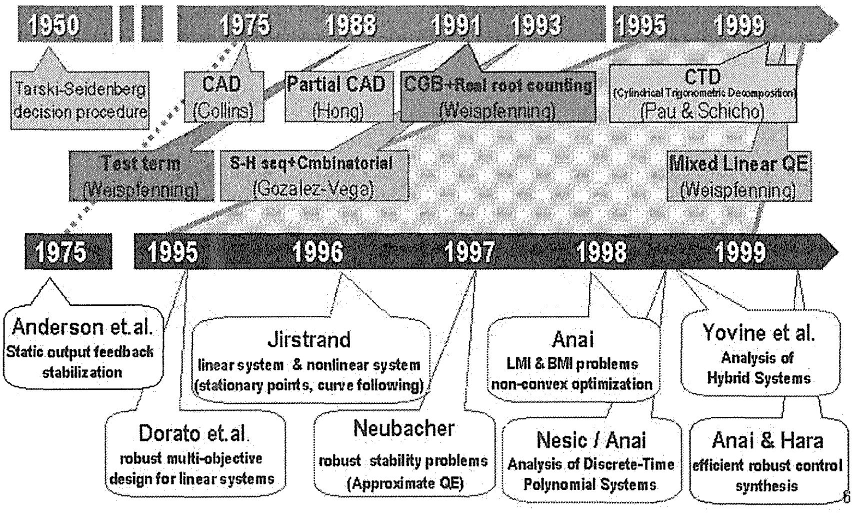

Figure 2: History of QE and its applications to Control Theory

developed (see [12],[13],[29], [39]) and implemented on computers (see [22],[34],[35]). By virtue of the considerable developments of both algorithms and software in QE methods, we explore the application of the QE theory to control problems of great practical interest.

4 Robust Control System Design

Multi-objective design and robust control synthesis are of great practical interest and main concerns in the control system design. However, in general, they are hard to solve and there are no analytical solutions. Recently, for such problems, the methods based on Quantifier Elimination (QE) were proposed by several researchers (see [16][25][31][4]).

For example, in [16] it is shown that how certain robust multi-objective design problems can be reduced to QE problems and actually solved by using "QEPCAD". QEPCAD is a symbolic computation package for QE based on the Cylindrical Algebraic Decomposition (CAD) algorithm presented by G.E.Collins [12]. In [25] it is shown that, in feedback design of linear time-invariant systems, robustness and several performance specifications (H_∞ norm constraint, gain and phase margins) on the close-loop system can also be solved as QE problems by using QEPCAD.

In this article, we consider this kind of problem, in particular, focus on a robust control system design methods based on QE. QE based approach is really effective for such problems. However, unfortunately the size of the problems which can be solved by QE based approach is limited, because the computational complexity of the general QE algo-

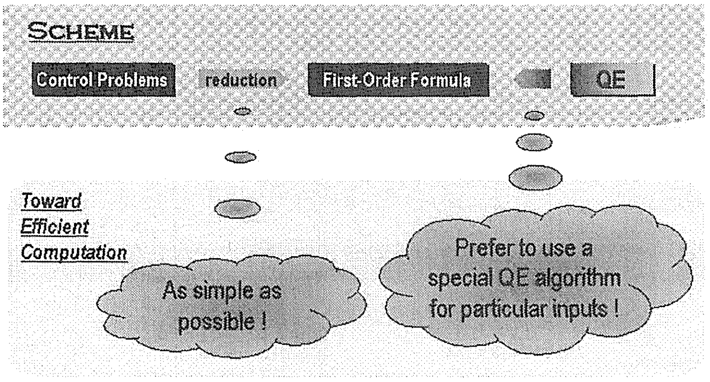

Figure 3: Scheme for solving Control Problems by QE

rithm based on CAD algorithm is doubly exponential in the number of quantified variables (including parameter variables).

In applications of QE to control problems so far, QE method is applied to the first-order formulas derived from the control problems by a direct translation. For the efficient computation, it is important to reduce the target problems to a first-order formula as simple as possible. Furthermore, it is preferable to use special QE algorithm which is effective for a particular input. (See Fig. 3.) Hence, we should try to translate the control system design problem to a formula to which a special QE algorithm is applicable. As one of such formulas, there is a *"Sign Definite Condition (SDC)"* for robust control system design problems.

A parameter space design method is known to be one of the useful tools to deal with multi-objective design problems. A parameter space approach for robust control system design is developed by reducing important design specifications such as H_∞ norm constraint, stability margins *etc*, which are frequently used as indices of the robustness, to sign definite condition. See [21][26][27]. The sign definite condition is a very simple (first-order) formula and suited for a QE procedure in view of computational efficiency. Moreover, In [21] it is also proposed that SDC is checked by using Routh-Hurwitz like criterion proposed by D.Šiljak for positive realness [37]. A parameter space approach based on SDC using D.Šiljak's criterion is essentially equivalent to performing QE for the particular inputs

$$\forall x > 0, \ f(x) > 0 \tag{2}$$

where $f(x)$ is a polynomial with real coefficients. So this method is regarded as a special QE algorithm for the particular input first-order formula (2) and more efficient than the general QE algorithm based on CAD algorithm. However in the method using D.Šiljak's criterion, there remains some issues related to singular cases (see [18]) and specialization of parameters.

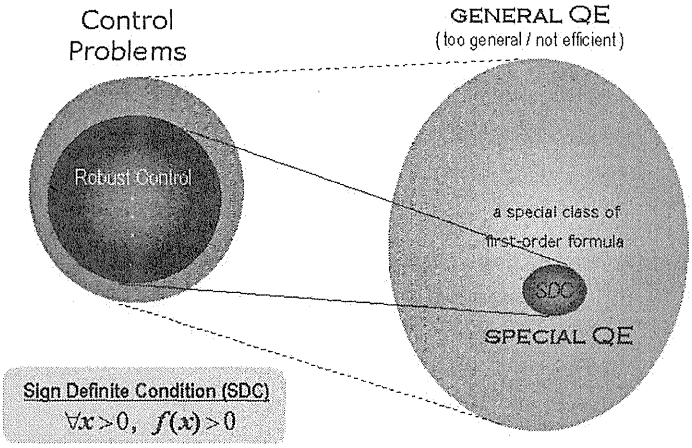

Figure 4: Relevance of our approach

Hence, in this paper, we propose a parameter space approach for robust control system design based on a special QE method for SDC using Sturm-Habicht sequence. A combinatorial algorithm to solve the particular QE problem $\forall x,\ f(x) > 0$ based on Sturm-Habicht sequence is proposed by L.González-Vega *et.al.*[19]. We utilize their algorithm with some modification for a sign definite condition (2). The method proposed here is more efficient than the method using Routh-Hurwitz like criterion by D.Šiljak and moreover has a good specialization property.

5 Sign definite condition (SDC)

In this paper we use $\mathbf{R}$ and $\mathbf{Q}$ for the fields of real numbers and rational numbers, respectively.

Definition 1

Let $f(x)$ be a polynomial in x over $\mathbf{R}$ i.e. $f(x) \in \mathbf{R}[x]$. $f(x)$ is sign definite in the interval $x \in [a, b]$ such that $a < b$ $(\in \mathbf{R})$, denoted by $f(x) \in \mathrm{N}_0[a, b]$, if $f(x)$ preserves its sign in $[a, b]$, or does not cross zero in $[a, b]$.

Note that in actual computation we consider the polynomial $f(x)$ over $\mathbf{Q}$. This restriction is needed since we utilize a computer algebra system. In this paper we, in particular, consider the parametric case that is the coefficients of $f(x)$ contain some real parameters, say, $p_1, \cdots, p_s$. Strictly speaking, this means $f(x)$ is a polynomial over the rational function fields $\mathbf{R}(p_1, \cdots, p_s)$ *i.e.* $f(x) \in \mathbf{R}(p_1, \cdots, p_s)[x]$.

The sign definition condition have emerged as the important problem in a parameter space approach for robust control system design. The specifications such as

- H_∞ norm constraint,
- frequency restricted norm constraint,
- gain and phase margin constraint, and
- pole location,

that are frequently used as indices of robustness of feedback control systems, are reduced to sign definite condition (see [21][26][27][37]). This fact makes it appealing to look into the SDC.

5.1 H_∞ norm constraint

Among the specifications that can be reduced to SDC, here we show how H_∞ norm constraint is transformed to SDC (see [21]). First we have the following lemma:

Lemma 2
[10] A stable transfer function $G(s) = C(sI - A)^{-1}B + D$ with degree n satisfies

$$||G(s)||_\infty < \gamma$$

if and only if the following conditions hold;

i) $D^TD < \gamma^2 I$,
ii) Hamilton matrix

$$H = \begin{bmatrix} A & 0 \\ C^TC & -A^T \end{bmatrix} - \begin{bmatrix} B \\ C^TD \end{bmatrix} \times (\gamma^2 I - D^TD)^{-1}[-D^TC \ B^T]$$

has no eigenvalues on imaginary axis.

Since the characteristic polynomials h of Hamilton matrices are even polynomials, *i.e.*,

$$h(s^2) = |sI - H| = \sum_{i=0}^{n} h_i x^{2i},$$

this condition is equivalent that h has no root in pure imaginary number and on the origin. Let $s^2 = x$ then the condition is that $h(x) = \sum_{i=0}^{n} h_i x^i$ has no negative real roots and no root on the origin. Finally we have the sign definite condition:

$$f(x) = (-1)^n \, h(x) > 0, \quad \forall x \geq 0.$$

Moreover, frequency restricted norm, a generalization of H_∞ norm, defined by

$$||G||_{[\omega_1,\omega_2]} = \sup_{\omega_1 \leq \omega \leq \omega_2} \bar{\sigma}(G(j\omega))$$

can be also reduced to SDC:

$$f(x) \in N_0[-\omega_2^2, -\omega_1^2]$$

where $\bar{\sigma}(G(j\omega))$ is the maximal singular value of G.

Example 1

We consider a PI control system shown in Fig.1. The compensator is fixed as $C(s) = k + \frac{m}{s}$.

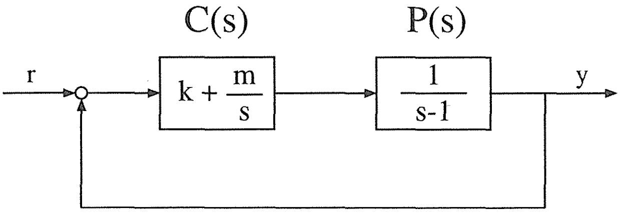

Figure 5: PI control system

The complementary sensitivity function is given by

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{ks + m}{s^2 + (k-1)s + m}. \tag{3}$$

Now we consider the specifications

$$||T(s)||_{[\omega_t, \infty]} < \gamma_t. \tag{4}$$

From the characteristic polynomial of the Hamilton matrix concerning with complementary sensitivity, the specification (4) is reduced to SDC:

$$f_t(x) = b_2x^2 + b_1x + b_0 \ \in \mathrm{N}_0[0, +\infty] \tag{5}$$

where

$$\begin{aligned} b_2 &= 1, \\ b_1 &= -2\omega_t^2 + 2m - (1-k)^2 + \tfrac{k^2}{\gamma_t^2}, \\ a_0 &= \omega_t^4 - (2m - (1-k)^2 + \tfrac{k^2}{\gamma_t^2})\omega_t^2 + m^2(1 - \tfrac{1}{\gamma_t^2}). \end{aligned}$$

Hereafter, without loss of generality, it is enough to consider the problem

$$f(x) \in \mathrm{N}_0[\ 0, +\infty\ ], \tag{6}$$

because the condition $f(x) \in \mathrm{N}_0[a, b]$ can be translated to the condition $f(z) \in \mathrm{N}_0[0, +\infty]$ by a bilinear transformation

$$z = -\frac{x - a}{x - b}.$$

In [21][26][27], it is shown that SDC can be readily checked by the following lemma based on the Routh-Hurwitz like criterion proposed by D.Šiljak [37]:

Lemma 3

[26] Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbf{R}[x]$. $f(x)$ is sign definite in $x \in [0, +\infty]$ if and only if

$$V[f(x)] = n$$

holds, where V is the number of sign changes of the most left column of the Modified Routh Array defined by

$$\begin{array}{ccccc} (-1)^n a_n & (-1)^{n-1} a_{n-1} & \cdots & -a_1 & a_0 \\ (-1)^n n a_n & (-1)^{n-1}(n-1) a_{n-1} & \cdots & -a_1 & \\ \vdots & & & & \\ a_0 & & & & \end{array}$$

Remark 1

We note that the first two rows of Routh array above are formed by the coefficients of the polynomial $f(-x)$ and $f'(-x)$. And following rows in Routh array are formed by the coefficients of the polynomials remainder sequence generated by Euclidean divisions. This, in general, implies that construction of modified Routh array for $f(x)$

The first two rows of Routh array above are formed by the coefficients of the polynomial $f(-x)$ and $f'(-x)$, and following rows in Routh array are formed by the coefficients of the polynomial remainder sequence generated by Euclidean divisions. Here we enumerate the issues when we use the Routh type criterion.

- In the computation of the remainder sequence by using exact arithmetic, the size of the (rational) coefficients of the polynomials appearing in the sequence grows exponentially in the degree of the polynomial.
- In the case where the coefficients contain some parameters, there remains the problem concerning specialization of parameters; Since rational functions may appear in the sequence due to Euclidean division procedure, "division by 0" may occur by substitution of parameters by real numbers. Then we have to recompute completely for the special values of the parameters. (see an example in §7).
- Moreover, separately from the regular case, we have to take care the singular cases which occur when (i) an element of the first column become zero (not all the elements in the corresponding row are zero), (ii) all the elements in a row of the array vanish simultaneously. (See Remark 4 in [37], or [18] for details.)

6 Algorithm

Now we present a robust control system design method based on a more efficient special QE algorithm for SDC using Sturm-Habicht sequence. Usage of Sturm-Habicht sequence

- ∘ resolves the exponential growth of coefficients,
- ∘ clears away the specialization problem,
- ∘ makes us free from the care for singular cases (*i.e.* we can deal with all the cases uniformly),

by virtue of subresultants instead of remainders by Euclidean divisions. Furthermore, Sturm-Habicht sequence has good worst-case computational complexity (See [20] for details.)

6.1 Sturm-Habicht sequence computation

Let $f(x) \in \mathbf{R}[x]$ with degree n. Sturm-Habicht sequence of a polynomial $f(x)$ is defined as the *subresultant* sequence starting from $f(x)$ and $f'(x)$ modulo some specified sign changes. (See Definition 10,11 in Appendix.) We have the following theorem [20]:

Theorem 4 (Structure theorem)
For every $k \in \{0, 1, \cdots, n-1\}$, let $H_k = SH_k(f)$ and $h_k = st_k(f)$ for short. And let $h_n = 1$. Then for every $j \in \{1, \cdots, n-1\}$ such that $h_{j+1} \neq 0$ and $deg(H_j) = r \leq j$, we have

1. *if $r < j-1$ then $H_{j-1} = \cdots = H_{r-1} = 0$,*
2. *if $r < j$ then $h_{j+1}^{j-r} H_r = \delta_{j-r} LC(H_j)^{j-r} H_j$,*
3. *$h_{j+1}^{j-r+2} H_{r-1} = \delta_{j-r+2} Prem(H_{j+1}, H_j)$.*

where $LC(A)$ stands for the leading coefficient of a polynomial A and $Prem(A, B)$ is a remainder obtained by division of $LC(B)^{n-m+1}A$ by B for polynomials A, B with degree n, m, respectively.

Sturm-Habicht sequence of a polynomial f is constructed according this theorem and then we need $O(n^2)$ algebraic operations in $\mathbf{Q}(p_1, \cdots, p_s)$.

6.2 Checking SDC

Let the Sturm-Habicht sequence of f be $\{SH_j(f)\}_{j=0,\cdots,n} = \{g_0, \cdots, g_s\}$. Then for $\alpha \in \mathbf{R} \cup \{-\infty, +\infty\}$ we define $W_{SH}(f;\alpha)$ as the number of sign variations in the list $\{g_0(\alpha), \cdots, g_s(\alpha)\}$. And let $W_{SH}(f;\alpha,\beta) = W_{SH}(f;\alpha) - W_{SH}(f;\beta)$. For every j, the principal j-th Sturm-Habicht coefficient is defined as the coefficient of degree j of $SH_j(f)$. We denote the principal j-th Sturm-Habicht coefficient by $st_j(f)$ and the constant term of $SH_j(f)$ by $ct_j(f)$.

The sign definiteness of f in the interval $[0,+\infty]$ is equivalent to that f has no real roots in $[0,+\infty]$. Hence, an equivalent condition to the sign definition condition in $[0,+\infty]$ is obtained according to the following proposition (*cf.* Theorem 12 in Appendix):

Proposition 5
A polynomial $f(x)$ is sign definite in $[0,+\infty]$ if and only if $W_{SH}(f;0,+\infty)=0$.

By definitions we have

$$\begin{aligned} W_{SH}(f;0,+\infty) &= W_{SH}(f;0)-W_{SH}(f;+\infty) \\ &= V(\{ct_n(f),\cdots,ct_0(f)\})\ -V(\{st_n(f),\cdots,st_0(f)\}) \qquad (*) \end{aligned}$$

The last formula $(*)$ gives us how we count the number $W_{SH}(f;0,+\infty)$ concretely. Since $ct_0(f)=st_0(f)$, we need only $2(n+1)-1=2n+1$ sign evaluations.

If we have Sturm-Habicht sequence for f, we construct the (quantifier-free) equivalent condition for SDC of f by the following procedure. The obtained conditions are of the form of the union of semi-algebraic sets.

1. consider all the 3^{2n+1} (at most) possible sign conditions over the polynomials
$$ct_i(f)\text{'s and } st_i\ (f)\text{'s},$$
2. choose all sign conditions which satisfy
$$W_{SH}(f;0)-W_{SH}(f;+\infty)=0$$
according to $(*)$,
3. construct semi-algebraic sets generated by
$$ct_i(f)\text{'s and } st_i(f)\text{'s}$$
for each selected sign conditions and combine them as a union.

Remark 2
Once we execute this algorithm for the generic polynomial with degree n

$$F_n(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0,$$

the result can be used for any other polynomials with degree n by substituting the coefficients c_i by those of an input polynomial. (In the case of F_2, see a example in §7.) So, the results for the generic cases should be stored in a database (or table) to be called upon, whenever needed. This greatly improves the total efficiency of our methods.

6.3 Simplification

The result through above procedure obviously tends to be large and complicated, and hence we should reduce the result as simple as possible. Some possible simplifications are as follows:

- Manual simplifications by deleting some sign conditions trivially false (*i.e.* empty) or decreasing the number of unions by using the well-known rules;

  $$< \cup > \to \neq, \quad < \cup = \to \leq, \quad > \cup = \to \geq$$

  are indicated in [19].

- We, fortunately, have some sophisticated softwares for automatic formula simplification which are implemented on a QE package "REDLOG" [3] and another QE package on a computer algebra system "Risa/Asir" [5].

7 Example

Here we demonstrate our method by applying it two examples. All the computations were done by using a computer algebra system Risa/Asir and the results were all obtained immediately on a PC with Pentium 200MHz CPU.

Example 2 (sensitivity analysis of a PI control system)
We consider a PI control system shown in Fig.5. The structure of the compensator is fixed as $C(s) = k + \frac{m}{s}$. *The sensitivity and complementary sensitivity functions are given by*

$$S(s) = \frac{1}{1 + P(s)C(s)} = \frac{s^2 - s}{s^2 + (k-1)s + m} \tag{7}$$

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{ks + m}{s^2 + (k-1)s + m} \tag{8}$$

The goal is to determine the possible range of the parameters k *and* m *which satisfy the specifications*

$$||S(s)||_{[0,\omega_s]} < \gamma_s, \tag{9}$$

$$||T(s)||_{[\omega_t,\infty]} < \gamma_t, \tag{10}$$

where $||G||_{[\omega_1,\omega_2]}$ *is a norm defined for a restricted frequency domain* $[\omega_1, \omega_2]$ *i.e.*

$$||G||_{[\omega_1,\omega_2]} = \sup_{\omega_1 \leq \omega \leq \omega_2} \bar{\sigma}(G(j\omega))$$

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[3] REDLOG is developed at University of Passau (Germany) on a computer algebra system REDUCE, see [14]. It is based on the virtual substitution method of parametric test points proposed by V. Weispfenning [38].

[5] Risa/Asir is developed at Fujitsu labs, see [32], anonymous -ftp via: `endeavor.fujitsu.co.jp:/pub/isis/asir`. Virtual substitution method are implemented on Risa/Asir.

if we denote the maximal singular value of G *by* $\bar{\sigma}(G(j\omega))$. *As shown in [27], the both specifications (9) and (4) are reduced to the sign definite conditions. The specification (9) is equivalent to the following SDC:*

$$f_s(x) = a_2x^2 + a_1x + a_0 \ \in \mathrm{N}_0[0, +\infty] \tag{11}$$

where

$$\begin{array}{rcl} a_2 & = & -\omega_s^4 - (2m\alpha + (1-k\alpha)^2)\omega_s^2 + m^2\alpha, \\ a_1 & = & (2m\alpha + (1-k\alpha)^2)\omega_s^2 - 2m^2\alpha, \\ a_0 & = & m^2\alpha \end{array}$$

with $\alpha = \frac{\gamma_s^2}{(1-\gamma_s^2)}$. *And the specification (4) is equivalent to the following:*

$$f_t(x) = b_2x^2 + b_1x + b_0 \ \in \mathrm{N}_0[0, +\infty] \tag{12}$$

where

$$\begin{array}{rcl} b_2 & = & 1, \\ b_1 & = & -2\omega_t^2 + 2m - (1-k)^2 + \frac{k^2}{\gamma_t^2}, \\ a_0 & = & \omega_t^4 - (2m - (1-k)^2 + \frac{k^2}{\gamma_t^2})\omega_t^2 + m^2(1 - \frac{1}{\gamma_t^2}). \end{array}$$

Consequently, what we do to obtain the possible range of k, m *such that (11),(5) is determining the SDC for the generic polynomial with degree 2:*

$$F_2(x) = c_2x^2 + c_1x + c_0 \ \in \mathrm{N}_0[0, +\infty]$$

Sturm-Habicht sequence $\{SH_j(F_2)\}_{j=2,1,0}$ *of* $F_2(x)$ *consists of*

$$\begin{array}{rcl} SH_2(F_2) & = & c_2x^2 + c_1x + c_0, \\ SH_1(F_2) & = & 2c_2x + c_1, \\ SH_0(F_2) & = & c_2c_1^2 - 4c_0c_2^2. \end{array}$$

Then immediately we have

$$\begin{array}{rcl} \{ct_i\}_{i=2,1,0} & = & \{c_0, c_1, (c_2c_1^2 - 4c_0c_2^2)\} \\ \{st_i\}_{i=2,1,0} & = & \{c_2, 2c_2, (c_2c_1^2 - 4c_0c_2^2)\} \end{array}$$

Hence we check whether the number $W_{SH}(F_2; 0, +\infty)$ *is equal to* 0 *or not according to the formula* $(*)$ *for* 3^4 *sign conditions* $\{-, 0, +\}^4$ *over the sequence*

$$\{c_0, c_1, c_2, (c_2c_1^2 - 4c_0c_2^2)\}.$$

Finally, in the case of (11), we reaches the results; $f_s(x) \in \mathrm{N}_0[0, +\infty]$ *if and only if*

$$\begin{array}{c} [\ a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge (a_2a_1^2 - 4a_0a_2^2) > 0\ ] \cup \\ [\ a_0 > 0 \wedge a_1 > 0 \wedge a_2 > 0 \wedge (a_2a_1^2 - 4a_0a_2^2) < 0\ ] \cup \\ \vdots \end{array}$$

$$\cup \; [\; a_0 < 0 \wedge a_1 < 0 \wedge a_2 < 0 \wedge (a_2 a_1^2 - 4 a_0 a_2^2) < 0 \;].$$

If we substitute ω_s, γ_s *with the appropriate values, then this result gives the possible range of the parameters* k, m *as a union of the semi-algebraic sets. The possible range is visualized easily by plotting the semi-algebraic sets on* $k - m$ *plane.*

In the case of (5), since $c_2 = b_2 = 1$ *we check whether the number* $W_{SH}(f_t; 0, +\infty) = 0$ *or not for* 3^3 *sign conditions over the sequence* $\{c_0, c_1, (c_1^2 - 4c_0)\}$.

Remark 3

For a certain class of plants with structured uncertainties, robust performance problem can be reduced to SDC by utilizing Kharitonov's theorem (see [21][26]) and hence is solved by the our method. For example, in [21][26], it is shown that for the same PI control system as in Fig.5 with a plant with structured uncertainties, norm constraints can be decomposed to a conjunction of SDCs and the stability margins constraint is satisfied if and only if the the Kharitonov systems associated to the open loop system satisfy the constraints.

Remark 4

Note that no rational polynomials appears in Sturm-Habicht sequence. On the other hand, modified Routh array of $F_2(x)$ *is given as follows:*

$$\begin{array}{ccc}
c_2 & -c_1 & c_0 \\
2c_1 & -c_0 & \\
\frac{-c_0 c_2 + 2c_1^2}{2c_1} & c_0 & \\
\frac{c_2 c_0^2 - 6 c_1^2 c_0}{-c_2 c_0 + 2c_1^2} & & \\
c_0 & &
\end{array}$$

In the most left column, rational functions appear due to Euclidean division. This leads to bad specialization property i.e. "division by 0" by specialization. For example, if $c_1 = 0$, *the denominator of* $\frac{-c_0 c_2 + 2c_1^2}{2c_1}$ *vanishes and specialization is impossible.*

Example 3 (a generic quartic polynomial F_4))

This is the first non-trivial case.

$$F_4 = c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

Determine the SDC for the generic polynomial with degree 4. Sturm-Habicht sequence

$\{SH_j(F_4)\}_{j=4,3,2,1,0}$ of $F_2(x)$ are given as

$$\begin{array}{rcl}
SH_4(F_4) & = & c_4x^4+c_3x^3+c_2x^2+c_1x+c_0,\\
SH_3(F_4) & = & 4c_4x^3+3c_3x^2+2c_2x+c_1,\\
SH_2(F_4) & = & (-8c_4^2c_2+3c_4c_3^2)x^2+(-12c_4^2c_1+2c_4c_3c_2)x+c_4c_3c_1-16c_0c_4^2,\\
SH_1(F_4) & = & (-36c_4^3c_1^2+(28c_4^2c_3c_2-6c_4c_3^3)c_1-8c_4^2c_2^3+2c_4c_3^2c_2^2+32c_0c_4^3c_2-12c_0c_4^2c_3^2)x\\
 & = & +3c_4^2c_3c_1^2+(-4c_4^2c_2^2+c_4c_3^2c_2-48c_0c_4^3)c_1+32c_0c_4^2c_3c_2-9c_0c_4c_3^3,\\
SH_0(F_4) & = & -27c_4^3c_1^4+(18c_4^2c_3c_2-4c_4c_3^3)c_1^3+(-4c_4^2c_2^3+c_4c_3^2c_2^2+144c_0c_4^3c_2\\
 & & -6c_0c_4^2c_3^2)c_1^2+(-80c_0c_4^2c_3c_2^2+18c_0c_4c_3^3c_2-192c_0^2c_4^3c_3)c_1\\
 & & +16c_0c_4^2c_2^4-4c_0c_4c_3^2c_2^3-128c_0^2c_4^3c_2^2+144c_0^2c_4^2c_3^2c_2-27c_0^2c_4c_3^4+256c_0^3c_4^4.
\end{array}$$

Then we have

$$\begin{array}{rcl}
\{st_i\}_{i=4,3,2,1,0} & = & \{c_4, 4c_4, -8c_4^2c_2+3c_4c_3^2,\\
 & & -36c_4^3c_1^2+(28c_4^2c_3c_2-6c_4c_3^3)c_1-8c_4^2c_2^3+2c_4c_3^2c_2^2+32c_0c_4^3c_2-12c_0c_4^2c_3^2,\\
 & & SH_0(F_4)\}\\
\{ct_i\}_{i=4,3,2,1,0} & = & \{c_0, c_1, c_4c_3c_1-16c_0c_4^2,\\
 & & 3c_4^2c_3c_1^2+(-4c_4^2c_2^2+c_4c_3^2c_2-48c_0c_4^3)c_1+32c_0c_4^2c_3c_2-9c_0c_4c_3^3, SH_0(F_4)\}
\end{array}$$

Hence we consider the set of all sign conditions $\{\varepsilon_7,\cdots,\varepsilon_0\}$ ($\varepsilon_i \in \{-,0,+\}$) for $\{S_7,\cdots, S_0\}$ which satisfy that the number $W_{SH}(F_4;0,+\infty)$ is equal to 0. Here S_i's are given by

$$\begin{array}{rcl}
S_7 & = & c_4,\\
S_6 & = & 3c_3^2c_4-8c_2c_4^2 \quad = c_4(3c_3^2-8c_2c_4),\\
S_5 & = & -36c_4^3c_1^2+(28c_2c_3c_4^2-6c_3^3c_4)c_1+32c_0c_2c_4^3+(-12c_0c_3^2-8c_2^3)c_4^2+2c_2^2c_3^2c_4,\\
 & = & 2c_4(-18c_4^2c_1^2+(14c_2c_3c_4-3c_3^3)c_1+16c_0c_2c_4^2+(-6c_0c_3^2-4c_2^3)c_4+c_2^2c_3^2)\\
S_4 & = & SH_0(F_4) \qquad = c_4\cdot S_0',\\
S_3 & = & c_0,\\
S_2 & = & c_1,\\
S_1 & = & c_3c_4c_1-16c_0c_4^2 \;= c_4(c_3c_1-16c_0c_4),\\
S_0 & = & 3c_3c_4^2c_1^2+(-48c_0c_4^3-4c_2^2c_4^2+c_2c_3^2c_4)c_1+32c_0c_2c_3c_4^2-9c_0c_3^3c_4,\\
 & = & c_4(3c_3c_4c_1^2+(-48c_0c_4^2-4c_2^2c_4+c_2c_3^2)c_1+32c_0c_2c_3c_4-9c_0c_3^3).
\end{array}$$

where

$$\begin{aligned}
S_0' = {} & -27c_4^2c_1^4+(18c_2c_3c_4-4c_3^3)c_1^3+(144c_0c_2c_4^2+(-6c_0c_3^2-4c_2^3)c_4+c_2^2c_3^2)c_1^2+\\
& (-192c_0^2c_3c_4^2-80c_0c_2^2c_3c_4+18c_0c_2c_3^3)c_1+256c_0^3c_4^3-128c_0^2c_2^2c_4^2+\\
& (144c_0^2c_2c_3^2+16c_0c_2^4)c_4-27c_0^2c_3^4-4c_0c_2^3c_3^2
\end{aligned}$$

Note that $\varepsilon_3 \in \{+,-\}$ since here we assume that $c_0 \neq 0$. Moreover from the algebraic properties of Sturm-Habicht sequence it it impossible that more than two consecutive zeros appear in the sequence. This also implies that $c_4 \neq 0$. Then we have the necessary and sufficient conditions, the union of the following 561 semialgebraic sets, such that F_4 satisfies sign definite condition :

$$\begin{array}{c}
[\, S_7 > 0 \wedge S_6 > 0 \wedge S_5 > 0 \wedge S_4 > 0 \wedge S_3 > 0 \wedge S_2 > 0 \wedge S_1 > 0 \wedge S_0 > 0 \,] \cap \\
[\, S_7 > 0 \wedge S_6 < 0 \wedge S_5 > 0 \wedge S_4 > 0 \wedge S_3 > 0 \wedge S_2 > 0 \wedge S_1 < 0 \wedge S_0 > 0 \,] \cap \\
\vdots \\
[\, S_7 < 0 \wedge S_6 < 0 \wedge S_5 < 0 \wedge S_4 < 0 \wedge S_3 < 0 \wedge S_2 < 0 \wedge S_1 < 0 \wedge S_0 < 0 \,].
\end{array}$$

Furthermore, this formula can be simplified by deleting trivially empty semialgebraic sets based on the followings:

- $S_1 = 0, \quad S_2 = 0 \implies c_1 = 0, c_0 = 0 \implies$ *This is contrary to* $c_0 \neq 0$.
- $S_6 = 0 \implies c_2 = \frac{3c_3^2}{8c_4} \implies S_5 = -(16c_4^2c_1 - c_3^3)^2 \leq 0$
- $S_1 = 0 \implies c_3 = \frac{16c_0c_4}{c_1} \implies S_0 = \frac{-c_4^2(c_2c_1^2-96c_0^2c_4)^2}{c_1^3} \implies S_0, S_2$ *have different sign.*

Finally we have the union of the 477 semialgebraic sets, such that F_4 *satisfies the sign definite condition (Total computation time on Risa/Asir is 65.26 seconds).*

8 Computational Complexity

Our approach consists of two parts: reduction to SDC and special QE computation. The dominant part of our approach is QE part. In particular, the construction of Sturm-Habicht sequence occupies the total computation time. Here we show some experimental results to demonstrate the tractability of our proposed method for practical control problems. All the computations were done by using a computer algebra system *Risa/Asir* and were executed on a PC with Pentium 200MHz CPU.

8.1 Generic polynomials

By using QEPCAD[9) , we can immediately solve the SDC for generic polynomials $F_n = \sum_{i=0}^{n} c_i x^i$, *i.e.*, $\forall x \; (\, x > 0 \to F_n > 0 \,)$ up to $n = 3$. However we could not solve the QE problems by QEPCAD for $n \geq 4$ due to the lack of memory.

On the other hand, we can solve it for generic polynomials up to $n = 8$ in our method as shown in Table 1. Table 1 shows the timing data to compute Sturm-Habicht sequence for generic polynomials $F_n(x)$. Once we compute Sturm-Habicht sequence of $F_n(x)$, the result can be used for another polynomials with degree n by substituting the coefficients c_i by those of an input polynomial. The results for the generic cases should be stored in a database to be called upon, whenever needed. This greatly improves the total efficiency. In the case of polynomials with many parameters it seems to be better that we compute Sturm-Habicht sequence in this way.

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9) These computation by QEPCAD are executed on Sun Ultra Sparc 1 Model 140.

n	time (sec)	n	time (sec)
2	0.002	6	1.533
3	0.006	7	34.120
4	0.028	8	> 3600
5	0.121	9	—

Table 1: Sturm-Habicht sequence computation for generic polynomials

8.2 PID-controller synthesis

Table 2 shows the timing data to compute Sturm-Habicht sequence of the polynomials $f_t(z)$, for which we check the SDC in analyzing sensitivity of PI control systems with compensators $C(s) = k + \frac{m}{s}$ and PID control systems with compensators $C(s) = k + \frac{m}{s} + \frac{d \cdot s}{1+0.1s}$. PI and PID control systems have same structure as Fig.1 and the compensator has 2 and 3 design parameters, respectively. As a target specification, here we consider the frequency restricted norm constraint for complementary sensitivity function : $||T(s)||_{[20,+\infty]} < -10$. This is equivalent to a SDC $f_t(z) > 0, \forall z > 0$. The numerators of the plants $p(s)$ are fixed as 1 and the denominators for each degree are given randomly. Noted that the computation of $f_t(z)$ is achieved immediately.

degree of $p(s)$ w.r.t. s	PI (sec)	PID (sec)
2	0.001	0.3709
3	0.029	1.931
4	0.111	9.807
5	0.459	35.840
6	1.528	145.700
7	4.718	443.200
8	13.090	1346.000
9	35.630	3644.000
10	82.700	7689.000
11	266.600	—
12	443.200	—
13	1176.000	—
14	1838.000	—
15	4333.000	—

Table 2: Sturm-Habicht sequence computation for PI and PID control systems

As a practical example, we quote the flexible beam example in [17]. The plant transfer function is given by

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}.$$

We consider the PID control system for this plant with a same controller as above and the same frequency restricted norm constraint for complementary sensitivity $||T(s)||_{[20,+\infty]} < -10$. Then $f_t(z)$ is obtained in 0.55 sec and Sturm-Habicht for $f_t(z)$ is computed in 115.50 sec.

8.3 Combinatorial part

There are several possibilities to improve the combinatorial part. We can prune the impossible sign combinations before counting the number of sign changes owing to the followings;

- $\langle a \rangle$ In the case of positive sign definite condition, we have to add one more condition such that "(head coefficient of f) > 0". This implies that $st_n > 0$ and $st_{n-1} > 0$. Hence all possible sign conditions are reduced to $3^{2(n-1)}$.
- $\langle b \rangle$ From the algebraic properties of Sturm-Habicht sequence, it it impossible that more than two consecutive zeros appear in the sequence.
- $\langle c \rangle$ When we determine the design parameters, we usually do not choose the parameter values on the boundaries of possible ranges of parameters. This implies that for actual design we do not have to check the sign combinations including 0 (except identically 0 case). Hence we should consider 2^{2n} sign combinations.

For example, in the case of generic polynomials with degree 4 there are totally $3^8 = 6561$ sign combinations to verify the number of sign changes. After pruning impossible sign combinations by $\langle a \rangle$, $\langle b \rangle$, and checking the number of sign changes, we have 561 feasible sign combinations. Furthermore, this formula can be simplified by deleting trivially empty semialgebraic sets manually. Finally we have 477 feasible sign combinations. For practical control problems, the number of possible sign combinations can become rather small as in §10. For $g(\omega)$ in §10, whose degree is 4, finally we have only one sign combination.

8.4 Summary

Here we summarize the computational complexity of our approach based on the computational results above.

Tractability : Our approach is practically applicable to the systems up to order 15 for the case of the number of design parameters in fixed-structure controller is 2 (e.g. PI control systems), and to the systems up to order 10 for the case of the number of design parameters the is 3 (e.g. PID control systems). In the case that controller has more than 3 parameters, our approach is practically applicable to the systems up to order 7 by using stored general forms.

Applicability : Our approach outputs a disjoint union $\mathcal{R}$ of semi-algebraic sets R_i which describes the possible range of design parameters Θ; $\mathcal{R} = \bigcup_{i=1}^{n} R_i$. And the obtained results are applicable to

- visualization of possible region of design parameters by a projection to 2 or 3 dimensional space,
- pre-processing (reduction to sub-problems) for numerical optimization such that

  $$\min_{\Theta \in \mathcal{R}} F(\Theta) = \min_{i} \left\{ \min_{\Theta \in R_i} F(\Theta) \right\}$$

  where $F(\Theta)$ is an objective function in Θ,
- reduction of the VC-dimension for randomized algorithm

9 Mechanical system design for positive-realness

Here we consider applying our method to mechanical system design (for positive-realness) to examine the tractability of our approach. As shown in [23] it is appropriate to design a mechanical system such that the transfer function from the force input to the velocity output is *"positive real (PR)"*. In this section, we consider a class of mechanical systems and show the methods to obtain possible ranges of design parameters for which a given system satisfies the positive real condition.

First we define the positive-real transfer functions as follows.

Definition 6
A square transfer function $G(s)$ is called positive real (PR) if

$$G(s) + G(s)^* \geq 0, \quad \forall Re(s) \geq 0 \tag{13}$$

holds where $G(s)^*$ denotes its complex conjugate transpose.

For a scalar transfer function, positive real function is defined as follows:

Definition 7
A real function

$$G(s) = \frac{q(s)}{p(s)} \tag{14}$$

with relatively prime polynomials $p(s)$ and $q(s)$ is called (strictly) positive real *if and only if*

(i) the polynomial

$$f(s) = p(s) + q(s) \tag{15}$$

is Hurwitz;

(ii) and

$$Re[G(i\omega)] > 0 \tag{16}$$

for all real ω.

Here we establish the positive real property of a given transfer function according to Definition 7. The condition (i) is checked by using Lienard-Chipart criterion:

Theorem 8 (Lienard-Chipart criterion)
Let $f(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, a_0 > 0$ *be a given polynomial with real coefficients. Define the Hurwitz determinant of order* $1 \leq i \leq n$ *as*

$$D_i = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & & \\ a_0 & a_2 & a_4 & \cdots & & \\ 0 & a_1 & a_3 & \cdots & & \\ 0 & a_0 & a_2 & a_4 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & & & a_i \end{vmatrix}, \quad a_k = 0 \ \ for \ \ k > n$$

Then f *is a Hurwitz polynomial if and only if*

$$a_n > 0, a_{n-1} > 0, a_{n-4} > 0, \cdots ; \ D_2 > 0, D_4 > 0, D_6 > 0, \cdots$$

As for the condition (ii), we first convert (ii) to the following equivalent condition. $Re[G(i\omega)] > 0$ for all real ω if and only if

$$(iii) \qquad g(\omega) \equiv p_r(\omega)q_r(\omega) + q_i(\omega)p_i(\omega) > 0 \text{ for all real } \omega > 0,$$

where $G(s) = \frac{q(s)}{p(s)}$, $p(i\omega) = p_r(\omega) + ip_i(\omega)$ and $q(i\omega) = q_r(\omega) + iq_i(\omega)$. This type of conditions is called sign definite condition (SDC). The SDC is verified efficiently by using an algebraic method, a special quantifier elimination using Sturm-Habicht sequence.

As pointed out in [24] it also seems reasonable to design the mechanical system to achieve the PR property up to the desired control band width. We define the finite frequency positive-real transfer functions as follows.

Definition 9
A square transfer function $G(s)$ *is called positive real (PR) up to the frequency* ω_0 *if it has no poles in the open right half plane and satisfies*

$$G(j\omega) + G(j\omega)^* \geq 0, \quad \forall |\omega| \leq \omega_0 \tag{17}$$

When $\omega_0 = \infty$*, in particular,* $G(s)$ *is PR.*

The frequency restricted positive real condition is converted to a frequency restricted H_∞ norm condition via a bilinear transformation:

$$G(j\omega) + G(j\omega)^* > 0, \quad \forall \omega_1 \leq \omega \leq \omega_2$$

$$\Longleftrightarrow \ ||H||_{[\omega_1,\omega_2]} < 1 \tag{18}$$

where $H(s) = (G - I)(G + I)^{-1}$. Frequency restricted norm constraints are reduced to SDC (see [27]). Hence this is also checked by a special quantifier elimination using Sturm-Habicht sequence.

10 Integrated design examples

Here we show some computational results applying our method to practical integrated design examples, which demonstrate the tractability of our approach, We note that the first example can not be reduced to a convex optimization problem, and hence it is difficult to obtain the exact solution by numerical optimization.

We consider a swing-arm positioning mechanism for small disc storage devices shown in Fig.2 taken from [23]. It works basically as follows: when we apply a force input u to the point B, the swing-arm rotates around the pivot A with i the $x - y$ plane, and the sensing point C moves to a desired position. We design the shape of the swing-arm such that the resulting transfer function from u to $\dot{y}$ is positive real (PR). We employ the equation of motion, linearized around the equilibrium state, given in [23] by

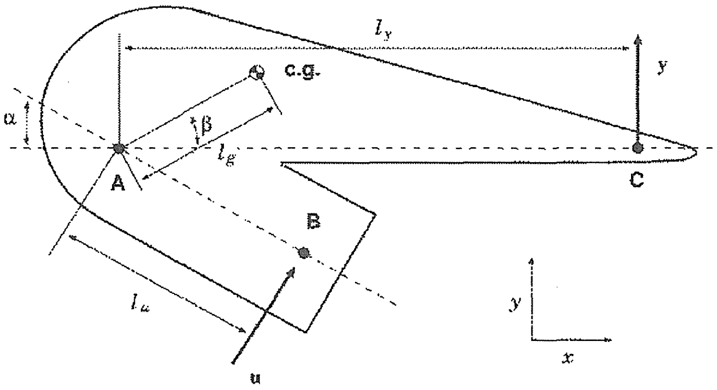

Figure 6: Geometry of the swing-arm

$$M\ddot{q} + D\dot{q} + Kq = bu, \quad y = cq$$

where

$$q = \begin{bmatrix} x_g \\ y_g \\ \gamma \end{bmatrix}, \quad M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix},$$

$$D = dS, \ K = kS$$

$$S = \begin{bmatrix} 1 & 0 & l_g sin(\beta) \\ 0 & 1 & -l_g cos(\beta) \\ l_g sin(\beta) & -l_g cos(\beta) & l_g^2 \end{bmatrix},$$

$$b = \begin{bmatrix} sin(\alpha) \\ cos(\alpha) \\ l_u - l_g cos(\alpha + \beta) \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & 1 & l_y - l_g cos(\beta) \end{bmatrix}.$$

and (x_g, y_g) $[m]$ is the displacements of the center of gravity and γ $[rad]$ is the angle between the x-axis and the line AC, measured counter clockwise.

In this equation, the flexibility of the pivot is modeled by two linear springs in the x and the y directions with small damping and assumed that the stiffness and the damping coefficients are the same for both directions. The values of the other swing-arm parameters are taken from [33] and shown in Table 3.

mass of swing-arm	m	0.033	kg
moment of inertia	J	1.7×10^{-5}	kg m^2
actuator point (angle)	α	—	deg
actuator point (length)	l_u	—	m
sensor point (length)	l_y	—	m
c.g. location (angle)	β	10	deg
c.g. location (length)	l_g	0.02	m
stiffness of pivot	k	1.5×10^6	N/m
damping of pivot	d	4.4	Ns/m

Table 3: Swing-arm parameters

10.1 Simultaneous design of an actuator point B and a sensing point C (nonlinear case):

The goal is to obtain simultaneously the region of actuator point B and a sensing point C yielding PR transfer functions. Thus, α, l_u and l_y are the design parameters. It is noted that the problem can not be reduced to a convex problem, and hence it is very difficult to find the exact region by numerical optimization. Instead, our approach can provide the exact region as will shown below. We define the new design parameter vector

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} sin(\alpha)/l_u \\ cos(\alpha)/l_u \end{bmatrix}$$

and the new control input $v = l_u u$. Then we have

$$bu = (b_1 + b_2\theta)v$$

where

$$b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ l_g sin(\beta) & = l_g cos(\beta) \end{bmatrix}.$$

Note that the transfer function $T_{\dot{y}u}(s)$ from u to $\dot{y}$ is PR if and only if the transfer function $T_{\dot{y}v}(s)$ from v to $\dot{y}$ is so. Hence we try to compute the region of θ for which $T_{\dot{y}v}(s)$ is PR, and then to find the corresponding region in the original parameters.

A state space realization for $T_{\dot{y}v}(s)$ is given by

$$\left[\begin{array}{c|c} A & B(\theta) \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & I & 0 \\ -M^{-1}K & -M^{-1}D & M^{-1}(b_1 + b_2\theta) \\ \hline 0 & c & 0 \end{array}\right].$$

Then we have $T_{\dot{y}v}(s) = Q(s)/P(s)$ where

$$\begin{aligned} Q(s) = {} & (((3781008l_y - 1163605212/15625)\theta_1 + (-21448944l_y + \\ & 15366537516/15625)\theta_2 + 1089000000l_y - 2859859200)s^3 + \\ & (171864000000000l_y\theta_1 + (-974952000000000l_y + 45300000000000)\theta_2 + \\ & 99019360000000000l_y - 974952000000000)s^2 + 13200000000000000000l_y s + \\ & 2250000000000000000000000l_y), \end{aligned}$$

$$\begin{aligned} P(s) = {} & 18513s^5 + 6853440s^4 + 2336984672000s^3 + 398640000000000s^2 + \\ & 67950000000000000000s. \end{aligned}$$

Then $f(s) = P(s) + Q(s)$ is Hurwitz if and only if

$$[D_4 > 0 \ \wedge \ D_2 > 0 \ \wedge \ A_4 > 0 \ \wedge \ A_2 > 0 \ \wedge A_0 > 0] \tag{19}$$

(see Appendix C for $f(s), D_4, D_2, A_4, A_2$ and A_0).

Next we compute Sturm-Habicht sequence of $g(\omega)$ and we have

$$\{ct_i\}_{i=4,3,2,1,0} = \{S_1, 0, -S_1, 0, S_1\}$$

$$\{st_i\}_{i=4,3,2,1,0} = \{S_3, S_3, S_2, S_4, S_1\}$$

(see Appendix C for $g(\omega)$ and S_i's). Additionally, we need the condition that the head coefficient of $g(\omega)$ is positive *i.e.* $S_3 > 0$ to ensure the positivity of $g(\omega)$. Finally, we have that the condition (ii) holds if and only if

$$\begin{aligned} & [S_1 > 0 \ \wedge \ S_2 < 0 \ \wedge \ S_3 > 0 \ \wedge \ S_4 > 0] \cup \\ & [S_1 > 0 \ \wedge \ S_2 < 0 \ \wedge \ S_3 > 0 \ \wedge \ S_4 < 0] \cup \\ & [S_1 > 0 \ \wedge \ S_2 > 0 \ \wedge \ S_3 > 0 \ \wedge \ S_4 < 0] \end{aligned} \tag{20}$$

Consequently, by superposing two possible regions (19) and (20) of design parameters θ_1, θ_2 and l_y in the parameter space, we have the feasible region of the design parameters for positive-realness.

10.2 Design of an actuator point B (linear case):

Here we fix the sensing point and let $l_y = 0.06[m]$. Now we obtain the region of actuator point B yielding PR transfer functions. Thus, α and l_u are the design parameters. In this case, $f(s)$ is Hurwitz if and only if

$$[D'_4 > 0 \ \wedge \ D'_2 > 0 \ \wedge \ A'_4 > 0 \ \wedge \ A'_2 > 0] \tag{21}$$

(see Appendix C for D'_4, D'_2, A'_4, A'_2, and A'_0).

As for the condition (ii), after removing the sign conditions which is obviously empty, we have necessary and sufficient conditions

$$[S'_1 > 0 \ \wedge \ S'_2 > 0 \ \wedge \ S'_3 > 0 \ \wedge \ S'_4 < 0] \tag{22}$$

(see Appendix C for S'_i's). Consequently, by superposing two possible regions (21) and (22) of design parameters θ_1 and θ_2 in the parameter space, we have the feasible region of θ_1 and θ_2 for positive-realness

$$\begin{array}{l} [D'_4 > 0 \wedge D'_2 > 0 \wedge A'_4 > 0 \wedge A'_2 > 0] \cup \\ [S'_1 > 0 \ \wedge \ S'_2 > 0 \ \wedge \ S'_3 > 0 \ \wedge \ S'_4 < 0] \end{array} \tag{23}$$

which is shown in Fig. 8 as a shaded cell. This region is transformed to the region in α and l_u and described in the $x - y$ plane as shown in Fig. 9. Integer lattice points in (23) are described in Fig. 9 as dots. And the region (23) corresponds to that below the dotted line. (All the computations needed here has been done in about one minute.)

11 Conclusion

In this paper, we explain roughly about current situation of the application of QE to control theory and, in order to aim at pracitical applicability, have proposed a method of robust control design based on SDC by a special QE method using Sturm-Habicht sequence. Our method, in particular, effective practically for multi-objective control design using low degree fixed-structure controller.

Our approach is more efficient than the method using Routh-Hurwitz like criterion and has a good specialization property. Moreover, compared with the matrix inequality approach based on numerical optimizations, our approach based on a special QE has several advantages such as applicability to parametric and nonlinear cases, possibility to obtain non-conservative results and less complexity for multi-objective design.

Moreover we have demonstrated our method by applying it to some examples and showed by computational experiments on a computer algebra system that our approach is practically appricable one.

Acknowledgments

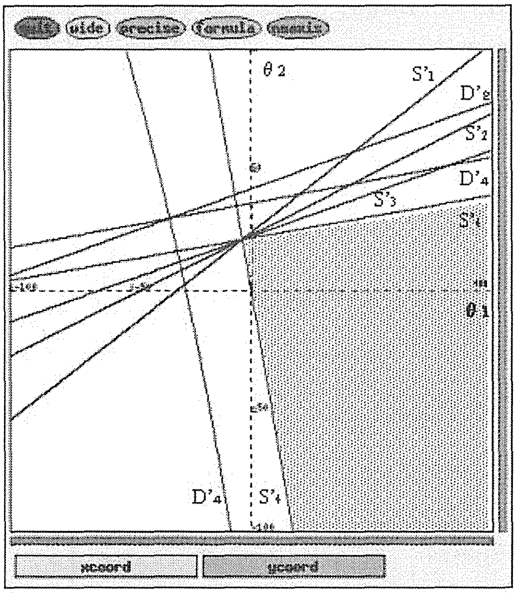

Figure 7: Admissible parameter space for PR

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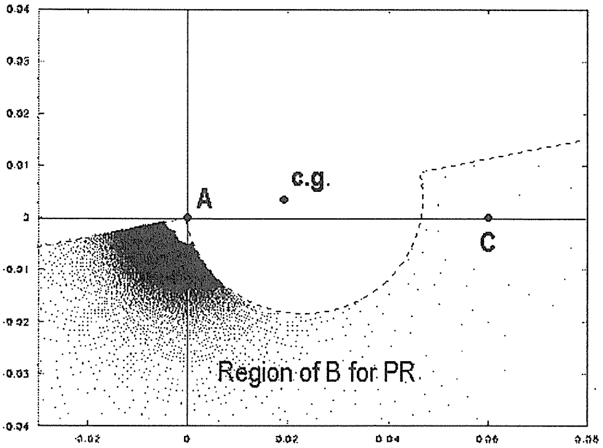

Figure 8: Region of actuation points for positive-realness

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Appendix

A. Quantifier elimination : QE deals with the *first-order formulas*, which consists of polynomial equations, inequalities, quantifiers ($\forall, \exists$) and Boolean operators ($\wedge, \vee, \neg, \rightarrow$

, etc). QE procedure is an algorithm to compute equivalent quantifier-free formula for a given first-order formula over the real closed field. For example, for the input

$$\forall x(x^2 + bx + c > 0),$$

QE outputs the equivalent quantifier-free formula; $b^2 - 4c < 0$. See [11] for the details about QE.

B. Sturm-Habicht sequence : We briefly show the definition of Sturm-Habicht sequence and the relation between Sturm-Habicht sequence and the number of real roots (see [20] for derails).

Definition 10
[20] Let P, Q be polynomials in $\mathbf{R}[x]$

$$P = \sum_{k=0}^{n} a_k x^k \quad Q = \sum_{k=0}^{m} b_k x^k$$

where n and m be non-negative integers and let $\ell = min(n, m)$. For $i = 0, 1, \cdots, \ell$ we define the subresultant associated to P, n, Q and m of index i as follows:

$$Sres_i(P, n, Q, m) = \sum_{j=0}^{i} M_j^i(P, Q)x^j$$

where $M_j^i(P, Q)$ is the determinant of the matrix composed by the columns $1, 2, \cdots, n + m - 2i - 1$ and $n + m - i - j$ in the matrix $s_i(P, n, Q, m)$:

$$s_i = \overbrace{\begin{pmatrix} a_n & \cdots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_n & \cdots & a_0 \\ b_n & \cdots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_n & \cdots & b_0 \end{pmatrix}}^{n+m-i} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m-i \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-i \end{array}$$

Definition 11
[20] Let P, Q be polynomials in $\mathbf{R}[x]$ with degrees n, m, respectively. Here n, m be non-negative integers. Let $v = n + m - 1$ and $\delta_k = (-1)^{\frac{k(k+1)}{2}}$ for every integer k. The Sturm-Habicht sequence associated to P and Q is defined as the list of the polynomials $\{SH_j(P, Q)\}_{j=0,\cdots,v+1}$ given by

- $SH_{v+1}(P, Q) = P$,

- $SH_v(P,Q) = P'Q$, *and*
- $SH_j(P,Q) = \delta_{v-j} Sres_j(P,p,P'Q,v)$

for every $j = 0,1,\cdots,v-1$ *where* $P' = \frac{dP}{dx}$. *When* $Q = 1$, $\{SH_j(P,1)\}_{j=0,\cdots,v+1}$ *is called the Sturm-Habicht sequence of* P.

Sturm-Habicht sequence can be used for real root counting as is Sturm sequence according to the following theorem [20]:

Theorem 12
Let $P(x) \in \mathbf{R}[x]$ *and* $\alpha,\beta \in \mathbf{R} \cup \{-\infty, +\infty\}$ *s.t.* $\alpha < \beta$. *Then* $W_{SH}(P;\alpha,\beta)$ *gives a number of real roots of* P *in* $[\alpha,\beta]$.

C. Results in Swing-arm example :

$$\begin{aligned} f(s) = {} & 289265625s^5 + ((59078250000l_y - 1163605212)\theta_1 + (-335139750000l_y + \\ & 15366537516)\theta_2 + 17015625000000l_y - 228054750000)s^4 + (7877100000000l_y\theta_1 + \\ & (-44685300000000l_y + 2076250000000)\theta_2 + 4537500000000000l_y + \\ & 364707002000000000)s^3 + (26853750000000000000l_y\theta_1 + (-15233625000000000000l_y + \\ & 7078125000000000000)\theta_2 + 15471775000000000000000l_y - 9004875000000000000)s^2 + \\ & (20625000000000000000000l_y + 10617187500000000000000)s + \\ & 35156250000000000000000000l_y, \\ g(\omega) = {} & ((36798364968750l_y - 1132769673882)\theta_1 + (-20875017178125 0l_y + \\ & 9499435599951)\theta_2 + 4632503906250000l_y - 208750171781250)w^4 + \\ & ((-33446517130950000000l_y + 6588914512950000000)\theta_1 + \\ & (18973577229585000000l_y - 8699577542810000000)\theta_2 - \\ & 42105436312500000000l_y + 18973577229585000000)w^2 + \\ & 760296796875000000000000l_y\theta_1 + (-43130200781250000000000l_y + \\ & 200399414062500000000000)\theta_2 + 957128906250000000000000l_y - \\ & 43130200781250000000000. \end{aligned}$$

$$
\begin{aligned}
D_4 = {} & (1211811197127300000 0l_y^4 + 12452279264358355699 2l_y^3 - 2457301932362396196l_y^2)\theta_1^3 + \\
& ((-20623127884751700000 0\theta_2 + 349296976605654000000 0)l_y^4 + \\
& (-21109966275687545551 68\theta_2 + 15867003689606143388160)l_y^3 + \\
& (12597424023706621268 4\theta_2 + 13638763301915311449 12)l_y^2 + \\
& (-12953937710749959000\theta_2 + 31484379177903875648 04)l_y - \\
& 14351667589739879576 1)\theta_1^2 + ((11699110795451310000 00\theta_2^2 - \\
& 39629915041618440000 000\theta_2 + 23589945225000000000 00)l_y^4 + \\
& (11928846259542507349 824\theta_2^2 - 17657157973896678120 5760\theta_2 + \\
& 80773895102420520000 000)l_y^3 + (-11902326700350474816 12\theta_2^2 + \\
& 93216262281909522127 68\theta_2 + 32262349665695682093 7920)l_y^2 + \\
& (33106639142119712400\theta_2^2 - 17655756543516011526 144\theta_2 + \\
& 47401282886558376993 9408)l_y - 1707202725111056250\theta_2^2 + 7939120994727513627 96\theta_2 \\
& -18104337448291872915 984)\theta_1 + (-22122281703841110000 00\theta_2^3 + \\
& 11240651017565046000 0000\theta_2^2 - 13382130217500000000 000\theta_2 + \\
& 45295593750000000000 0000)l_y^4 + (-22468884233376565299 744\theta_2^3 + \\
& 49104457097933114553 9840\theta_2^2 + 54759477299036400000 00\theta_2 + \\
& 40664705112075000000 00000)l_y^3 + (31466090928386338119 72\theta_2^3 - \\
& 96289201547661476611 312\theta_2^2 + 24545512539904570472 5440\theta_2 + \\
& 11392003692328013260 000000)l_y^2 + (-14599600419336908910 0\theta_2^3 + \\
& 23808122956829744415 96\theta_2^2 - 43748829208541155764 0144\theta_2 + \\
& 10157235104237487024 000000)l_y + 2254527089797573125\theta_2^3 + \\
& 4292498901749325686 1\theta_2^2 + 19906686424242796631 112\theta_2 - \\
& 37721631423833878950 0000 \\
D_2 = {} & (6409990125000l_y^2 - 126251165502l_y)\theta_1^2 + ((-727253257500000\theta_2 + \\
& 5538585937500000)l_y^2 + (40730167659972\theta_2 + 1888102200600000 0)l_y - 332773460250\theta_2 - \\
& 584538523939524)\theta_1 + (2062785161250000\theta_2^2 - 31419351562500000 \theta_2 + \\
& 106347656250000000 0)l_y^2 + (-190425929035980\theta_2^2 - 10607399780175000 00\theta_2 + \\
& 23690315859375000 00)l_y + 43945969032 50\theta_2^2 + 48926867629213332\theta_2 - \\
& 78684788175125000 \\
A_0 = {} & l_y \\
A_2 = {} & 42966l_y\theta_1 + (-243738\theta_2 + 24754840)l_y + 113250\theta_2 - 144078 \\
A_4 = {} & (447562500l_y - 8815191)\theta_1 + (-25389375000\theta_2 + 128906250000)l_y + 1164131630\theta_2 - \\
& 1727687500
\end{aligned}
$$

$$\boxed{(20)}$$

$$
\begin{aligned}
S_1 = {} & 2162622l_y\theta_1 + (-12268146l_y + 570025)\theta_2 + 272250000l_y - 12268146, \\
S_2 = {} & (608118493290l_y - 11979844569)\theta_1 + (-3449741314470l_y + 158174137142)\theta_2 + \\
& 76555338750000l_y - 3449741314470, \\
S_3 = {} & (33790968750l_y - 1040192538)\theta_1 + (-191689781250l_y + 8723081359)\theta_2 + \\
& 4253906250000l_y - 191689781250, \\
S_4 = {} & (-1315259047878465690 0l_y^2 - 28927429723109037582 0l_y + 13046970536127163251)\theta_1^2 \\
& +((14922432146897615340 0l_y^2 + 15830176410011743210 20l_y - 71642355796305376236)\theta_2 \\
& -33115290215757750000 00l_y^2 - 36267184751020046346 600l_y + 16409984326795956342 60)\theta_1 \\
& +(-42326069061822498810 0l_y^2 + 32891407629556053732 0l_y - 13962545539331773576)\theta_2^2 \\
& +(18785678458800825000 000l_y^2 - 81456566598242949762 00l_y + 32891407629556053732 0)\theta_2 \\
& -20844229276406250000 0000l_y^2 + 18785678458800825000 000l_y - 42326069061822498810 0.
\end{aligned}
$$

$$\boxed{(21)}$$

$$
\begin{aligned}
D'_4 = {} & 31610567683432500147\theta_1^3 + (-14386178878433366766 3\theta_2 + \\
& 93354332714994774269385)\theta_1^2 + (21290504087151505298 4\theta_2^2 - \\
& 46966754104041336897 4860\theta_2 + 19991935079026618520 967000)\theta_1 \\
& -10310746368577548820 4\theta_2^3 - 92612817898780616398 885\theta_2^2 - \\
& 94756306894848817778 56000\theta_2 + 47589102994385781193 7500000,
\end{aligned}
$$

$$\begin{aligned}
D'_2 &= 3875223629 97\theta_1^2 + (-12676878191 67\theta_2 + 14206542644886900)\theta_1 + \\
&\quad 9876693539 78\theta_2^2 - 396215692702167 00\theta_2 + 1682140565153125000, \\
A'_2 &= 214 83\theta_1 - 274 94\theta_2 + 11176770, \\
A'_4 &= 1803855 9\theta_1 - 3592308 7\theta_2 + 6006687500.
\end{aligned}$$

$\boxed{(22)}$

$$\begin{aligned}
S'_1 &= 3243933\theta_1 - 4151594\theta_2 + 101671350, \\
S'_2 &= 1225363251142\theta_1 - 2440517086631\theta_2 + 5717895052650, \\
S'_3 &= 9872655587\theta_1 - 27783055516\theta_2 + 63544593750, \\
S'_4 &= -1089209155865471015 76\theta_1^2 + (5968977555263349294 36\theta_2 \\
&\quad -1367385392148199841 3400)\theta_1 + 1062140138044062176 51\theta_2^2 \\
&\quad -2304922021055354781300t2 - 1162805926020012202500.
\end{aligned}$$