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An algorithm for computing the residue of a rational function via $\mathcal{D}$-modules

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In a previous paper [4], we studied residues of a rational function from a view point of D-modules. We considered the system of linear differential equations for the algebraic local cohomology class associated to a given rational function. In particular, we gave there a description of the kernel space of the residue map induced by the cohomology class in terms of adjoint differential operators.

In this paper, we present algorithms for computing residues of a rational function according to the results obtained in [4]. By exploiting properties of adjoint differential operators, we reduce the computation of residues of a rational function to that of a rational function which has simple poles.

In the first section, we recall some facts about D-modules and the algebraic local cohomology groups for the one dimensional case. In the second section, we briefly recall the main results obtained in [4]. In the third section, we describe two algorithms for computing residues of a rational function. We utilize a formula for a squarefree denominator and properties of a logarithmic derivative in each algorithms.

1 D-module and Algebraic local cohomology

Let X be the complex plane $\mathbf{C}$, $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Let A be a set of finitely many points $z = \alpha_1, \ldots, \alpha_n$ on X. The sheaf $\mathcal{O}_X\langle *A\rangle/\mathcal{O}_X$ can be understood as the sheaf of principal parts of the meromorphic functions at A. Let $\mathcal{D}_X$ be the sheaf of rings on X of linear differential operators of finite order with holomorphic coefficients. Then $\mathcal{O}_X\langle *A\rangle/\mathcal{O}_X$ is naturally endowed with a structure of $\mathcal{D}_X$-module.

Moreover, we have the following isomorphism.

$$\mathcal{H}^1_{[A]}(\mathcal{O}_X) \simeq \mathcal{O}_X\langle *A\rangle/\mathcal{O}_X. \tag{1}$$

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Let us illustrate some fundamental properties of the algebraic local cohomology group as $\mathcal{D}_X$-module by using the Dirac's delta function $\delta(z) = \frac{1}{z} \bmod \mathcal{O}_X$, which belongs to the algebraic local cohomology group $\mathcal{H}^1_{[0]}(\mathcal{O}_X)$ with support at $z = 0$. ¿From $(\frac{d}{dz})^k \frac{1}{z} = (-1)^k \frac{k!}{z^{k+1}}$, we see that $\mathcal{H}^1_{[0]}(\mathcal{O}_X)$ is generated by $\delta(z)$ over $\mathcal{D}_X$. The annihilators of $\delta(z)$ is generated by the multiplication operator z as an ideal over $\mathcal{D}_X$. We arrive at the following representation :

$$\mathcal{H}^1_{[0]}(\mathcal{O}_X) = \mathcal{D}_X / \mathcal{D}_X z.$$

Moreover, we have the following fact.

FACT

$\mathcal{H}^1_{[0]}(\mathcal{O}_X)$ is simple as $\mathcal{D}_X$-module.

To see this fact, by example, we put $\sigma(z) = \frac{1}{z^2} + \frac{3}{z} \bmod \mathcal{O}_X$. Then we have $\delta(z) = z\sigma(z)$ and $\sigma(z) = (-\frac{d}{dz}+3)\delta(z)$. Thus, the cohomology class $\sigma(z)$ also generates $\mathcal{H}^1_{[0]}(\mathcal{O}_X)$. Moreover the annihilator ideal of $\sigma(z)$ over $\mathcal{D}_X$ is generated by $z\frac{d}{dz} - 3z + 2$ and z^2. Thus we have

$$\mathcal{H}^1_{[0]}(\mathcal{O}_X) = \mathcal{D}_X / \mathcal{D}_X (z\frac{d}{dz} - 3z + 2) + \mathcal{D}_X z^2.$$

In general, the principal part of a meromorphic function defines an element of the first algebraic local cohomology group with support at the poles and that, it is characterized by the annihilators.

2 Construction and properties of annihilator ideal

Let $q(z) = (z - \alpha_1)^{r_1} \cdots (z - \alpha_n)^{r_n}$ and $w(z) = 1/q(z)$. Denote by m the residue class $w(z) \bmod \mathcal{O}_X \in \mathcal{O}_X\langle *A\rangle / \mathcal{O}_X$, where $A = \{z \in \mathbf{C} | z = \alpha_j, j = 1, \ldots, n\}$. Then, in view of the isomorphism (1), m is an element of the algebraic local cohomology group $\mathcal{H}^1_{[A]}(\mathcal{O}_X)$ with support at A. Denote by $\mathcal{A}nn$ the set of annihilators of m as $\mathcal{D}_X$-module, $\mathcal{A}nn = \{R \in \mathcal{D}_X | Rm = 0\}$. This is a sheaf of left ideals of $\mathcal{D}_X$.

Put $A_j = \{\alpha_j\}$, $j = 1, \ldots, n$. Then the algebraic local cohomology group $\mathcal{H}^1_{[A]}(\mathcal{O}_X)$ has a direct sum decomposition by algebraic local cohomology groups with support at each A_j, i.e.,

$$\mathcal{H}^1_{[A]}(\mathcal{O}_X) = \mathcal{H}^1_{[A_1]}(\mathcal{O}_X) \oplus \cdots \oplus \mathcal{H}^1_{[A_n]}(\mathcal{O}_X).$$

Consequently, for given m, there exist local cohomology classes $m_j \in \mathcal{H}^1_{[A_j]}(\mathcal{O}_X)$, $j = 1, \ldots, n$ so that $m = m_1 + \ldots + m_n$.

The following Theorem asserts that the left ideal $\mathcal{A}nn$ characterizes m_j, up to constants.

Theorem 1

([4]) *Let $\mathcal{A}nn$ be the annihilator ideal of the algebraic local cohomology class m. Then we have*

$$\{f \in \mathcal{H}^1_{[A_j]}(\mathcal{O}_X) | Rf = 0, {}^\forall R \in \mathcal{A}nn\} = \{cm_j | c \in \mathbf{C}\}.$$

Moreover, $\mathcal{A}nn$ is generated by the following differential operators P and Q with polynomial coefficients.

$$P = (\prod_j (z-\alpha_j))\frac{d}{dz} + \sum_j r_j (\prod_{\ell \neq j}(z-\alpha_\ell)),$$

$$Q = (z-\alpha_1)^{r_1} \cdots (z-\alpha_n)^{r_n}.$$

That is, the principal part m_j at each pole A_j is characterized as a solution of the following homogeneous differential equations:

$$Pf = Qf = 0, \quad f \in \mathcal{H}^1_{[0]}(\mathcal{O}_X).$$

Denote by $q'(z)$ the derivative of $q(z)$. Then $q(z)/GCD(q(z), q'(z))$ is the squarefree part of $q(z)$, where $GCD(q(z), q'(z))$ is the greatest common divisor of $q(z)$ and $q'(z)$. Thus, the differential operator P in Theorem 1 is written in the following form:

Proposition 2

$$P = \frac{q(z)}{GCD(q(z), q'(z))}\frac{d}{dz} + \frac{q'(z)}{GCD(q(z), q'(z))}.$$

For a differential operator $R = \sum_j a_j(z)(\frac{d}{dz})^j$ with holomorphic coefficients $a_j(z)$, we can define the formal adjoint operator R^* of R by $R^* = \sum_j (-\frac{d}{dz})^j a_j(z)$. Denote by Ω_X the sheaf of holomorphic differential forms on X. For $\psi(z)dz \in \Omega_X$, we define the action of a differential operator $R \in \mathcal{D}_X$ from right by

$$(\psi(z)dz)R = (R^*\psi(z))dz.$$

Then, Ω_X has the structure of right $\mathcal{D}_X$-module. From Proposition 1, we have the following result.

Proposition 3

$$P^* = -\frac{q(z)}{GCD(q(z), q'(z))}\frac{d}{dz} + \frac{q(z)}{GCD(q(z), q'(z))}\frac{GCD(q(z), q'(z))'}{GCD(q(z), q'(z))}. \quad (2)$$

For a holomorphic differential form $\psi(z)dz \in \Omega_X$, let us consider residues of the differential form $\psi(z)w(z)dz$ at A_j. For this purpose, we look at the following linear mapping $\mathrm{Res}_{A_j}(\cdot, m)$ from Ω_X to $\mathbf{C}$ defined by

$$\begin{array}{rccc} Res_{A_j}(\cdot,m): & \Omega_X & \to & \mathbf{C} \\ & \psi(z)dz & \mapsto & Res_{A_j}(\psi(z)dz,m), \end{array}$$

where $m = w(z) \bmod \mathcal{O}_X$ and $\mathrm{Res}_{A_j}(\psi(z)dz, m) = \mathrm{Res}_{A_j}(\psi(z)mdz)$ which is equal to the residue of a meromorphic differential form $(\psi(z)/q(z))dz$ at A_j.

Since $Rm = 0$ holds for any $R \in \mathcal{A}nn$, we have

$$\mathrm{Res}_{A_j}((R^*\psi(z))dz, m) = \mathrm{Res}_{A_j}(\psi(z)dz, Rm) = 0.$$

Theorem 4
([4]) *Put* $K = \{\varphi(z)dz \in \Omega_X | Res_{A_j}(\varphi(z)dz, m) = 0, j = 1, 2, \ldots, n\}$. *Then we have*

$$K = \{(R^*\phi(z))dz | R \in \mathcal{A}nn, \phi(z)dz \in \Omega_X\}.$$

Put $r = \deg q(z)$. Let $p_j(z)$ be the image of $z^j (j = 0, 1, \ldots, r-n-1)$ by P^* in $\Gamma(X, \Omega_X)/\Gamma(X, Im(Q^*))$. For $K = \{\varphi(z)dz \in \Omega_X | Res_{A_j}(\varphi(z)dz, m) = 0, j = 1, 2, \ldots, n\}$, we have

Corollary 5

$$\Gamma(X, K)/\Gamma(X, Im(Q^*)) \cong Span\{p_0(z), \ldots, p_{r-n-1}(z)\}.$$

That is, a differential form $\varphi(z)dz$ which satisfy $Res_{A_j}(\varphi(z)dz, m) = 0$ and $\deg \varphi(z) \leq r-1$ can be written in a linear combination of $p_0(z)dz$, ..., $p_{r-n-1}(z)dz$.

3 Algorithm

We derive algorithms for computing residues of a rational function by using the results presented in th last section.

Let $u(z) = h(z)/q(z)$ be a rational function where $h(z)$ and $q(z)$ are polynomials in $\mathbf{Q}[z]$ with $\deg h(z) < \deg q(z) = r$. Denote by n the number of different zeros of $u(z)$. Let $q_s(z)$ be the square free part of $q(z)$. The degree of $q_s(z)$ is equal to n.

Now we introduce three vector spaces:

$$E = \{e(z) \in \mathbf{Q}[z] | \deg e(z) < r,\ e(z)/q(z) \text{ has at most simple poles}\},$$

$$F = \{f(z) \in \mathbf{Q}[z] | \deg f(z) < r,\ Res_{A_j}(\frac{f(z)}{q(z)}dz) = 0,\ j = 1, \ldots, n\},$$

$$G = \{g(z) \in \mathbf{Q}[z] | \deg g(z) < r\}.$$

The dimensions of these spaces are $\deg E = n$, $\deg F = r - n$ and $\deg G = r$, respectively. We use the vector space G to represent the quotient space $\mathbf{Q}[z]/\langle q(z)\rangle$ which is identified with

$$\{\varphi(z)dz|\varphi(z) \in \mathbf{Q}[z]\}/\{(q(z)\phi(z))dz|\phi(z) \in \mathbf{Q}[z]\}.$$

We denote by $srem(s_1, s_2)$ the remainder of the division of a polynomial s_1 by a polynomial s_2. Let $p_j(z) = srem(P^* z^j, q(z))$.

Then by using the formula (2) in the last section, we get

$$p_j(z) = srem(-q_s(z)jz^{j-1} + q_s(z)\frac{GCD(q(z), q'(z))'}{GCD(q(z), q'(z))}z^j, q(z)).$$

Then we have the next proposition.

Proposition 6

1. $G = E \oplus F$.
2. $F = Span\{p_0(z), p_1(z), \ldots, p_{r-n-1}(z)\}$.

Assume that we have $h(z) = h_E(z) + h_F(z)$, where $h_E(z) \in E$ and $h_F(z) \in F$. Then we have

$$\mathrm{Res}_{A_j}(\frac{h(z)}{q(z)}dz) = \mathrm{Res}_{A_j}(\frac{h_E(z)}{q(z)}dz).$$

We use it to reduce the computation to a simple case.

3.1 Algorithm($A1$)

Recall that $q(z)/GCD(q(z), q'(z))$ is the squarefree part $q_s(z)$ of $q(z)$. Then we have the following formula:

$$\frac{b(z)}{q_s(z)} = \frac{GCD(q(z), q'(z)) \cdot b(z)}{q(z)}, \tag{3}$$

with $b(z) \in \mathbb{Q}[z]$. Since $\{GCD(q(z), q'(z))z^\ell,\ \ell = 0, \ldots, n-1\}$ constitutes a basis of the vector space E, we can express $h(z)$ in the form

$$h(z) = h_E(z) + h_F(z),\ h_E(z) \in E \text{ and } h_F(z) \in F,$$

where $h_E(z) = GCD(q(z), q'(z)) \cdot b(z)$ with $\deg b(z) < n$ and $h_F(z) = \sum_{k=0}^{r-n-1} a_k p_k(z)$. ¿From Theorem 2 and the formula (3), we have

$$\begin{aligned}\mathrm{Res}(\frac{h(z)}{q(z)}dz) &= \mathrm{Res}(\frac{h_F(z)}{q(z)}dz)\\ &= \mathrm{Res}(\frac{b(z)}{q_s(z)}dz)\end{aligned}$$

Since $b(z)/q_s(z)$ is a rational function with poles of order one, the residue at $z = \alpha \in A$ is equal to $b(\alpha)/q_s'(\alpha)$. Let I be the ideal of $\mathbf{Q}[z,t]$ generated by $q_s(z)$ and $b(z) - tq_s'(z)$ where t is a new indeterminate. Let $g_1(t)$ be a generator of the ideal $I \cap \mathbf{Q}[t]$. Then residues of $u(z)$ satisfy the equation $g_1(t) = 0$.

Algorithm($A1$)

Input: polynomials $q(z), h(z) \in \mathbf{Q}[z]$ with $\deg h(z) < \deg q(z)$

$q_s(z) \leftarrow \dfrac{q(z)}{GCD(q(z), q'(z))}$

$r \leftarrow \deg q(z)$

$n \leftarrow \deg q_s(z)$

for $j = 0$ to $r - n - 1$

$\quad p_j(z) \leftarrow srem(-q_s(z) j z^{j-1} + q_s(z) \dfrac{GCD(q(z), q'(z))'}{GCD(q(z), q'(z))} z^j, q(z))$

$H \leftarrow h(z) - \sum_{j=0}^{r-n-1} a_j p_j(z) - GCD(q(z), q'(z)) \sum_{\ell=0}^{n-1} b_\ell z^\ell$

$(a_0, \ldots, a_{r-n-1}, b_0, \ldots, b_{n-1}) \leftarrow$ solve(coefficient(H, z^k) $= 0, 0 \leq k \leq r-1$)

$I \leftarrow \langle q_s(z), \sum_{\ell=0}^{n-1} b_\ell z^\ell - t \cdot q_s'(z) \rangle$

$\{g_1(t), g_2(t,z)\} \leftarrow$ the Gröbner basis of I with respect to the lexicographic order $z \succ t$

Output: $b(z)/q_s(z)$, $g_1(t)$

Note that the Grönber basis of I with respect to the lexicographic order $z \succ t$ consists of two polynomials $g_1(t)$ and $g_2(t,z)$ where $g_1(t)$ is an univariate polynomial of t of degree n and $g_2(t,z)$ is a polynomial of (t,z).

Localization : Let $q(z) = q_1^{\gamma_1}(z) \cdots q_N^{\gamma_N}(z)$ be the squarefree factorization. Let us compute residues at $q_j(z) = 0$ for some j. Since $q_j(z)$ and $\prod_{\ell \neq j} q_\ell(z)$ are linearly independent, we can compute polynomials $A(z)$ and $B(z)$ such that $A(z) q_j(z) + B(z) \prod_{\ell \neq j} q_\ell(z) = 1$ by using the extended Euclidean algorithm. Then we have

$$\frac{1}{\prod_{\ell \neq j} q_\ell(z)} = B(z) + A(z) \frac{q_j(z)}{\prod_{\ell \neq j} q_\ell(z)}.$$

For $z = \beta$ so that $q_j(\beta) = 0$, we have $1/\prod_{\ell \neq j} q_\ell(\beta) = B(\beta)$. Since $q_s'(z) = q_j'(z) \prod_{\ell \neq j} q_\ell(z) + q_j(z)(\prod_{\ell \neq j} q_\ell(z))'$, the residue of $u(z)$ at $z = \beta$ is given by $(b(\beta)B(\beta))/q_j'(\beta)$. Thus we obtain the equation for residues as follows:

Let I_j be the ideal of $\mathbf{Q}[z,t]$ generated by $q_j(z)$ and $b(z)B(z) - tq_j'(z)$ where t is a new indeterminate t. Let $g_{1j}(t)$ be a generator of the ideal $I \cap \mathbf{Q}[t]$. Then residues of $u(z)$ at $q_j(z) = 0$ satisfy the equation $g_{1j}(t) = 0$.

We can also obtain the localization of ($A1$) by using the primary decomposition of $\{g_1(t), g_2(t,z)\}$ in Algorithm($A1$).

3.2 Algorithm($A2$)

In Algorithm ($A2$), we take advantage of the fact that the logarithmic derivative of $q(z)$ has only simple poles. Let $e_\ell(z)$ be the remainder of the division of $q'(z)z^\ell$ by $q(z)$ for $\ell = 0, \ldots, n-1$. Then, by Corolary 1 of Theorem 2, we have $\{p_0(z), \ldots, p_{r-n-1}(z)$, $e_0(z), \ldots, e_{n-1}(z)\}$ as the basis of G. Thus, we can decompose $h(z)$ into

$$h(z) = \sum_{j=0}^{r-n-1} a_j p_j(z) + \sum_{\ell=0}^{n-1} b_\ell e_\ell(z). \tag{4}$$

Then we have

$$\mathrm{Res}(\frac{h(z)}{q(z)}dz) = \mathrm{Res}(\frac{\sum_{\ell=0}^{n-1} b_\ell e_\ell(z)}{q(z)}dz),$$

which equals to $\mathrm{Res}((\frac{q'(z)}{q(z)} \sum_{\ell=0}^{n-1} b_\ell z^\ell)dz)$.

Let $q(z) = q_1^{\gamma_1}(z) \cdots q_N^{\gamma_N}(z)$ be the squarefree factorization. Let $b_j(z)$ the remainder of the division of $\sum_{\ell=0}^{n-1} b_\ell z^\ell$ by $q_j(z)$, $j = 1, \ldots, N$. The residue of $u(z)$ at β with $q_j(\beta) = 0$ is equal to $\gamma_j b_j(\beta)$. Thus, we can compute residues in the following manner:

Let I_j be the ideal of $\mathbf{Q}[z,t]$ generated by $q_j(z)$ and $\gamma_j b_j(z) - t$. The Gröbner basis of I_j with respect to the lexicographic order $z \succ t$ is given by two polynomials $g_{j1}(t)$ and $g_{j2}(t,z)$ where $g_{j1}(t)$ is a polynomial of t with $\deg g_{j1}(t) = \deg q_j(z)$ and $g_{j2}(t,z)$ is a polynomial of t and z. Then $g_{j1}(t) = 0$ is the equation for residues of $u(z)$ at $q_j(z) = 0$.

Algorithm($A2$)

Input $q(z), h(z) \in \mathbf{Q}[z]$ with $\deg h(z) < \deg q(z)$

$q_1^{\gamma_1}(z) \cdots q_N^{\gamma_N}(z) \leftarrow$ squarefree factorization of $q(z)$

$q_s(z) \leftarrow \dfrac{q(z)}{GCD(q(z), q'(z))}$

$r \leftarrow \deg q(z)$

$n \leftarrow \deg q_s(z)$

for $j = 0$ to $r - n - 1$

  $p_j(z) \leftarrow srem(-q_s(z) j z^{j-1} + q_s(z)\dfrac{GCD(q(z), q'(z))'}{GCD(q(z), q'(z))})z^j, q(z))$

for $\ell = 1$ to N

  $e_\ell(z) \leftarrow srem(q'(z)z^\ell, q(z))$

$H \leftarrow h(z) - \sum_{j=0}^{r-n-1} a_j p_j(z) + \sum_{\ell=0}^{n-1} b_\ell e_\ell$

$(a_0, \ldots, a_{r-n-1}, b_0, \ldots, b_{n-1}) \leftarrow$ solve(coefficient(H, z^k), $0 \le k \le r-1$)

for $j = 1$ to N

  $b_j(z) \leftarrow srem(\sum_{\ell=0}^{n-1} b_\ell z^\ell, q_j(z))$

  $I_j \leftarrow \langle q_j(z), \gamma_j b_j(z) - t \rangle$

  $\{g_{j1}(t), g_{j2}(t,z)\} \leftarrow$ Gröbner basis of I_j with respect to the lexicographic order $z \succ t$

Output $\{g_{11}(t) = 0, g_{12}(t,z) = 0\}, \ldots, \{g_{N1}(t) = 0, g_{N2}(t,z) = 0\}$

4 Example

Let us compute residues of $u(z) = \dfrac{(z^3+z+1)^3}{z^4(z^2+1)^3}$. Put $m = 1/q(z) \bmod \mathcal{O}_X$ for $q(z) = z^4(z^2+1)^3$. Then the annihilator ideal $\mathcal{A}nn$ of m is generated by

$$P = (z^3+z)\frac{d}{dz} + 10z^2 + 4 \quad \text{and} \quad Q = z^{10} + 3z^8 + 3z^6 + z^4.$$

We have

$$P^* = (-z^3-z)\frac{d}{dz} + 7z^2 + 3 \quad \text{and} \quad Q^* = z^{10} + 3z^8 + 3z^6 + z^4.$$

The squarefree part of $q(z)$ is $q_s(z) = z(z^2+1)$. Then $\dim F (= 10-3) = 7$. The Corolary 1 of Theorem 2 implies that $\{p_k(z),\ k = 0,\dots,6\}$ is a basis of F, where $p_0(z) = 7z^2+3$, $p_1(z) = 6z^3+2z$, $p_2(z) = 5z^4+z^2$, $p_3(z) = 4z^5$, $p_4(z) = 3z^6 - z^4$, $p_5(z) = 2z^7 - 2z^5$, $p_6(z) = z^8 - 3z^6$.

Along $(A1)$

We have $GCD(q(z), q'(z)) = z^7 + 2z^5 + z^3$. Then

$$\begin{aligned}(z^3+z+1)^3 &= \frac{1}{3}p_0(z) + \frac{3}{2}p_1(z) + \frac{2}{3}p_2(z) + \frac{9}{2}p_3(z) - -\frac{31}{24}p_4(z) + 3p_5(z) - -\frac{11}{8}p_6(z)\\ &\quad + GCD(q(z), q'(z)) \cdot (z^2 + \frac{11}{8}z - 5)\end{aligned}$$

Let I be the ideal generated by $(z^2+1)z$ and $(z^2 + \frac{11}{8}z - 5) - t \cdot (3z^2+1)$. The Gröbner basis of I with respect to the lexicographic order $z \succ t$ is given by

$$\{-256t^3 + 256t^2 + 5255t - 12125, 181555z + 32768t^2 + 67472t - 481840\}.$$

Thus we have

$$-256t^3 + 256t^2 + 5255t - 12125 = -(t+5)(256t^2 - 1536t + 2425) = 0$$

as the equation for residues.

Since the primary decomposition of I is given by

$$\langle t+5, z\rangle, \langle 256t^2 - 1536t + 2425, 11z + 16t - 48\rangle,$$

we have $\mathrm{Res}_{z=0}(u(z)dz) = -5$.

Along $(A2)$

Since $q'(z) = 10z^9 + 24z^7 + 18z^5 + 4z^3$, we have

$$\begin{aligned} e_0(z) &= 10z^9 + 24z^7 + 18z^5 + 4z^3,\\ e_1(z) &= -6z^8 - 12z^6 - 6z^4,\\ e_2(z) &= -6z^9 - 12z^7 - 6z^5.\end{aligned}$$

From

$$\begin{aligned}(z^3+z+1)^3 &= \frac{1}{3}p_0(z)+\frac{3}{2}p_1(z)+\frac{2}{3}p_2(z)+\frac{9}{2}p_3(z)--\frac{31}{24}p_4(z)+3p_5(z)--\frac{11}{8}p_6(z)\\ &\quad +(-\frac{9}{4}e_2(z)-\frac{11}{48}e_1(z)-\frac{5}{4}e_0(z)),\end{aligned}$$

we have $b(z) = -\frac{9}{4}z^2 - \frac{11}{48}z - \frac{5}{4}$. Since the point $z = 0$ is the pole of order 4 and $b_1(z) = -\frac{5}{4}$, we have

$$\mathrm{Res}_{z=0}(u(z)dz) = 4\cdot(-\frac{5}{4}) = -5$$

as the residue of $u(z)$ at $z = 0$.

The points $z^2+1=0$ are poles of order 3 of and $b_2(z) = -\frac{11}{48}z+1$. Denote by I_2 the ideal generated by z^2+1 and $3(-\frac{11}{48}z+1)-t$. The Gröbner basis of I_2 with respect to the lexicographic order $z \succ t$ is given by

$$\{-256t^2+1536t-2425, -11z-16t+48\}.$$

Thus we have $-256t^2+1536t-2425=0$ as the equation for residues at $z^2+1=0$.

5 Summary

For a given rational function, we gave two algorithms ($A1$) and ($A2$) for computing residues. As a rational function which has only simple poles, we utilized a rational function with a squarefree denominator in ($A1$) and a logarithmic derivative in ($A2$). By generalizing the Algorithm ($A2$), we have constructed an algorithm for computing the Grothendieck residue in [5].

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