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CONSTRUCTION OF FLAT TORI IN THE 3-SPHERE AND ITS APPLICATIONS

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Introduction

In 1975, Yau [16, p.87] posed the problem of the classification of flat tori in the unit 3-sphere S^3. Concerning this problem, there is a method for constructing flat surfaces in S^3 which was obseved by Lawson ([12]). Let $p : S^3 \to S^2$ be the Hopf fibration, and let γ be a curve in S^2. Then the inverse image $p^{-1}(\gamma)$ is a flat surface in S^3. If the curve γ is closed, the inverse image $p^{-1}(\gamma)$ is a flat torus in S^3 and it is called the *Hopf torus* corresponding to γ ([10]). On the other hand, there is another method for constructing flat surfaces in S^3 which was obtained by Bianchi[1] and Sasaki[11]. Let α and β be curves in S^3 whose torsions satisfy $\tau_\alpha = 1$ and $\tau_\beta = -1$. Using the group structure on S^3, define a surface $F : \mathbb{R}^2 \to S^3$ by

$$F(s_1, s_2) = \alpha(s_1)\beta(s_2).$$

Then F is a flat surface such that the s_i-curves are the asymptotic curves of the surface ([13, p.139 - 163]). Infinitely many complete flat surfaces in S^3 are constructed by this method.

Recently, using the Hopf fibration, the author obtained a method for constructing closed curves in S^3 with $\tau = \pm 1$. Combining this result with the method of Bianchi and Sasaki, the author established a method for constructing all the flat tori isometrically immersed in S^3 (Theorem 1). Applying this method, we obtain some interesting results on flat tori in S^3. For example, by using the Arf invariant for knots, we see that every embedded flat torus in S^3 is invariant under the antipodal map of S^3 (Theorem 4). This implies a rigidity theorem for the Clifford tori in S^3 (Theorem 5). Furthermore, we obtain the classification of undeformable flat tori in S^3 (Theorem 8).

The outline of this article is as follows. In Section 1 we explain the method for constructing all the flat tori isometrically immersed in S^3. In Section 2 we deal with embedded flat tori in S^3, and in Section 3 we explain the result of the classification of undeformable flat tori in S^3.

1. Construction of flat tori in S^3

We first explain a method for constructing closed curves in S^3 with $\tau = \pm 1$. Let $\mathbb{H}$ denote the set of all quaternions, and let $\mathbb{R}^4$ be the 4-dimensional Euclidean space identified with $\mathbb{H}$ as follows:

$$(x_1, x_2, x_3, x_4) \longleftrightarrow x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}.$$

Then the unit spheres S^2 and S^3 are given by

$$S^2 = \{x \in \operatorname{Im}\mathbb{H} : |x| = 1\}, \quad S^3 = \{x \in \mathbb{H} : |x| = 1\}.$$

Note that the unit sphere S^3 has a group structure induced by the multiplicative structure of $\mathbb{H}$. Let US^2 denote the unit tangent bundle of S^2 identified with a subset of $S^2 \times S^2$ as follows:

$$US^2 = \{(x, v) \in S^2 \times S^2 : \langle x, v\rangle = 0\},$$

where the canonical projection $p_1 : US^2 \to S^2$ is given by $p_1(x, v) = x$. Let $p_2 : S^3 \to US^2$ be a double covering map defined by $p_2(a) = (a\mathbf{i}a^{-1}, a\mathbf{j}a^{-1})$. Then the Hopf fibration $p : S^3 \to S^2$ is given by $p = p_1 \circ p_2$. We now consider a regular curve $\gamma : \mathbb{R} \to S^2$ with a period $l > 0$, and define a curve $\hat{\gamma} : \mathbb{R} \to US^2$ by

$$\hat{\gamma}(s) = (\gamma(s), \gamma'(s)/|\gamma'(s)|).$$

We denote by $I(\gamma)$ the element of the homology group $H_1(US^2)$ represented by the closed curve $\hat{\gamma} : [0, l] \to US^2$. Let $c : \mathbb{R} \to S^3$ be a lift of the curve $\hat{\gamma}$ with respect to the covering p_2. Since $H_1(US^2) \cong \mathbb{Z}_2$ and the double covering p_2 satisfies the relation $p_2(-a) = p_2(a)$, we obtain

$$c(s+l) = \begin{cases} c(s) & I(\gamma) = 0, \\ -c(s) & I(\gamma) = 1. \end{cases} \tag{1}$$

On the other hand we obtain

$$c(s)^{-1}c'(s) = \frac{1}{2}|\gamma'(s)|(\mathbf{k} + k(s)\mathbf{i}),$$

where $k(s)$ denotes the geodesic curvature of $\gamma(s)$. This implies that the torsions of the curves c and c^{-1} satisfy

$$\tau_c = 1, \quad \tau_{c^{-1}} = -1. \tag{2}$$

Combining the observations above with the method of Bianchi and Sasaki, we obtain a method for constructing all the flat tori isometrically immersed in S^3.

Definition . A *periodic admissible pair* (p.a.p.) is a pair of periodic regular curves $\gamma_i : \mathbb{R} \to S^2$, $i = 1, 2$, such that
(a) $k_1(s_1) > k_2(s_2)$ for all $s_1, s_2 \in \mathbb{R}$,
(b) $|\gamma_i'(s)|\sqrt{1 + k_i(s)^2} = 2$,
where $k_i(s)$ denotes the geodesic curvature of $\gamma_i(s)$.

Let $\Gamma = (\gamma_1, \gamma_2)$ be a p.a.p., and let c_i be a lift of $\hat{\gamma}_i$ with respect to the covering p_2. Using the group structure of S^3, we define the map $F_\Gamma : \mathbb{R}^2 \to S^3$ by

$$F_\Gamma(s_1, s_2) = c_1(s_1)c_2(s_2)^{-1}. \tag{3}$$

Then it follows that the map F_Γ is an immersion and induces a flat Riemannian metric g_Γ on $\mathbb{R}^2$. We now consider the group

$$G(\Gamma) = \{\varphi \in \mathrm{Diff}(\mathbb{R}^2) : F_\Gamma \circ \varphi = F_\Gamma\}.$$

Since each element of $G(\Gamma)$ is a parallel translation of $\mathbb{R}^2$, we identify the group $G(\Gamma)$ with a subgroup of $\mathbb{R}^2$. Then we obtain a flat torus $M_\Gamma = (\mathbb{R}^2, g_\Gamma)/G(\Gamma)$, and an isometric immersion

$$f_\Gamma : M_\Gamma \to S^3$$

satisfying the relation $f_\Gamma \circ \pi_\Gamma = F_\Gamma$, where $\pi_\Gamma : \mathbb{R}^2 \to M_\Gamma$ denotes the canonical projection.

Theorem 1 ([5, 7]). *Let $f : M \to S^3$ be an isometric immersion of a flat torus M into the unit sphere S^3. Then there exist a p.a.p. Γ and a covering map $\rho : M \to M_\Gamma$ such that $A \circ f = f_\Gamma \circ \rho$ for some isometry $A : S^3 \to S^3$.*

Remark 1. If the image of f_Γ contains a great circle of S^3, either γ_1 or γ_2 is a circle in S^2. So it follows that there exists a p.a.p. Γ such that the image of f_Γ is not congruent to any Hopf torus in S^3.

Remark 2. Let l_i be the period of γ_i. Then

$$\text{generators of } G(\Gamma) = \begin{cases} (l_1, 0),\ (0, l_2) & \text{if } I(\gamma_1) = 0,\ I(\gamma_2) = 0, \\ (2l_1, 0),\ (0, l_2) & \text{if } I(\gamma_1) = 1,\ I(\gamma_2) = 0, \\ (l_1, 0),\ (0, 2l_2) & \text{if } I(\gamma_1) = 0,\ I(\gamma_2) = 1, \\ (l_1, l_2),\ (l_1, -l_2) & \text{if } I(\gamma_1) = 1,\ I(\gamma_2) = 1. \end{cases} \tag{4}$$

Remark 3. Consider two curves a_1 and a_2 in M_Γ given by

$$a_1(s) = \pi_\Gamma(s, 0), \quad a_2(s) = \pi_\Gamma(0, s). \tag{5}$$

Then it follows from (4) that they are simple closed curves in M_Γ.

Remark 4. Weiner [14, 15] studied the Gauss map of a flat torus in S^3, and obtained a method for constructing all the flat tori in S^3 which is different from ours. For the Gauss map of the immersion $f_\Gamma : M_\Gamma \to S^3$, see Section 7 of [7]

In the rest of this section we deal with the intrinsic structure of M_Γ. For each p.a.p. $\Gamma = (\gamma_1, \gamma_2)$, we set

$$L(\gamma_i) = \int_0^{l_i} |\gamma_i'(s)| ds, \quad K(\gamma_i) = \int_0^{l_i} k_i(s)|\gamma_i'(s)| ds,$$

where l_i denotes the period of γ_i. Then it follows that the intrinsic structure of the flat torus $M_\Gamma = (\mathbb{R}^2, g_\Gamma)/G(\Gamma)$ is determined by the following data:

$$\{I(\gamma_i), K(\gamma_i), L(\gamma_i)\}_{i=1,2}.$$

More precisely, we obtain

Theorem 2 ([8]). *Let $W(\Gamma)$ be a lattice of $\mathbb{R}^2$ generated by the following vectors.*

$$\begin{cases} \vec{v}_1,\ \vec{v}_2 & \text{if} \quad I(\gamma_1)=0,\ I(\gamma_2)=0, \\ 2\vec{v}_1,\ \vec{v}_2 & \text{if} \quad I(\gamma_1)=1,\ I(\gamma_2)=0, \\ \vec{v}_1,\ 2\vec{v}_2 & \text{if} \quad I(\gamma_1)=0,\ I(\gamma_2)=1, \\ \vec{v}_1 \pm \vec{v}_2 & \text{if} \quad I(\gamma_1)=1,\ I(\gamma_2)=1, \end{cases}$$

where $\vec{v}_i = \frac{1}{2}(K(\gamma_i), L(\gamma_i))$. Then the flat torus M_Γ is isometric to $\mathbb{R}^2/W(\Gamma)$.

2. Embedded flat tori in S^3

In this section, using the method explained in Section 1, we study the embedded flat tori in S^3. Let $\Gamma = (\gamma_1, \gamma_2)$ be a p.a.p., and let $I(\Gamma) = (I(\gamma_1), I(\gamma_2))$.

Theorem 3 ([7]). *If $f_\Gamma : M_\Gamma \to S^3$ is an embedding, then $I(\Gamma) = (1,1)$.*

Proof (Outline). Assume that $I(\Gamma) \neq (1,1)$. Using the embedding f_Γ, we identify M_Γ with a subset of S^3. Let a_1 and a_2 be the simple closed curves in M_Γ given by (5), and let a_i^+ be a simple closed curve in $S^3 - M_\Gamma$ obtained by pushing the curve a_i a very small amount along a unit normal vector field ξ of M_Γ. Then we obtain the links $\{a_1, a_1^+\}$ and $\{a_2, a_2^+\}$ in S^3, and it follows that the linking numbers of these links satisfy

$$\text{(6)} \qquad \mathrm{lk}(a_1, a_1^+) \equiv \mathrm{lk}(a_2, a_2^+) \equiv 1, \quad \mod 2.$$

We now consider a disk $D \subset M_\Gamma$ which does not intersect the union $a_1 \cup a_2$, and let K be a knot in S^3 given by $K = \partial D$. Since $I(\Gamma) \neq (1,1)$, it follows from (4) that $\{a_1, a_2\}$ is a canonical basis of the homology group $H_1(V)$, where V is a Seifert surface of the knot K given by $V = M_\Gamma - D$. So, by using [4, Chapter10], we see that the Arf invariant of K is given by

$$\mathrm{Arf}(K) \equiv \mathrm{lk}(a_1, a_1^+)\,\mathrm{lk}(a_2, a_2^+), \quad \mod 2.$$

Hence, (6) implies $\mathrm{Arf}(K) = 1$. On the other hand, since $K = \partial D$, we obtain $\mathrm{Arf}(K) = 0$. This is a contradiction. □

It follows from Theorem 3 and (1) that if f_Γ is an embedding, then the image of f_Γ is invariant under the antipodal map of S^3. Hence, Theorem 1 implies the following

Theorem 4 ([7]). *If $f : M \to S^3$ is an isometric embedding of a flat torus M, then the image $f(M)$ is invariant under the antipodal map of S^3.*

As an application of this theorem, we obtain a rigidity theorem for the Clifford tori in S^3. For positive numbers R_1 and R_2 satisfying $R_1^2 + R_2^2 = 1$, let $S^1(R_1) \times S^1(R_2)$ denote the Clifford torus in S^3 given by

$$S^1(R_1) \times S^1(R_2) = \{x \in \mathbb{R}^4 : x_1^2 + x_2^2 = R_1^2,\ x_3^2 + x_4^2 = R_2^2\},$$

and let $i : S^1(R_1) \times S^1(R_2) \to S^3$ denote the inclusion map.

Theorem 5 ([3]). *If $f : S^1(R_1) \times S^1(R_2) \to S^3$ is an isometric embedding, then there exists an isometry A of S^3 such that $f = A \circ i$.*

Proof (Outline). We can show that if $f : S^1(R_1) \times S^1(R_2) \to S^3$ is an isometric immersion whose extrinsic diameter is equal to π, then there exists an isometry A of S^3 such that $f = A \circ i$. So the assertion follows from Theorem 4. □

Remark 5. RecentlyCDadok-Sha [2] obtained the same result as we have proved in Theorem 4. Their proof is different from ours.

Remark 6. In Theorem 4 the word "embedding" cannot be replaced by "immersion". In fact, there exists an isometric immersion of a flat torus into S^3 whose image is not invariant under the antipodal map of S^3 ([7, Theorem 4.4]). However the author does not know whether the extrinsic diameter of any flat torus isometrically immersed in S^3 is equal to π. If this is true, the conclusion of Theorem 5 is valid for every isometric immersion $f : S^1(R_1) \times S^1(R_2) \to S^3$.

3. The classification of undeformable flat tori in S^3

An isometric immersion $f : M \to S^3$ is said to be *deformable* if there exists a nontrivial isometric deformation of f. As a corollary of Theorem 5, it follows that the inclusion map $i : S^1(R_1) \times S^1(R_2) \to S^3$ is not deformable ([6]). In this section we give the classification of undeformable flat tori isometrically immersed in S^3.

Theorem 6 ([8]). *Let $f : M \to S^3$ be an isometric immersion of a flat torus M into S^3. If the mean curvature of the immersion f is not constant, then f is deformable.*

Proof (Outline). It follows from Theorem 1 that there exist a p.a.p. $\Gamma = (\gamma_1, \gamma_2)$ and a covering map $\rho : M \to M_\Gamma$ such that

$$f = A \circ f_\Gamma \circ \rho,$$

where A denotes an isometry of S^3. So it is sufficient to show that f_Γ is deformable. On the other hand, we see that the mean curvature of f_Γ is constant if and only if both γ_1 and γ_2 are circles in the unit sphere S^2. So, by the assumption, either γ_1 or γ_2 is not a circle. This ensures the existence of a nontrivial deformation of Γ which preserves the data $\{I(\gamma_i), K(\gamma_i), L(\gamma_i)\}_{i=1,2}$. Hence, Theorem 2 implies a nontrivial isometric deformation of f_Γ. □

We now consider an isometric immersion $f : M \to S^3$ of a flat torus M into S^3 with constant mean curvature. In this case, it is easy to see that f is congruent to the immersion

$$F/G : \mathbb{R}^2/G \to S^3, \tag{7}$$

where F is a covering map of $\mathbb{R}^2$ onto a Clifford torus $S^1(R_1) \times S^1(R_2)$ defined by

$$F(x_1, x_2) = \left(R_1 \cos \frac{x_1}{R_1},\ R_1 \sin \frac{x_1}{R_1},\ R_2 \cos \frac{x_2}{R_2},\ R_2 \sin \frac{x_2}{R_2} \right),$$

and G is a subgroup of the covering transformation group of F such that $\mathbb{R}^2/G$ is compact. The covering transformation group of F, which consists of parallel translations of $\mathbb{R}^2$, is generated by the vectors $\vec{e}_1 = (2\pi R_1,\ 0)$ and $\vec{e}_2 = (0, 2\pi R_2)$. So the group G is generated by

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2, \quad \vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2, \tag{8}$$

where $a_i, b_i \in \mathbb{Z}$ and $a_1b_2 - a_2b_1 \neq 0$.

Theorem 7 ([9]). *The following statements* (*a*) *and* (*b*) *are equivalent.*
(*a*) $F/G : \mathbb{R}^2/G \longrightarrow S^3$ *is not deformable,*
(*b*) $g.c.d.(a_1 + a_2, b_1 + b_2) = g.c.d.(a_1 - a_2, b_1 - b_2) = 1.$

Proof (Outline). (a) $\Rightarrow$ (b). Assume that $g.c.d.(a_1 + a_2, b_1 + b_2) = n \geq 2$. Let G_0 denote the covering transformation group of F, which is generated by $\vec{e}_1$ and $\vec{e}_2$, and let $W(n)$ be a subgroup of G_0 given by

$$W(n) = \{n_1\vec{e}_1 + n_2\vec{e}_2 : n_1 + n_2 \in n\mathbb{Z}\}.$$

Then $G \subset W(n)$ and the immersion $F/W(n)$ is congruent to the Hopf torus $p^{-1}(\gamma)$, where γ denotes a n-fold circle in S^2. Since the Hopf torus $p^{-1}(\gamma)$ is deformable for $n \geq 2$, we see that the immersion $F/W(n)$ is deformable. So it follows from $G \subset W(n)$ that the immersion F/G is deformable. Similarly, the immersion F/G is deformable if $g.c.d.(a_1 - a_2, b_1 - b_2) \neq 1$.

(b) $\Rightarrow$ (a). Let $f_t : \mathbb{R}^2/G \longrightarrow S^3$ be an isometric deformation of F/G, and let F_t be the isometric deformation of $F : \mathbb{R}^2 \longrightarrow S^3$ induced by f_t. Then each F_t is invariant under the group G. Furthermore we can show that each F_t is $\sigma(G)$-invariant, where σ denotes an automorphism of G_0 satisfying

$$\sigma(\vec{e}_1) = \vec{e}_2, \quad \sigma(\vec{e}_2) = \vec{e}_1.$$

On the other hand, the assumption (b) implies $G + \sigma(G) = G_0$, and so F_t is G_0-invariant. Hence we obtain an isometric deformation F_t/G_0 of the isometric embedding $F/G_0 : \mathbb{R}^2/G_0 \longrightarrow S^3$. Since the embedding is congruent to the inclusion map $i : S^1(R_1) \times S^1(R_2) \longrightarrow S^3$, it follows from Theorem 5 that for each t there exists an isometry A_t of S^3 such that $F_t/G_0 = A_t \circ (F/G_0)$. Hence $f_t = F_t/G = A_t \circ (F/G)$, and so the immersion F/G is not deformable. □

By Theorems 6 and 7, we obtain the following classification of undeformable flat tori in S^3.

Theorem 8. *Let* $f : M \longrightarrow S^3$ *be an isometric immersion of a flat torus* M *into* S^3. *Then the immersion* f *is not deformable if and only if it is congruent to the immersion* F/G *defined by* (7) *such that the group* G *is generated by* $\vec{a}$ *and* $\vec{b}$ *satisfying* (8) *and* $g.c.d.(a_1 + a_2, b_1 + b_2) = g.c.d.(a_1 - a_2, b_1 - b_2) = 1.$

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