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SOME LIOUVILLE TYPE INEQUALITIES AND ITS APPLICATIONS TO GEOMETRIC PROBLEMS

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Before going to give our talk let us consider an easy global property in Calculus. We consider a function f with one variable on the open interval (a,b). As is well known, if it satisfies

$$f''(x) = 0$$

on (a,b) and if it has a maximum on (a,b), namely, if there is a point x_0 on (a,b) at which $f(x_0) \geq f(x)$ for any point x on (a,b), then f is constant. This property is usually called *Maximum Principle* in Calculus.

Now let us denote by U an open connected set in an m-dimensional Euclidean space $\mathbb{R}^m$ and $\{x^j\}$ a Euclidean coordinate. We denote by L a differential operator defined by

$$L = \sum a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum b^j \frac{\partial}{\partial x^j},$$

where a^{ij} and b^j are smooth functions on U for any indices. When the matrix (a^{ij}) is positive definite and symmetric, it is called *a second order elliptic differential operator*. We assume that L is an elliptic differential operator. The Maximum Principle is explained as follows:

Maximum Principle.

For a smooth function f on U if it satisfies

$$Lf \geqq 0$$

and if there exists a point in U at which it attains the maximum, namely, if there exists a point x_0 in U at which $f(x_0) \geqq f(x)$ for any point x in M, then the function f is constant.

In Riemannian Geometry, this property is reformed as follows. Let (M, g) be a Riemannian manifold with Riemannian metric g. Then we denote by $\triangle$ the Laplacian associated with the Riemannian metric g. A function f is said to be *subharmonic* or *harmonic* if it satisfies

$$\triangle f \geqq 0 \quad \text{or} \quad \triangle f = 0.$$

The maximum principle on Riemannian manifolds is as follows:

Maximum Principle.

For a subharmonic function f on a Riemannian manifold M if there exists a point in M at which it attains the maximum, then the function f is constant.

In other words, we have a litter different Maximum Principle:

Maximum Principle.

On a compact Riemannian manifold M a subharmonic function f on M is constant.

This property is to give a certain condition for a subharmonic function to be constant. When we give an attention to the facts which are relative to this kind of Maximum Principles, we see the classical theorem of Liouville type.

Liouville's theorem.

(1) *Let f be a subharmonic function on $\mathbb{R}^2$. If it is bounded, then it is constant.*

(2) *Let f be a harmonic function on $\mathbb{R}^m (m \geqq 3)$. If it is bounded, then it is constant.*

As is already stated, each of these Maximum Principles plays an important role in each branch of Mathematics. Actually *Generalized Maximum Principles* which are later introduced are also similarly important properties to Maximum Principle in a compact Riemannian manifold or more important ones than them.

In particular, a similar property on a complete Riemannian manifold was treated by Nishikawa [13], who determined space-like hypersurfaces in a Lorentz space. His Liouville type theorem in a complete Riemannian manifold says

Theorem A. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If a C^2-positive function f satisfies*

$$\triangle f \geq 2f^2,$$

where $\triangle$ denotes the Laplacian on M, then f vanishes identically.

The purpose of this talk is to prove the fact that

$$\triangle f \geq k f^n \Longrightarrow f = 0$$

for any $n \in \mathbb{R}$, $n \geq 1$. In order to solve this kind of Liouville type problem, we want to investigate all of situations for *any positive real number* n not less than 1 (See [7],[8],[12], [15] and [16]). For this problem we want to arrange all the results concerned with this fact.

Firstly we will show that all the situation greater than 2 could be arrived at the Theorem of Nishikawa [13] in above. Next we treat for the case $n = 2$. By using a new method due to Omori and Yau's maximum principle in [14] and [17], we give another proof of this case (See [15]).

Now we will show another type of Liouville's theorem for $1 < n < 2$ by using some generalized maximum principles due to Choi, Kwon and the present author (See [7],[8] and [11]) as follows:

Theorem B. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be any polynomial of the variable f with constant coefficients such that*

$$F(x) = c_0 x^{n_0} + c_1 x^{n_1} + \cdots + c_k x^{n_k} + c_{k+1},$$

where $n_0 > 1$, $n_0 > n_1 > \cdots > n_k$, $c_0 > 0$ and $c_0 > c_{k+1}$. If a C^2-function f satisfies

$$\triangle f \geq F(f),$$

then we have

$$F(f_0) \leq 0,$$

where f_0 denotes the supermum of the given function f.

Then we will show its applications of Theorem B to some geometric problems given in [1],[3],[5],[6],[8],[10] and [11]. In order to do this let us introduce the following.

Let M' be an $(n+1)$-dimensional Lorentz manifold and let M be a space-like hypersurface of M'. For a point x in M let $\{e_0, e_1, ..., e_n\}$ be a local field of orthogonal frames of M' around of x in such a way that, restricted to M, the vectors $e_1, ..., e_n$ are tangent to M and the other is normal to M. Accordingly, $e_1, ..., e_n$ are space-like vectors and e_0 is a time-like one. For a linearly independent vectors u and v in the tangent space T_xM' by which the non-degenerate plane section is spanned, we denote by $K'(u, v)$ the sectional curvature of the plane section in M' and by R' or $Ric'(u, u)$ the Riemannian curvature tensor on M or the Ricci curvature in the direction of u in M', respectively.

Let us denote by ∇' the Riemannian connection on M'. We assume that the ambient space M' satisfies the following conditions; For some constants c_1, c_2 and c_3

$$K'(u, v) = -\frac{c_1}{n}$$

for any space-like vector u and any time-like vector v,

$$K'(u, v) \geq c_2$$

for any space-like vectors u and v,

$$|\nabla' R'| \leq \frac{c_3}{n}.$$

When M' satisfies the above three kind of curvature conditions, it is said simply for M' to satisfy the (*) condition.

Remark 1. It can be easily seen that if $c_3 = 0$, then the ambient space M' is locally symmetric.

Remark 2. If M' is a Lorentz space form $M_1^{n+1}(c)$ of index 1 and of constant curvature c, then it satisfies the condition (*), where $-\frac{c_1}{n} = c_2 = c$.

Now as a first application of Theorem B of Liouville type inequality for $1 < n < 2$ we introduce the following

Theorem 1. ([8]) Let M' be an $(n+1)$-dimensional Lorentz manifold which satisfies the condition (*) and let M be a complete space-like hypersurface with constant mean curvature. If M is not maximal and if it satisfies

$$2nc_2 + c_1 > 0,$$

then there exist a positive constant a_1 depending on c_1, c_2, c_3, h and n such that $h_2 \geq -a_1$.

Of course much more generalized conditions than the above curvature condtions (*) will be discussed in this talk. Also as an another application of Theorem B for $1 < n < 2$ we assert the following

Theorem 2. ([11]) Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p$ (> 0). Then it satisfies

$$h_2 \geqq \frac{1}{2}npc,$$

where the equality holds if and only if $p = 1$ and M is globally congruent to a complex quadric Q^n in $CH_1^{n+1}(c)$.

Finally we want to discuss for the case $n = 1$, that is, $\triangle f \geq kf$ for a function f bounded from above. Then in such a case we are able to show that the function f vanishes identically. Moreover, we will show that there exist a counter example for this type. Namely, there is a smooth unbounded function f which can not satisfy the above inequality for $n = 1$ as follows:

Theorem 3. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If the non-negative function f is bounded from above and satisfies*

(**) $$\triangle f \geq kf, \text{ for a positive constant } k,$$

then f vanishes identically.

Proof. For a constant $a > 0$ let us put $F = (f+a)^{-\frac{1}{2}}$ a smooth positive function. Then we are able to apply a generalized maximum principle due to Omori [14] and Yau [17]. For any $\epsilon > 0$ there exists a point p in M such that

$$|\nabla F|(p) < \epsilon, \triangle F(p) > -\epsilon, F(p) < infF + \epsilon.$$

Then it follows from these properties that we have

$$\epsilon(3\epsilon + 2F(p)) > F(p)^4 \triangle f(p) \geq 0.$$

Thus for a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon_m \to 0$ as $m \to \infty$, there is a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ satisfies the above formula and converges to F_0, by taking a subsequence, if necessary, because the sequence $\{F(p_m)\}$ is bounded. From the definition of the infimum and the above formula we have $F_0 = infF$ and hence $f(p_m) \to f_0 = supf$. It follows that we have

$$\epsilon_m\{3\epsilon_m + 2F(p_m)\} > F(p_m)^4 \triangle f(p_m)$$

and the left hand side converges to 0 because the function F is bounded. Thus we get

$$F(p_m)^4 \triangle f(p_m) \to 0 \quad (m \to \infty).$$

As is already seen, the Ricci-curvature is bounded from below i.e., so is any λ_B. Since $r = 2\Sigma_B \lambda_B$ is constant, λ_B is bounded from above. Hence $F = (f+a)^{-\frac{1}{2}}$ is bounded from below by a positive constant. From the above formula it follows that $\triangle f(p_m) \to 0$ as $m \to \infty$. Then by (**) we have that

$$\triangle f(p_m) \geq k f(p_m) \geq 0.$$

Thus we have $f(p_m) \to 0 = inff$. Since $f(p_m) \to supf$, $supf = inff = 0$. Hence $f = 0$ on M. This completes the proof of Theorem 3. □

Remark 3. As a Remark we want to show that there exists an example of a smooth function f satisfying (**) but not bounded from above. Let us consider a function f defined by $f(x_1, ..., x_k) = cosh(ax_1)$ on $\mathbb{R}^k$ for some positive constant a.

Then it can be easily seen that the function f satisfies $\triangle f = a^2 f$. So naturally it satisfies the inequality (**). But this function f can not be bounded from above. So the condition that the boundness from above for the function f in Theorem 3 is essential.

After the above preparation, we will make its applications to some geometric problems of semi-Kaehler manifolds given in [12], [15] and [16]. Among them we show the following

Theorem 4. ([16]) Let M be an $n(\geq 3)$-dimensional complete Kaehler manifold with constant scalar curvature r. Assume that the totally real bisectional curvature is bounded from above by a constant b. If the scalar curvature satisfies

$$r > \frac{2n^2 - 3n + 2}{n - 1} b,$$

then M is globally congruent to a complex projective space $\mathbb{C}P^n$.

Example 4. In the complex quadric Q^n in a complex projective space $\mathbb{C}P^{n+1}$ of constant holomorphic curvature c, it is seen that the totally real bisectional curvature B satisfies $0 \leq B \leq \frac{c}{2}$ and the scalar curvature $r = n^2 c$. Hence $b = \frac{c}{2}$ and $r = 2n^2 b$. On the other hand, in a complex projective space $\mathbb{C}P^n$ we see $B = b = \frac{c}{2}$ and $r = n(n+1)c = 2n(n+1)b$. Then it can be easily seen that

$$2n^2 < (2n^3 - 3n + 2)/(n - 1) < 2n(n + 1).$$

Remark 5. The above estimation for the scalar curvature r is best possible. This means that if the equality holds in the estaimation of the scalar curvature in Theorem 4, there is an example of complete Kaehler manifolds M which is not Einstein. *In other words, the equality holds if and only if the infimum $a(M)$ of the totally real bisectinal curvatures of M is equal to zero.* This means that there is a complete Kaehler manifold with constant scalar curvature and non-negative totally real bisectional curvature $B(u, v) \geq 0$ but not Einstein as follows (see [12]):

Now let us consider a product Kaehler manifold

$$M = \mathbb{C}P^{n^1}(c_1) \times \mathbb{C}P^{n^2}(c_2).$$

Then its totally real bisectional curvature is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{c_1}{2} & \text{if } A = a, B = b, \\ 0 & \text{if } A = a, B = s, \\ R_{\bar{r}rs\bar{s}} = \frac{c_2}{2} & \text{if } A = r, B = s, \end{cases}$$

where indices $A, B(A \neq B), ...; 1, ..., n_1, n_1 + 1, ..., n_2$, and $a, b, ..; 1, ..., n_1, r, s, ..; n_1 + 1, ..., n_2$. So it can be easily seen that its totally real bisectional curvatures are lower bounded from $a(M) = 0$.

And its Ricci-tensor is given by the following

$$\begin{aligned} S_{A\bar{B}} &= \Sigma_C R_{\bar{B}AC\bar{C}} = \Sigma_a R_{\bar{B}Aa\bar{a}} + \Sigma_r R_{\bar{B}Ar\bar{r}} \\ &= \begin{cases} \frac{n_1+1}{2} c_1 \delta_{bc} & \text{if } B = c, A = b, \\ 0 & \text{if } B = s, A = b, \\ \frac{n_2+1}{2} c_2 \delta_{ts} & \text{if } B = s, A = t. \end{cases} \end{aligned}$$

Thus for the case where $(n_1 + 1)c_1 \neq (n_2 + 1)c_2$ it follows that

$$M = \mathbb{C}P^{n^1}(c_1) \times \mathbb{C}P^{n^2}(c_2)$$

is a complete Kaehler manifold with constant scalar curvature $r = n_1(n_1 + 1)c_1 + n_2(n_2 + 1)c_2$ but not Einstein.

In differential geometry, we have some Riemannian analogues of the classical Liouville theorem, which are closely related to some kinds of Liouville type inequalities. They played respectively important roles in their branches. For examples, see Cheng and Nakagawa [3], Ki and the present author [12] and Nishikawa [13].

From this point of view it seems to be of interest for us to investigate that under what kind of geometric conditions other than the upper boundness of the function satisfying (**) the following holds or not:

Problem. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If a C^2-nonnegative function f satisfies*

$$\triangle f \geq kf,$$

where $\triangle$ denotes the Laplacian on M and k is any positive constant, then f vanishes identically.

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