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ON DIFFEORMORPHISMS OF A HYPERBOLIC MANIFOLD

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1. Introduction

Let M be a complete connected Riemannian manifold of dimension n. We denote by $d(x, y)$ the distance between two points x and y of M, and by $i(x)$ the injectivity radius at x. If $d(x, y) < i(x)$, then there exists a unique geodesic connecting x and y. Hence, if f is a diffeomorphism of M that satisfies the condition

$$(*) \qquad d(x, f(x)) < i(x), \qquad \text{for all } x \in M,$$

then f is smoothly homotopic to the identity.

We shall investigate when a diffeomorphism of M that satisfies (*) is smoothly isotopic to the identity. We will have the following theorem:

Theorem 1. *Let M be a compact, connected and oriented Riemannian manifold with constant curvature -1, and f an orientation-preserving diffeomorphism that satisfies the condition (*). Set $W(x) = exp_x^{-1} f(x)$. If $\max |\nabla_v W| \leq 1$ holds for any $x \in M$ and any unit tangent vector v at x, then f is smoothly isotopic to the identity.*

As an immediate consequence , we have

Corollary 1. *Let M be as in the theorem. Then the group of orientation-preserving diffeomorphisms of M is locally contractible.*

The local contractibility of the group of diffeomorphisms of hyperbolic spaces has been studied by Earle and Eells for 2-dimensional case, and by Hatcher for 3-dimensional case, cf. Thurston[3], in different ways.

2. Outline of the proof

Let M be a complete, connected smooth Riemannian manifold of dimension $n(\geq 2)$. The readers refer to the book [1] for Riemannian geometric materials. Let$f$ be a diffeomorphism of M that satisfies the condition (*) in the introduction. Then we have a vector field W on M defined by

$$W(x) = \exp_x^{-1}(f(x))$$

for $x \in M$. It is well-defined since $\exp_x : \{ v \in T_xM : |v| < i(x)\} \to M$ is injective. Let $x \in M$ be a point with $f(x) \neq x$, and $c : [0, 1] \to M$ the unique geodesic connecting $x = c(0)$ to $f(x) = c(1)$, parametrized proportionally to arc-length. For a unit tangent vector v at x, let $\alpha : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\alpha(0) = x$ and $\dot\alpha(0) = v$. We consider a variation of c, $F : [0,1] \times (-\epsilon, \epsilon) \times T_x M \to M$ defined by $F(t, s; v) = \exp_{\alpha(s)} tW(\alpha(s))$, where $W(\alpha(s)) = \exp^{-1}_{\alpha(s)}(f(\alpha(s)))$. We see that

$$Y_v(t) := \left.\frac{\partial F(t,s;v)}{\partial s}\right|_{s=0} = (dF_t)_x(v)$$

is the Jacobi field along c. Also

$$\left.\frac{\partial F(0,s;v)}{\partial s}\right|_{s=0} = v$$

and

$$\left.\frac{\partial F(1,s:v)}{\partial s}\right|_{s=0} = (df)_x(v).$$

Since $[\partial/\partial t, \partial/\partial s] = 0$, it follows that $\nabla_{\dot c} Y_v(0) = \nabla_v W$, where ∇ denotes the covariant derivative.

The above observation and the inverse function theorem yield the following:

Let M be a complete connected Riemannian manifold with constant curvature -1, and f a diffeomorphism of M that satisfies the condition (*). Set $W(x) = \exp_x^{-1} f(x)$. Suppose that $\max |\nabla_v W| \leq 1$ holds for any $x \in M$ and any unit tangent vector v at x, where $|V|$ denote the norm of V. Define $F_t : M \to M$ by $F_t(x) = exp_x tW(x)$. Then F_t is a local diffeomorphism for each t, where $F_0 =$ identity and $F_1 = f$.

The above result gives us a local information about diffeomorphisms on M. Applying this, together with topological consideration, we obtain the theorem in the introduction. The detail will appear elsewhere.

References

[1] J.Cheeger and D.G.Ebin *Comparison Theorems in Riemannian Geometry* North Holand, American Elsevier, 1975.

[2] C.J.Earle and J.Eells,Jr A fiber bundle description of Teichmuller theory. J. Diff. Geom. **3**(1969), 19-43.

[3] W.P.Thurston The Topology and Geometry of 3-manifolds. (preprint), 1978, Princeton.