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SCHRÖDINGER FLOW AND ITS APPLICATIONS IN INTEGRABLE SYSTEMS

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Dedicated to Professor Katsuhiro Shiohama on his sixtieth birthday

ABSTRACT. By use of Schrödinger flow, we present a complete unified geometric interpretations of the nonlinear Schrödinger equations. The gauge equivalent structure of the matrix nonlinear Schrödinger equation is also mentioned. Furthermore, according to the correspondence principle in quantum dynamics, we display the gauge equivalent structures for the discrete nonlinear Schrödinger equation and the discrete matrix nonlinear Schrödinger equation.

I. Introduction

This paper is organized as follows. In section II we give a brief description of Schrödinger flow of maps into a symplectic manifold and some useful examples. In section III we show that the nonlinear Schrödinger equation $i\phi_t + \phi_{xx} + 2\kappa|\phi|^2\phi = 0$ for $\kappa = 1$ and -1 are respectively gauge equivalent to the Schrödinger flow of maps into S^2 and H^2 and the matrix nonlinear Schrödinger equation is gauge equivalent to the Schrödinger flow of maps into Grassmannian $G_{k,m}$. In section IV, by using new Lax pairs, we show that the continuous limits of the gauge equivalent structures of the discrete nonlinear Schrödinger equation for $\kappa = 1$ and -1 are exactly the classical ones of the nonlinear Schrödinger equation for $\kappa = 1$ and -1 respectively; we also demonstrate the discrete counterpart of the gauge equivalent structure for the matrix nonlinear Schrödinger equation and, finally, in section V we give a example to illustrate our results.

II. SCHRÖDINGER FLOW OF MAPS INTO H^2

Motivated by the Heisenberg ferromagnet model, the study of Schrödinger flow of maps into a symplectic manifold has attracted more attention recently (for examples, see [3,7]). First of all, let's give a brief summary of Schrödinger flow of maps into a symplectic manifold (N, ω). Let J be an almost complex structure on N such that

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

is a Riemannian metric on N. Now we work on the symplectic Banach manifold of mapping space $X = C^k(M, N)$ for some $k \geq 1$ with the following induced symplectic form:

$$\Omega(u)(v, w) = \int_{R^1} \omega(u)(v, w), \quad \forall u \in X; \forall v, w \in T_u(X)$$

from ω, where M is a Riemannian manifold. Then the following inner product on the tangent bundle TX:

$$< v, w >_u = \int_{R^1} g(u)(v, w), \quad \forall u \in X; \forall v, w \in T_u X \tag{1}$$

admits $J(u)$ as its compatible almost complex structure. Thus if we denote by $\nabla F(u)$ the gradient of a function $F(u)$ on X with repsect to the inner product (1), then the corresponding Hamiltonian vector field $V_{F(u)}$ can be expressed explicitly as $V_{F(u)} = J(u)\nabla F(u)$.

Definition 1. *The Schrödinger flow of maps from M into N is defined by the following Hamiltonian system of the energy functional $E(u)$ on X:*

$$u_t = J(u)\nabla E(u).$$

Recall that the energy $E(u)$ of $u : (M, h) \to (N, g)$ is defined by

$$E(u) = \int_{R^1} e(u)$$

where in a local coordinates

$$e(u) = \frac{1}{2} g_{jk}(u) \frac{\partial u^j}{\partial x_\alpha} \frac{\partial u^k}{\partial x_\beta} h^{\alpha\beta}.$$

It is easy to verify that the gradient $\nabla E(u)$ is exactly the tension field $\tau(u)$ of map u. In a local coordinates

$$\tau^l(u) = \Delta_M u^l + \Gamma^l_{jk}(u) \frac{\partial u^j}{\partial x_\alpha} \frac{\partial u^k}{\partial x_\beta} h^{\alpha\beta},$$

where Γ^l_{jk} are the Christoffel symbols on the target manifold N. So the Schrödinger flow of maps from M into N can also be re-written as:

$$u_t = J(u)\tau(u). \tag{2}$$

Remark 1. *Schrödinger flow of maps can be regarded as a twin equation of heat flow* $u_t = \tau(u)$. *They are respectively geometric generalizations of the linear Schrödinger equation and heat equation in mathematical physics.*

Example 1. When $M = R^1$ and $N = C$, the complex plane, (2) is nothing else but the linear Schrödinger equation $u_t = iu_{xx}$.

Example 2 ([7]). If we let $M = R^1$ and $N = S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$, the unit sphere in Euclidean 3-space, then it is easy to verify that the Schrödinger flow of maps to $N = S^2$ is actually the equation of Heisenberg ferromagnet model:

$$u_t = u \times u_{xx}, \tag{3}$$

where $\times$ denotes the vector product in R^3.

Example 3 ([3]). Now we set $R^{2,1} = \{(x_1, x_2, x_3) : ds^2 = dx_1^2 + dx_2^2 - dx_3^2\}$ to be the Minkowski 3-space and let $H^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 - x_3^2 = -1, x_3 > 0\}$ be a unit sphere in $R^{2,1}$, then H^2 is actually the hyperbolic 2-space. For arbitrary two vector $\mathbf{a}, \mathbf{b} \in R^{2,1}$, we introduce a pseudo cross product $\mathbf{a}\dot{\times}\mathbf{b}$ according to the Minkowski metric by

$$\mathbf{a}\dot{\times}\mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, -(a_1b_2 - a_2b_1)).$$

Obviously, we have $\mathbf{a}\dot{\times}\mathbf{a} = 0$ too.

One may verify directly by a computation that the Schrödinger flow (2) of maps from R^1 into $N = H^2$ becomes:

$$u_t = u\dot{\times}u_{xx}. \tag{4}$$

where $u = (u_1, u_2, u_3)$ with $u_1^2 + u_2^2 - u_3^2 = -1$ and $u_3 > 0$. We also call Eq.(4) as the Minkowski Heisenberg ferromagnet model (M-HF model).

Example 4 ([17]). Let $M = R^1$ and $N = G_{k,m}$, where $G_{k,m}$ is a Grassmannian manifold in the unitary group $U(m)$. If we regard $G_{k,m}$ as the adjoint $U(m)$-orbit at $a = \begin{pmatrix} \frac{i}{2}I_k & 0 \\ 0 & -\frac{i}{2}I_{m-k} \end{pmatrix}$ in the Lie algebra $u(m)$ of $U(m)$, i.e., $G_{k,m} = \{U^{-1}aU | U \in U(m)\}$. Then, by a straightforward computation, the Schrödinger flow of maps from R^1 to $G_{k,m}$ is expressed as follows,

$$\gamma_t = [\gamma, \gamma_{xx}], \quad \gamma \in G_{k,m}. \tag{5}$$

III. Gauge Equivalence — Classical version

Heat flow is an important object in differential geometry, especially in the study of harmonic maps. However, it seems, as we shall see, that Schrödinger flow has its main applications in mathematical physics.

The nonlinear Schrödinger equation (NLS):

$$i\phi_t + \phi_{xx} + 2\kappa|\phi|^2\phi = 0 \tag{6}$$

is a representative example in the theory of integrable systems and it arises in physics from varied backgrounds, such as in plasma physics and nonlinear optics. The κ in the equation is a real constant. In the linear limit, $\kappa = 0$, Eq.(6) goes over into the Schrödinger equation for the wave function of a particle. If $\kappa \neq 0$, the sign of κ in Eq.(6) distinguishes the equation between attractive ($\kappa > 0$) and repulsive ($\kappa < 0$) type. Without loss of generality, we will denote by NLS$^+$ and NLS$^-$ the Eq.(6) with $\kappa = 1$ and $\kappa = -1$ respectively.

In 1979, Zakharov and Takhtajan proved in [19] that the NLS$^+$ is gauge equivalent to the HF model, i.e. the Schrödinger flow of maps into S^2. Now we have

Theorem 1. *(Ding [3]) The nonlinear Schrödinger equation (6) for $\kappa = 1$ (i.e., NLS$^+$) and -1 (i.e., NLS$^-$) are gauge equivalent to the Schrödinger flow of maps from R^1 to S^2 and H^2 respectively.*

Proof. Based on the result in [19], what remains to prove is that the NLS$^-$ is gauge equivalent to the Schrödinger flow into H^2 (i.e. Eq.(4)). Put:

$$p_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and set

$$\widetilde{S} = u_1 p_1 + u_2 i p_2 + u_3 i p_3, \quad \text{for} \quad u \in H^2.$$

Obviously, $\widetilde{S}^2 = -I$, $\mathrm{tr}\widetilde{S} = 0$, the diagonal of $\widetilde{S}$ is a real matrix and the offdiagonal of $\widetilde{S}$ is a purely imaginary matrix. Using the commutive relations: $[p_1, p_2] = -2p_3$, $[p_1, p_3] = -2p_2$ and $[p_2, p_3] = -2p_1$, we obtain, by a direct computation, that system (4) can be rewritten as:

$$\widetilde{S}_t = \frac{1}{2}[\widetilde{S}, \widetilde{S}_{xx}], \tag{7}$$

and Eq.(7) permits a Lax pair as follows:

$$\begin{cases} \widetilde{F}_x(x,t,\lambda) = \widetilde{L}(x,t,\lambda)\widetilde{F}(x,t,\lambda) \\ \widetilde{F}_t(x,t,\lambda) = \widetilde{M}(x,t,\lambda)\widetilde{F}(x,t,\lambda), \end{cases} \tag{8}$$

where $\widetilde{L} = i\lambda\widetilde{S}$, $\widetilde{M} = 2\lambda^2\widetilde{S} + i\lambda\widetilde{S}\widetilde{S}_x$ and λ is a spectral parameter.

Next, let's recall that the NLS$^-$ has no light soliton solutoins but dark soliton solutions, so we put $\phi = \psi e^{-2i\rho^2 t}$, where ρ is a positive real constant, and get a equivalent equation for ϕ:

$$i\phi_t + \phi_{xx} - 2(|\phi|^2 - \rho^2)\phi = 0. \tag{9}$$

As pointing out in [8] or [3], we need to add the following finite density boundary condition:

$$\begin{cases} \phi \longrightarrow \rho, & \text{as} \quad x \longrightarrow +\infty, \\ \phi \longrightarrow \rho e^{i2\beta}, & \text{as} \quad x \longrightarrow -\infty, \end{cases}$$

in solving (9), where β is a constant. A direct computation shows that (9) permits a Lax pair as follows:

$$\begin{cases} F_x^{'}(x,t,\lambda) = L^{'}(x,t,\lambda)F^{'}(x,t,\lambda) \\ F_t^{'}(x,t,\lambda) = M^{'}(x,t,\lambda)F^{'}(x,t,\lambda) \end{cases} \tag{10}$$

where $L^{'} = \lambda\sigma_3 + U(x,t)$, $M^{'} = -i2\lambda^2\sigma_3 - i2\lambda U(x,t) + i\{U^2(x,t) - \rho^2 + U_x(x,t)\}\sigma_3$ and

$$U(x,t) = \begin{pmatrix} 0 & \phi(x,t) \\ \bar{\phi}(x,t) & 0 \end{pmatrix}.$$

Now one can verify that the NLS$^-$ is gauge equivalent to the Schrödinger flow to H^2 (4) by the following gauge transformation (see [3] for details):

$$F^{'}(x,t,\lambda) = G(x,t)F(x,t,\lambda), \tag{11}$$

where $G(x,t)$ is a fundamental solution to (10) at $\lambda = 0$, $F^{'}$ and F are solutions to (10) and (8) respectively. This completes the proof of the gauge equivalence between the NLS$^-$ equation and the Schrödinger flow of maps from R^1 into H^2. $\square$

The following matrix nonlinear Schrödinger equation (MNLS),

$$iq_t + q_{xx} + 2qq^*q = 0, \quad (k \text{ or } m \geq 2) \tag{12}$$

was first studied geometrically by Fordy and Kulish in [9], where q is a map from R^2 to the space $M_{(m-k)\times k}$ of $(m-k)\times k$ complex matrices, $1 \leq k \leq m-1$ and Q^* denotes the complex transposed conjugate matrix of q. Note that when q is a 1×1 matrix, Eq.(12) goes to the NLS$^+$. For this equation (12), Terng and Uhlenbeck proved

Theorem 2. *(Terng and Uhlenbeck [17]) The matrix nonlinear Schrödinger equation (12) is gauge equivalent to the Schrödinger flow of maps into Grassmannian $G_{k,m}$ (5).*

Proof. If we set $S = Q^{-1}\sigma_3 Q$, where $\sigma_3 = \begin{pmatrix} I_k & 0 \\ 0 & -I_{m-k} \end{pmatrix}$ and $Q \in U(m)$, then the Schrödinger flow into Grassmannian (5) is equivalent to

$$S_t = \frac{1}{2i}[S, S_{xx}], \tag{13}$$

which has the following Lax pair:

$$\psi_x = \lambda S\psi, \quad \psi_t = (-i2\lambda^2 S + i\lambda S_x S)\psi. \tag{14}$$

On the other hand, the matrix nonlinear schrödinger equation (12) admits a Lax pair as follows:

$$\phi_x = (\lambda\sigma_3 + U)\phi, \quad \phi_t = [-i2\lambda^2\sigma_3 - 2i\lambda U + i(U^2 + U_x)\sigma_3]\phi, \tag{15}$$

where $U = \begin{pmatrix} 0 & \bar{q}(x,t) \\ q(x,t) & 0 \end{pmatrix}$. One may verify straightforwardly that Eqs.(12) and (13) are gauge equivalent to each other by the following transformation:

$$\phi(x,t,\lambda) = G(x,t)\psi(x,t,\lambda), \tag{16}$$

where $G(x,t)$ is a fundamental solution to (15) at $\lambda = 0$, ϕ and ψ are solutions to (15) and (14) respectively. □

IV. Gauge Equivalence — Discrete Version

Since nonlinear integrable differential-difference equations are of fundamental important for the study of classical integrable systems, the study of nonlinear integrable differential-difference equations has received considerable attentions in recent years (see, for examples, [2,10]). The following discrete nonlinear Schrödinger equation (DNLS),

$$i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) + \kappa|q_n|^2(q_{n+1} + q_{n-1}) = 0 \tag{17}$$

where κ is a constant, was introduced by Ablowitz and Ladik [1] who constructed the discrete version of the AKNS system. Without loss of generality, we denote by DNLS$^+$ and DNLS$^-$ the DNLS (17) with $\kappa = 1$ and -1, respectively. The bright soliton solutions to the DNLS$^+$ and the dark soliton solutions to the DNLS$^-$ are constructed respectively in [1] and [12,13,18] by using the inverse scattering method, or in [15] by Bäcklund-Darboux transformations. The DNLS (1), being interesting from the mathematical viewpoint, has also a rather wide area of physical applications.

In 1982, Ishimori showed in [11] that the DNLS$^+$ is gauge equivalent to the discrete HF model (DHF), which reads,

$$d\mathbf{S}_n/dt = -\frac{2\mathbf{S}_{n+1}\times\mathbf{S}_n}{1+\mathbf{S}_{n+1}\cdot\mathbf{S}_n} + \frac{2\mathbf{S}_n\times\mathbf{S}_{n-1}}{1+\mathbf{S}_n\cdot\mathbf{S}_{n-1}} \tag{18}$$

where $\mathbf{S}_n = (s_n^1, s_n^2, s_n^3) \in R^3$ with $|\mathbf{S}|^2 = (S_n^1)^2 + (s_n^2)^2 + (s_n^3)^2 = 1$, and $\cdot$ and $\times$ denote the inner and the cross product in R^3. In this section, we are interested in 1) whether the DNLS$^-$ with a similar gauge equivalent structure exists, and, according to the correspondence principle in quantum dynamics, 2) whether the gauge equivalent structure of the DNLS$^-$ (resp. DNLS$^+$) is just the *discretization* of the classical one of the NLS$^-$ (resp. NLS$^+$) employed in [3] (resp. [19]). We will give affirmative answers to the above two questions, by using new Lax pairs for the DNLS$^+$, the DNLS$^-$ and the DHF below, and show why the Ishimori's gauge transformation in [11] is not the discretization of a classical one between the NLS$^+$ and the HF model.

IV.A Lax pairs and their continuous limits. For our purpose in this section, we hope to choose Lax pairs for the DNLS$^+$, the DNLS$^-$, the DHF and the DM-HF such that the continuous limits of them are just Lax pairs of their corresponding classical integrable systems. The following new Lax pairs are exactly such ones.

Example 1 The DNLS$^+$:

$$i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) + |q_n|^2(q_{n+1} + q_{n-1}) = 0. \tag{19}$$

A new Lax pair of this equation is

$$\phi_{n+1} = L_n\phi_n, \quad d\phi_n/dt = M_n\phi_n \tag{20}$$

in which

$$\begin{gathered} L_n = \begin{pmatrix} z & \bar{q}_n z^{-1} \\ -q_n z & z^{-1} \end{pmatrix} \\ M_n = i\begin{pmatrix} 1 - z^2 + z - z^{-1} - \bar{q}_n q_{n-1} & -\bar{q}_n + \bar{q}_{n-1} z^{-2} \\ -q_n + q_{n-1} z^2 & -1 + z^{-2} + z - z^{-1} + q_n \bar{q}_{n-1} \end{pmatrix} \end{gathered} \tag{21}$$

where z is a spectral parameter and the overbar denotes complex conjugate. Some Lax pairs of (19), which are different from the present one can be refered to [8,11-13].

For the above Lax pair, as usual (see, for example, [1]), the continuous limit ($\Delta x \to 0$; Δx being the discretization parameter) of (20) is

$$\phi_x = L\phi, \quad \phi_t = M\phi \tag{22}$$

with $L = \lambda\sigma_3 + U$, $M = -i2\lambda^2\sigma_3 - 2i\lambda U + i(U^2 + U_x)\sigma_3$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & \bar{q} \\ -q & 0 \end{pmatrix}$, after the substitution

$$z \to 1 + \lambda\Delta x, \quad q_n \to q\Delta x, \quad n\Delta x \to x, \quad t\Delta x^2 \to t \tag{23}$$

(λ is a parameter) and setting $\phi_n \sim \phi$, expanding $q_{n\pm1} \sim \Delta x(q \pm \Delta x q_x + (\Delta x^2/2)q_{xx} \pm \cdots)$. It is direct to verify that the integrability condition of (22) yields simply the NLS$^+$ equation: $iq_t + q_{xx} + 2|q|^2 q = 0$. We would like to point out that the term $z - z^{-1}$ in the second expression of (21) plays a very important role in calculating the continuous limit. Because this term is absent in the Lax pair used in [11], hence it is impossible for Ishimori's gauge transformation to be the discretization of a classical one between the NLS$^+$ and the HF model.

Example 2 The DNLS$^-$. Because the DNLS$^-$ has no bright solitons, we put $q_n = r_n e^{-2i\rho^2 t}$, where ρ is a positive real constant, and get an equivalent equation for r_n. The following is that equation in which r_n is replaced by q_n,

$$i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) - |q_n|^2(q_{n+1} + q_{n-1}) + 2\rho^2 q_n = 0. \tag{24}$$

As pointed out in [8], in order to solve (24), we should propose the nonzero boundary conditions or conventionally the finite density boundary conditions: $\lim_{n\to-\infty} q_n = \rho$ and $\lim_{n\to+\infty} q_n = \rho e^{i\theta}$, here ρ designates the background pulse amplitude and θ is a complete phaseshift. This equation permits the following new Lax pair,

$$\phi_{n+1} = L_n\phi_n, \quad d\phi_n/dt = M_n\phi_n \tag{25}$$

in which

$$\begin{aligned} L_n &= \begin{pmatrix} z & \bar{q}_n z^{-1} \\ q_n z & z^{-1} \end{pmatrix}, \\ M_n &= i\begin{pmatrix} 1 - z^2 + z - z^{-1} + \bar{q}_n q_{n-1} - \rho^2 & -\bar{q}_n + \bar{q}_{n-1} z^{-2} \\ q_n - q_{n-1} z^2 & -1 + z^{-2} + z - z^{-1} - q_n \bar{q}_{n-1} + \rho^2 \end{pmatrix}. \end{aligned}$$

Some other Lax pairs of (24) can be found in [12,13].

In a similar way, the continuous limit of (25) reads

$$\phi_x = L\phi, \quad \phi_t = M\phi \tag{26}$$

with $L = \lambda\sigma_3 + U$, $M = -i2\lambda^2\sigma_3 - 2i\lambda U + i(U^2 - \rho^2 + U_x)\sigma_3$ and $U = \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix}$, after the substitution

$$z \to 1 + \lambda\Delta x, \quad q_n \to q\Delta x, \quad n\Delta x \to x, \quad t\Delta x^2 \to t, \quad \rho \to \Delta x^2 \rho. \tag{27}$$

It is direct to verify that the integrability condition of (26) yields simply the NLS$^-$ equation: $iq_t + q_{xx} - 2(|q|^2 - \rho^2)q = 0$.

Example 3 The DHF (18). For this model, we convert it into the matrix form as follows,

$$dS_n/dt = i\frac{[S_{n+1}, S_n]}{1 + S_{n+1} \cdot S_n} - i\frac{[S_n, S_{n-1}]}{1 + S_n \cdot S_{n-1}}, \quad S_n = \begin{pmatrix} s_n^1 & s_n^2 - is_n^3 \\ s_n^2 + is_n^3 & -s_n^1 \end{pmatrix} \tag{28}$$

where $S_{n+1} \cdot S_n$ is defined as the inner product of the vectors $\mathbf{S}_{n+1}$ and $\mathbf{S}_n$ in R^3. (28) allows the following new Lax pair,

$$\psi_{n+1} = \widetilde{L}_n\psi_n, \quad d\psi_n/dt = \widetilde{M}_n\psi_n \tag{29}$$

in which $\widetilde{L}_n = \frac{z+z^{-1}}{2}I + \frac{z-z^{-1}}{2}S_n$ and $\widetilde{M}_n = (1 - \frac{z^2+z^{-2}}{2})\frac{i(S_n+S_{n-1})}{1+S_n\cdot S_{n-1}} + i(z - z^{-1})I - \frac{z^2-z^{-2}}{2}\frac{i(I+S_{n-1}S_n)}{1+S_n\cdot S_{n-1}}$, where I denotes the unit 2×2 matrix. Similarly, after the substitution $z \to 1 + \lambda\Delta x$, $\Delta x \to 0$, $n\Delta x \to x$, $S_n \to S$ and $\psi_n \to \psi$, the continuous limit of (29) is

$$\psi_x = \widetilde{L}\psi, \quad \psi_t = \widetilde{M}\psi \tag{30}$$

with $\widetilde{L} = \lambda S$ and $\widetilde{M} = -i2\lambda^2 S + i\lambda S_x S$ and $S = \begin{pmatrix} s^1 & s^2 - is^3 \\ s^2 + is^3 & -s^1 \end{pmatrix}$ satisfying $S^2 = I$. The integrability condition of (30) reads $S_t = \frac{1}{2i}[S, S_{xx}]$, which is just the matrix form of the HF model [8]. Some other Lax pairs of (18) or (28) can be referred to [15,16], where one can also find soliton solutions to (18) or (28).

Example 4 The discrete M-HF model (DM-HF),

$$d\mathbf{S}_n/dt = -\frac{2\mathbf{S}_{n+1}\dot{\times}\mathbf{S}_n}{1 - \mathbf{S}_{n+1} \cdot \mathbf{S}_n} + \frac{2\mathbf{S}_n\dot{\times}\mathbf{S}_{n-1}}{1 - \mathbf{S}_n \cdot \mathbf{S}_{n-1}} \tag{31}$$

where $\mathbf{S}_n = (s_n^1, s_n^2, s_n^3) \in R^{2+1}$ with $|\mathbf{S}|^2 = (s_n^1)^2 + (s_n^2)^2 - (s_n^3)^2 = -1$ and $s_n^3 > 0$, and $\cdot$ and $\dot{\times}$ denote the pseudo inner and the pseudo cross product in R^{2+1}. This new differential-difference equation is deduced (see below) from the discretization of the Schrödinger flow of maps into H^2 (i.e. the M-HF model) $S_t = S\dot{\times}S_{xx}$ (see [12] for details). The matrix form of this model is,

(32) $$dS_n/dt = -\frac{[S_{n+1}, S_n]}{1 - S_{n+1}\cdot S_n} + \frac{[S_n, S_{n-1}]}{1 - S_n \cdot S_{n-1}}, \quad S_n = \begin{pmatrix} s_n^1 & i(-s_n^2 + s_n^3) \\ i(s_n^2 + s_n^3) & -s_n^1 \end{pmatrix}$$

in which $S_{n+1} \cdot S_n$ denotes the pseudo inner product of the two vectors $\mathbf{S}_{n+1}$ and $\mathbf{S}_n$ in R^{2+1}. A complicated computation shows that Eq. (32) permits the following Lax pair,

(33) $$\psi_{n+1} = \widetilde{L}_n \psi_n, \quad d\psi_n/dt = \widetilde{M}_n \psi_n$$

with

$$\widetilde{L}_n = \frac{z + z^{-1}}{2} I + i\frac{z - z^{-1}}{2} S_n$$

,

$$\widetilde{M}_n = (\frac{z^2 + z^{-2}}{2} - 1)\frac{S_n + S_{n-1}}{1 - S_n \cdot S_{n-1}} + i(z - z^{-1})I - i\frac{z^2 - z^{-2}}{2}\frac{I - S_{n-1}S_n}{1 - S_n \cdot S_{n-1}}$$

. Similarly, the continuous limit of (33) is

(34) $$\psi_x = \widetilde{L}\psi, \quad \psi_t = \widetilde{M}\psi$$

where $\widetilde{L} = i\lambda S$ and $\widetilde{M} = 2\lambda^2 S - i\lambda S_x S$ and $S = \begin{pmatrix} s^1 & i(-s^2 + s^3) \\ i(s^2 + s^3) & -s^1 \end{pmatrix}$ satisfying $S^2 = -I$. The integrability condition of (34) is $S_t = \frac{1}{2}[S, S_{xx}]$, which is just the matrix form of the M-MF model (also see [3]).

IV.B Quantizations of gauge transformations. In this subsection, by using the Lax pairs displayed in the preceding subsection, we show that there is a gauge transformation between the DNLS$^-$ (resp. DNLS$^+$) and the DM-HF (resp. DHF) and demonstrate that the continuous limits of the gauge transformations are just their corresponding classical ones.

Theorem 3. *(Ding [5]) The DNLS$^-$ (resp. DNLS$^+$) is gauge equivalent to the DM-HF (resp. DHF) by the following gauge transformation,*

(35) $$\phi_n(t, z) = G_n(t)\psi_n(t, z),$$

where $G_n(t)$ is a fundamental solution to (25) (reps. (20)) at $z = 1$, $\phi_n(t, z)$ and $\psi_n(t, z)$ are solutions to (25) and (33) (resp. (19) and (29)) respectively. Furthermore, the continuous limit of (35) becomes a classical gauge transformation between the NLS$^-$ and the M-HF model (resp. the NLS$^+$ and the HF model).

Remark 2. *This theorem indicates that there is a unified reconciliation of the gauge equivalent structures of the DNLS and NLS according to the correspondence principle.*

After a thorough understanding of the gauge structures of the NLS, the DNLD (which is the discrete gauge equivalent corresponding to the NLS) and the MNLS, one would naturally like to find the discrete counterpart of the gauge equivalent structure of the MNLS according to the correspondence principle in quantum dynamics. For the matrix nonlinear Schrödinger equation, the parallel generalization of the DNLS$^+$ as that of the NLS$^+$ is naturally introduced as follows,

$$i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}q_n^*q_n + q_nq_n^*q_{n-1}) = 0, \tag{36}$$

which is called as the discrete (integrable) matrix nonlinear Schrödinger equation (DMNLS). The discrete (ingtegrable) equation of the Schrödinger flow into Grassmannian is as follows,

$$dS_n/dt = 4i(I + S_nS_{n-1})^{-1} - 4i(I + S_{n+1}S_n)^{-1} \tag{37}$$

is the discrete equation of the Schrödinger flow of maps into Grassmannian $G_{k,m}$ (13), where $I = I_m$ denotes the $m \times m$ unit matrix and S_n is of the form $U_n^{-1}\sigma_3U_n$ with U_n being an $m \times m$ unitary matrix and $\sigma_3 = \begin{pmatrix} I_k & 0 \\ 0 & -I_{m-k} \end{pmatrix}$. When $k = 1$ and $m = 2$, i.e. $G_{k,m} = CP^1 = S^2$, (36) reduces to the DHF (18). Now we have

Theorem 4. *(Ding [6]) A class of solutions to the following (integrable) discrete coupled matrix nonlinear Schrödinger equation (DCMNLS):*

$$\begin{cases} i(dq_n/dt) + (q_{n+1} + q_{n-1} - 2q_n) + (q_{n+1}r_nq_n + q_nr_nq_{n-1}) = 0 \\ -i(dr_n/dt) + (r_{n+1} + r_{n-1} - 2r_n) + (r_{n+1}q_nr_n + r_nq_nr_{n-1}) = 0, \end{cases} \tag{38}$$

is gauge equivalent to the discrete equation of the Schrödinger flow of maps into Grassmannian $G_{k,m}$ and the continuous limit of the realizing gauge transformation is exactly a classical gauge transformation between the MNLS (12) and the Schrödinger flow of maps into Grassmannian $G_{k,m}$ (13).

V. Example

As applications, we would like to give some explicit solutions to the DM-HF (resp. M-HF model) from dark soliton solutions to the DNLS$^-$ (resp. NLS$^-$) by gauge transformations and illustrate that the continuous limit of the gauge transformation between the DNLS$^-$ and the DM-HF is just a classical one between the NLS$^-$ and the M-HF model. For the purpose of simplicity in calculation, we consider the one-soliton potential to the NLS$^-$ (see, for example, [12]),

$$q_s(x) = -\rho \tanh \rho x.$$

This is the dark one-soliton to the NLS$^-$ with θ being taken equal to π. Such a solution is also called a black soliton in constrast to a grey one at $\theta \neq \pi$. The corresponding black one-soliton of the DNLS$^-$ is given by (also refer to [13])

$$q_{sn} = \rho \frac{1-h^n}{1+h^n}$$

in which h is defined by the relation $h^{1/2} - h^{-1/2} = 2\rho/(1-\rho^2)^{1/2}$, $h > 1$, here only the case of $\rho^2 < 1$ is now considered. A corresponding matrix solution to the Lax equation $G_{n+1} = L_n G_n$, $dG_n/dt = M_n G_n$ at $z = 1$ is $G_n = \begin{pmatrix} A_n(-p_n t - i) & B_n i \\ A_n(-p_n t + i) & B_n i \end{pmatrix}$, where $A_n = a\frac{2^{n-1}(1+h^n)}{(1+h)^n}$, $B_n = b\frac{(2h)^{n-1}}{(1+h)^{n-1}(1+h^{n-1})}$ and $p_n = \frac{4h^{n-1}(h-1)^2}{(1+h^n)(1+h^{n-1})(h+1)}$ (a and b are free constants such that $ab \neq 0$). By using this gauge transformation G_n, we have $S_n = -G_{n-1} i\sigma_3 G_n = \begin{pmatrix} -p_n t & iB_n/A_n \\ i(p_n^2 t^2 + 1)A_n/B_n & p_n t \end{pmatrix}$. Therefore, the DM-HF has following explicit solutions, which may also be regarded as its black one-soliton solutions,

$$\begin{aligned} s_n^1 &= -\frac{4h^{n-1}(h-1)^2}{(1+h^n)(1+h^{n-1})(h+1)} t \\ s_n^2 &= \frac{1}{2}\left[\frac{16\alpha h^{n-1}(h-1)^4}{(1+h^n)(1+h^{n-1})(h+1)^3} t^2 + \frac{\alpha(1+h^{n-1})(1+h^n)}{h^{n-1}(h+1)} - \frac{h^{n-1}(h+1)}{\alpha(1+h^n)(1+h^{n-1})}\right] \\ s_n^3 &= \frac{1}{2}\left[\frac{16\alpha h^{n-1}(h-1)^4}{(1+h^n)(1+h^{n-1})(h+1)^3} t^2 + \frac{\alpha(1+h^{n-1})(1+h^n)}{h^{n-1}(h+1)} + \frac{h^{n-1}(h+1)}{\alpha(1+h^n)(1+h^{n-1})}\right] \end{aligned}$$

in which $\alpha = a/b \neq 0$ is a free parameter.

Now after the substitution of relation (11), we have

$$S_n \to S = \begin{pmatrix} -2\rho^2 t/\mathrm{ch}^2 \rho x & i/2\alpha \mathrm{ch}^2 \rho x \\ i2\alpha \mathrm{ch}^2 \rho x (4\rho^4 t^2/\mathrm{ch}^4 \rho x + 1) & 2\rho^2 t/\mathrm{ch}^2 \rho x \end{pmatrix}.$$

Hence, the M-HF model has following explicit black one-soliton solutions, which are the continuous limit of the above black one-soliton solutions to the DM-HF,

$$\begin{aligned} s^1 &= -2\rho^2 t/\mathrm{ch}^2 \rho x, \\ s^2 &= 1/2\left[8\alpha\rho^4 t^2/\mathrm{ch}^2 \rho x + 2\alpha \mathrm{ch}^2 \rho x - 1/2\alpha \mathrm{ch}^2 \rho x\right], \\ s^3 &= 1/2\left[8\alpha\rho^4 t^2/\mathrm{ch}^2 \rho x + 2\alpha \mathrm{ch}^2 \rho x + 1/2\alpha \mathrm{ch}^2 \rho x\right]. \end{aligned}$$

Furthermore, the gauge transformation G_n, between an auxiliary solution of the DNLS$^-$ and an auxiliary solution of the DM-HF for the black one-soliton solutions, has the continuous limit: $G = \begin{pmatrix} a\mathrm{ch}\rho x(-i - 2\rho^2 t/\mathrm{ch}^2\rho x) & ib/2\mathrm{ch}\rho x \\ a\mathrm{ch}\rho x(i - 2\rho^2 t/\mathrm{ch}^2\rho x) & ib/2\mathrm{ch}\rho x \end{pmatrix}$. It is easy to verify that G solves $G_x = \begin{pmatrix} 0 & q_s \\ q_s & 0 \end{pmatrix} G$, $G_t = i \begin{pmatrix} q_s^2 - \rho^2 & -q_{sx} \\ q_{sx} & -q_s^2 + \rho^2 \end{pmatrix} G$ and fulfils the desired gauge relation $S = -G^{-1} i\sigma_3 G$.

It should be mentioned that the gauge equivalent structures of the analogous of the nonlinear Schrödinger equation in $2+1$ dimensions were discussed in [4,14]. However, many questions remain open and deserve further investigation in this respect. Examples are: whether one can find a Schrödinger-like equation which is gauge equivalent to the generalized Landau-Lifshitz equation; whether the discrete equation (37) of the Schrödinger flow of maps into Grassmannian is not in general gauge equivalent to the DMNLS (36), though we believe that, not like the fact displayed in [5] for the NLS, this is the case. Finally, we remark that the obtained result suggests that there might exist an interesting and intriguing geometric relationship between the CMNLS (resp. DCMNLS) equation and the MNLS (resp. DMNLS) equation. A better understanding of this will be left for the future study.

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