Josai Mathematical Monographs
vol 3 (2001), *pp.* 37-41

ON SURFACES WITH HOLOMORPHIC GAUSS MAP IN A CERTAIN SYMMETRIC SPACE

MASATOSHI KOKUBU

I would like to talk about some results on analogue of minimal surface theory in a certain symmetric space, which is a joint work [KTUY] with Masaro Takahashi, Masaaki Umehara and Kotaro Yamada.

1. Background

We shall quickly review the fundamental facts about the minimal surface theory in Euclidean 3-space, the CMC-1 surface theory in hyperbolic 3-space and their correspondence.

	$f\colon M \to \boldsymbol{R}^3$ minimal immersion	$f\colon M \to \boldsymbol{H}^3$ CMC-1 immersion
Def	mean curvature 0	mean curvature 1
$\Longleftrightarrow$	$\exists$ null curve $F\colon \widetilde{M} \to \boldsymbol{C}^3$ s.t. $f = 2\operatorname{Re} F$.	$\exists$ null curve $F\colon \widetilde{M} \to \mathrm{SL}_2\boldsymbol{C}$ s.t. $f = FF^*$
$\Longleftrightarrow$	f is given by $\operatorname{Re}\int \begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \omega$	$f = FF^*$ where $F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega$
$\Longleftrightarrow$	Gauss map $[\partial f]\colon M \to Q_1 \subset P(\boldsymbol{C}^3)$ is holomorphic.	hyperbolic Gauss map $[\partial f f^{-1}]\colon M \to Q_1 \subset P(\mathfrak{sl}_2\boldsymbol{C})$ is holomorphic.

1st f.f.	$(1+\lvert g\rvert^2)^2\omega\bar{\omega}$	$(1+\lvert g\rvert^2)^2\omega\bar{\omega}$
period	$\forall[\gamma] \in \pi_1(M)$, $\int_\gamma \begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \omega \in \sqrt{-1}\boldsymbol{R}^3$	The monodromy group of $dFF^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \Omega$ is included in SU_2.

M is an oriented surface (with the induced metric), hence a Riemann surface.
$\widetilde{M}$ denotes the universal cover of M.
$\boldsymbol{H}^3 = \mathrm{SL}_2\boldsymbol{C}/\mathrm{SU}_2$ is embedded in $\mathrm{SL}_2\boldsymbol{C}$ (known as Cartan embedding).

2. A generalization of the ambient space

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then a symmetric space N of non-compact type is associated with $\mathfrak{g}$ as follows:

Let $\mathfrak{h}$ be a compact real form of $\mathfrak{g}$. Let $\sigma_0\colon \mathfrak{g} \to \mathfrak{g}$ denote the complex conjugate with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{h}$. We denote by $\widetilde{G}$ the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and by $\tilde{\sigma}\colon \widetilde{G} \to \widetilde{G}$ the involution of $\widetilde{G}$ such that $(\tilde{\sigma})_e = \sigma_0$. Let $\widetilde{H}$ be the set of fixed points of $\tilde{\sigma}$. Then $\widetilde{H}$ is a connected Lie subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{h}$.

In these circumstances, we have a Riemannian symmetric pair

$$(\widetilde{G}, \widetilde{H}, \tilde{\sigma}, B|_{\sqrt{-1}\mathfrak{h}}),$$

where B is the Killing form of $\mathfrak{g}$. Hence we have a Riemannian symmetric space $N = \widetilde{G}/\widetilde{H}$.

The hyperbolic 3-space is a model of $N = \widetilde{G}/\widetilde{H}$. If we start with $\mathfrak{g} = \mathfrak{sl}_2\boldsymbol{C}$, the associated space N is $\boldsymbol{H}^3 = \mathrm{SL}_2\boldsymbol{C}/\mathrm{SU}_2$.

Without loss of generality, we may assume that the symmetric space N can be represented as

$$N(= \widetilde{G}/\widetilde{H}) = G/H \quad (G \subset \mathrm{SL}_n\boldsymbol{C}).$$

(If $\widetilde{G}$ is not a subgroup of $\mathrm{SL}_n\boldsymbol{C}$, we have only to replace $\widetilde{G}$ and $\widetilde{H}$ by $G = \mathrm{Ad}(\widetilde{G})$ and $H = \mathrm{Ad}(\widetilde{H})$, respectively.)

Moreover we usually consider the Cartan model

$$N = G/H \subset G(\subset \mathrm{SL}_n\boldsymbol{C}).$$

3. Surfaces with holomorphic right Gauss map

Let $N = G/H(\subset G)$ be a symmetric space defined in the previous section. Let $f\colon M \to N$ be an immersion of an oriented surface. Here we think of f to be a G-valued map. As usual, we regard M as a Riemann surface by the orientation and the induced metric.

Definition 3.1. We call the map ν defined by

$$\nu\colon p \in M \mapsto \left[(f^*\mu_R)^{1,0}|_p\right] \in P(\mathfrak{g}) \tag{3.1}$$

the **right Gauss map**. Here, μ_R denotes the right-invariant Maurer-Cartan on G and $P(\mathfrak{g})$ is the complex projective space of $\mathfrak{g}$.

A surface with holomorphic Gauss map in N is a generalization of a CMC-1 surface in $\boldsymbol{H}^3$ from the viewpoint of Gauss map. There are also some similar property at the other point:

1. **Bryant type formula.** Any immersion $f\colon M \to N = G/H$ of a surface with holomorphic right Gauss map is given by $f = FF^*$ where $F\colon \widetilde{M} \to G$ is a null curve.
2. **Local correspondence to minimal surfaces.** For any surface with holomorphic right Gauss map $f\colon M \to N = G/H$, there exists a minimal surface in $\boldsymbol{R}^{\dim G}$ which is locally isometric to M.
3. **Duality, Dual metric.** Given a surface with holomorphic right Gauss map $f = FF^*$, there is another surface with holomorphic right Gauss map defined by $f^\# = (F^\#)(F^\#)^*\colon \tilde{M} \to N = G/H$ where $F^\#\colon z \mapsto F(z)^{-1}$. We call $f^\#$ the **dual** of f. Though the dual surface $f^\#$ is defined on the universal cover $\widetilde{M}$ in general, the induced metric $ds^{2\#} = B(dFF^{-1}, \overline{dFF^{-1}})$ is well-defined on M.

The following consideration implies that the study of surfaces with holomorphic Gauss map in $N = G/H$ is not only the analogue but also the generalization of Euclidean minimal surface theory:

$N = G/H$ contains a Euclidean space $\mathbf{R}^r$ as a totally geodesic submanifold, where r is the rank of the Lie group G. Indeed, the submanifold consisting of diagonal matrices in $N = \mathrm{SL}_n\boldsymbol{C}/\mathrm{SU}_n(\subset \mathrm{SL}_n\boldsymbol{C})$ is isometric to $\boldsymbol{R}^{n-1}$ via the exponential map. It is easy to see that $f\colon M \to \mathbf{R}^r \subset N = G/H$ has holomorphic right Gauss map (as a surface in G/H) if and only if f is a conformal minimal immersion as a surface in $\boldsymbol{R}^r$.

4. Examples

Example 4.1. Let M be a Riemann surface and a, b, c be meromorphic functions on M. We set

$$F := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Then F is a null meromorphic map on M^2. So we can construct many surfaces having holomorphic right Gauss maps as the projection of such F. For example, if we set $M = \boldsymbol{C} \setminus \{0\}$ and $a(z) = 1/b(z) = z$, $c(z) = 1$, then it gives a complete surfaces with finite total curvature in $\mathrm{SL}(3, \boldsymbol{C})/\mathrm{SU}(3)$. For a suitable choice of orthonormal basis, one can easily check that the canonical correspondence $F_0 : \tilde{M}^2 \to \boldsymbol{C}^8$ is given by

$$F_0 := \int_{z_0}^{z} (db - adc, i(db - adc), da, -ida, dc, idc, 0, 0).$$

F is called a null curve if $B(F^{-1}dF, F^{-1}dF) \equiv 0$.

Example 4.2. Let $M = \boldsymbol{C} \setminus \{0\}$, and let $\widetilde{M}$ be the universal cover of M. Define a holomorphic map $F_{\mu,a,b} \colon \widetilde{M} \to \mathrm{SL}(3, \boldsymbol{C})$ by

$$F_{\mu,a,b}(z) = \begin{bmatrix} \sqrt{\frac{b^2+3\mu^2}{b^2-a^2}}\, z^{\mu+a} & 0 & \sqrt{\frac{a^2+3\mu^2}{b^2-a^2}}\, z^{\mu-b} \\ 0 & z^{-2\mu} & 0 \\ \sqrt{\frac{a^2+3\mu^2}{b^2-a^2}}\, z^{\mu+b} & 0 & \sqrt{\frac{b^2+3\mu^2}{b^2-a^2}}\, z^{\mu-a} \end{bmatrix}, \tag{4.1}$$

where μ, a and b are real constants such that $b^2 > a^2$. Then $F_{\mu,a,b}$ takes values in the 4-dimensional subgroup $(\boldsymbol{C}^* \times \mathrm{GL}(2, \boldsymbol{C})) \cap \mathrm{SL}(3, \boldsymbol{C})$, in particular, it is $\mathrm{SL}(2, \boldsymbol{C})$-valued if $\mu = 0$.

For $F = F_{\mu,a,b}$, the 1-form α_F is computed as follows:

$$\alpha_F = \begin{bmatrix} (\mu + \frac{ab-3\mu^2}{a+b}) z^{-1} & 0 & -\frac{\sqrt{(b^2+3\mu^2)(a^2+3\mu^2)}}{a+b} z^{-a-b-1} \\ 0 & -2\mu z^{-1} & 0 \\ \frac{\sqrt{(b^2+3\mu^2)(a^2+3\mu^2)}}{a+b} z^{a+b-1} & 0 & (\mu - \frac{ab-3\mu^2}{a+b}) z^{-1} \end{bmatrix} dz.$$

It is verified treat the case when f itself is single-valued on $\boldsymbol{C} \setminus \{0\}$, i.e.,

$$f = F_{\mu,a,b}(F_{\mu,a,b})^*, \qquad b - a \in \boldsymbol{Z} \setminus \{0\},$$

where $F_{\mu,a,b}$ is given by (4.1). Then $\alpha_F^\# = dF F^{-1}$ is computed as

$$\alpha_F^\# = \begin{bmatrix} \left(\mu + \frac{ab+3\mu^2}{b-a}\right) z^{-1} & 0 & -(a+b) z^{a-b-1} \\ 0 & -2\mu z^{-1} & 0 \\ (a+b) z^{b-a-1} & 0 & \left(\mu - \frac{ab+3\mu^2}{b-a}\right) z^{-1} \end{bmatrix} dz.$$

Indeed, $\alpha_F^\#$ is single-valued on $\boldsymbol{C} \setminus \{0\}$. Furthermore, $ds^{2\#} = \operatorname{tr} \alpha^\# (\alpha^\#)^*$ is

$$ds^{2\#} = \left\{ \frac{2(a^2+3\mu^2)(b^2+3\mu^2)}{(b-a)^2} + (a+b)^2 (|z|^{2a-2b} + |z|^{2b-2a}) \right\} |z|^{-2}\, dz\, d\bar{z}.$$

If $|a - b| = 1$, the dual total curvature of f is equal to -4π which satisfies equality in the Chern-Osserman type inequality (see the next section).

5. Total (dual) curvature

We denote by ds^2 (resp. K) the induced metric (resp. the Gauss curvature) of a surface with holomorphic right Gauss map. Since the metric ds^2 is locally isometric to the metric of an Euclidean minimal surface, the Gauss curvature K is non-positive. Similarly for the dual metric $ds^{2\#}$, the dual Gauss curvature $K^\#$ is non-positive.

Theorem 5.1. *Let $f\colon M \to N = G/H$ be a complete surface with holomorphic right Gauss map. Then the dual metric is also complete, and the following hold:*

$$\frac{1}{2\pi}\int_M K^{\#}dA^{\#} \in \boldsymbol{Z}$$
$$\leq \chi(M) - \#\{ends\},$$

where $\chi(M)$ is the Euler number of M.

It is known that $(1/4\pi)\int_M K^{\#}dA^{\#} \in \boldsymbol{Z}$ holds in the case of $N = \boldsymbol{H}^3$. However, it fails to hold in general N. (Example 4.2 gives a counterexample.)

On the other hand, the total curvature satisfies neither the integrity nor the Chern-Osserman type inequality.

From the intrinsic condition, we can see that the Cohn-Vossen inequality

$$\frac{1}{2\pi}\int_M KdA \leq \chi(M)$$

holds. Umehara and Yamada proved the inequality

$$\frac{1}{2\pi}\int_M KdA < \chi(M)$$

in the case of $N = \boldsymbol{H}^3$. It is still unknown whether there is a surface with holomorphic Gauss map in the other symmetric space $N = G/H$ which satisfies equality in the Cohn-Vossen inequality.

References

[KTUY] M. Kokubu, M. Takahashi, M. Umehara and K. Yamada, *An analogue of minimal surface theory in* $\mathrm{SL}(n, C)/\mathrm{SU}(n)$, preprint

Tokyo Denki University