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ACUTE TRIANGULATIONS OF SPHERE AND ICOSAHEDRON

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ABSTRACT. In this paper firstly we summarize the known results about acute triangulations. Next on the reound sphere for any even $n \geq 35$ we construct an acute triangulations with n triangles and on the icosahedron for any even $n \geq 24$ we construct an acute triangulation with n triangles.

1. Introduction

For compact surfaces with inner metric, triangulations are finite sets of triangles satisfying certain natural conditions: the intersection of any two of the triangles is either empty or consists of a vertex or of an edge. The edges of any triangle must be shortest paths. From the view point of approximating surface in Euclidearn space by polyhedron as using vertices of triangulations, the triangulations with many thin triangles or obtuse triangles are not so conveniemt.

Now, we raise the question of existence and minimality (least number of triangles used) of *acute triangulations*, for various types of surfaces. This means that the involved triangles must be acute, i.e. all their angles must be less than $\frac{\pi}{2}$.

We settle the the following problems.

Problem. Does there exist a natural number N such that every convex surface admits an acute triangulation consisting of at most N triangles? If yes, find the minimal such N.

This question does not seem to be easy. A solution to this problem restricted to smooth surfaces would also be of interest.

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Problem. Find the analogous numbers in the cases of other surfaces or polyhedrons.

In section 2 we summarize the known results about acute triangulations. In section 3 we consider the various acute triangulations on the standard sphere and the icosahedron. Sometimes in this paper we abbreviate an acute triangulation with n triangles as an n-acute-triangulation.

2. Survey of acute triangulations

Firstly we consider about the problem on a smooth surface homeomorphic to the sphere as the simplest example.

Theorem 1. *For any surface homeomorphic to the sphere with smooth metric, all acute triangulations contains at least 20 triangles.*

Proof. For each vertex v of acute triangulations on smooth surfaces the degree at v must be at least 5. Indeed, if there is a vertex v with its degree 4, then there is a triangle around v such that the inner angle at v is not acute. It immediately follows from the above discussion and Euler's formula that any acute triangulation of such a surface contains at least 20 triangles.

Next let's consider the five Platonic solid as the easiest example of polyhedron. Of course the regular tetrahedron, octahedron and icosahedron have their natural acute triangulations.

Theorem 2 ([2]). *The cube admits several acute triangulations with 24 triangles, and no acute triangulation with fewer triangles.*

One example of acute triangulation of the cube is indicateed as Figure 1. Concerning the dodecaheder we have easily the following.

Proposition 3. *There is an acute triangulation of the dodecahedron with 20 triangles.*

Proof. Consider the dual graph on dodecahedron, i.e. take 12 vertices on the center of each pentagon and connect as segments 2 vertices in pentagons which are adjacent, then we get 20 equilateral triangles such that each triangle contains just one vertex of dodecahedron and at each vertex (center of pentagon) there are 5 triangles. Then all the angles at vertex of the equilateral triangles are equal to $\frac{2\pi}{5}$, then these are acute triangles.

Remark. But it seems that the number is not best possible. We don't know whether there is an acute triangulation of the dodecaheder with less than 20 triangles or prove that it holds that 20 is the leasest number.

Theorem 4 ([2]). *There is an acute triangulaton of the regular icosahedron with 14 triangles.*

Remark. We expect that 14 is the best possible estimate, but it has not been proved yet.

Furthermore, it seems difficult, but interesting for us even the following problem, too.

Problem. Find the analogous number in the case of all tetrahedral surfaces (not only the regular).

Next we consider the case of flat torus.

Theorem 5. *For any flat torus there is an acute triangulaton with 18 triangles.*

Proof. Each flat torus corresponds to a $\mathbf{Z}^2$-lattice on the plane with the shortest distance with two lattice points 1. We can take a parallerogram P with vertices $(0,0)$ and $(1,0)$ and (a,b), where $a^2+b^2 \geq 1$ and $-\frac{1}{2} \leq a \leq \frac{1}{2}$ from the above $\mathbf{Z}^2$-lattice. Indeed take (a,b) as a lattice point with the second shortest distance from $(0,0)$ without lattice points on the x-axis.

In the case of $a \neq 0$, we can naturally divide the parallerogram to the similar 9 parallerogram which are congruent each other by the dividing a segment three equal parts. We can divide each small parallerogram to 2 acute triangles from the figure of parallerogram P (See Figure 2.).

In the case of $a = 0$, P is a rectangle, divide each segment three equal part. Join with $(0,0)$ and $(\frac{1}{3},b)$, and so on. Join with $(0,b)$ and $(\frac{2}{3},0)$, and so on. The intersection points lie on the lines $y=0, y=\frac{1}{3}, y=\frac{2}{3}$. Taking the above three lines, we get an acute triangulation with 18 triangles.

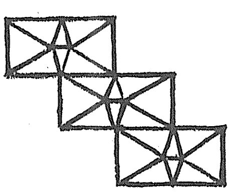

Figure 1.

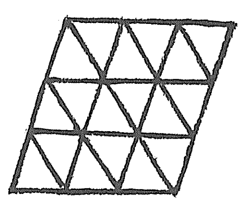

Figure 2.

In the case of flat torus 18-acute-triangulation seems not best possible. It is known that some flat torus attains 14-acute-triangulation and it is best possible number. But maybe it is impossible to construct 14-acute-triangulation on any flat torus.

In the case of standard torus (obtained by rotating a circle around a coplanar, nonintersecting line in the Euclidean 3-space), it seems that there is not the finite analogous number. It suggest us the following problem.

Problem. What are the smallest numbers t_α such that any standard torus with length ratio of the minimal cycles equal to α has an acute triangulation with at most t_α triangles?

Important and elementary surfaces with boundary which can be considered are planar sets. In this case there are some previous works.

Theorem 6 ([2], [6]). *Every triangle is triangulable with at most 7 acute triangles, and this bound is best possible.*

Theorem 7 ([2]). *Each rectangle is triangulable with 8 acute triangles, and this is the best possible estimate.*

In the case of square, it was discussed in [1]. They showed the same result and that the 8-acute-trianglations is unique in a sense. Examples of acute triangulations of an obtuse triangle and a rectangle are indicated as Figure 3 and Figure 4.

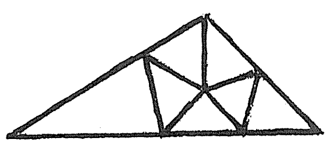

Figure 3.

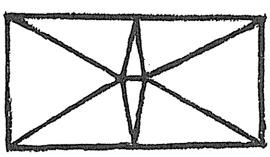

Figure 4.

Theorem 8. *Each convex quadrilateral is triangulable with at most 9 acute triangles.*

H. Maehara proved this theorem in his proof of the next theorem. (See [6].)

Remark. We don't know whether there is a quadrilateral which can not attaines 8-acute-triangulation or it holds that all quadrilateral can attaine 8-acute-triangulation.

Theorem 9 ([6]). *Every quadrilateral (even if non-convex) is triangulable with at most 10 acute triangles, and this bound is best possible when it is non-convex.*

Moreover H. Maehara discuss the in genaral polygon in [5].

Theorem 10 ([5]). *Every polygon admits an acute triangulation.*

It seems that it is interesting to discuss on acute triangulations on some polygon (even a triangle) on another surface (even on the standard sphere). But there are not any results.

3. In the case of sphere and icosahedron

Recently J. Hass and F. Morgan defined in [3] the geodesic net on surface as a graph embedded on a surface that consists of geodesic arcs whose tangent unit vectors sum to zero at each vertex. They formulate the following problem.

Problem. Let S be a smooth Riemannian 2-sphere and n be a natural number. Is there a geodesic net with vertices of degree 3 or 4 partitioning S into n regions?

In [4] A. Heppes solved this problem in the case of the standard sphere. Relateing the above problem, in this section we will consider the following problem in the case of the standard sphere and the icosahedron.

Problem. Let S be a compact surface and n a natural number. Is there an acute triangulations with n triangles on S?

Now we show the main results of this paper.

Theorem A. *For any even number n which is ≥ 20 and excluded 22, 28, 34, there is an acute triangulation of the round sphere with n triangles. Furthermore for any $n \leq 19$, odd number n or $n = 22$, there are not any acute triangulation of the round sphere with n triangles.*

Remark. In the case of $n = 28$ and 34 we don't know whether there is an acute triangulation of the round sphere with n triangles or not.

Proof. Firstly we show that any triangulation of sphere has even number triangles. Let $\#f$ (resp. $\#e$) be the number of triangles (resp. edges) of a triangulation. From the elementary equation $3\#f = 2\#e$ it follows that $\#f$ must be even.

Next we construct three triangulations with 20 acute triangles, 24 acute trinagles and 30 acute triangles. A 20-acute-triangulation is easily follows the icosahedron, a 24-acute-triangulation is follows from an acute triangulation of cube. Exactly it is constructed as follows. Take an acute triangulation of cube as Theorem 2. Settle the inscribed cuve and project the triangulation on round sphere. Around the point on round sphere which is the image of x by the projection,

A 40-acute-triangulation is constructed as follows. Let $a_1a_2a_3$ be a triangle of the triangulation induced from the inscribed tetrahedron. Denote the center of a triangle by g, the middle point of of opposite side of a_i by m_i. Take the small equilatral triangle $b_1b_2b_3$ such that b_i belongs on the segment ga_i. Draw the edges a_1b_1, a_2b_2, a_3b_3, m_1b_2, m_1b_3, m_2b_3, m_2b_1, m_3b_1, m_3b_2. Then we get an triangulation of the triangle $a_1a_2a_3$ with 10 acute triangles. By the same triangulations of the other three atriangles we get a triangulation of round sphere with 40 acute triangles.

Lemma. *If there is an acute triangulation with n triangles such that it includs one isosceles abc such that the neighboring 3 triangles abc', bca', cab' has symmetry with respect to the line through a and the middle point of bc (i.e. if $ab = ac$, then bca' is an isosceles with $a'b = a'c$ and abc' is congruent to acb'), then for any positive integer m there is an acute triangulation with $n + 6m$ triangles.*

Proof of Lemma. Take the foods of perpendiculars from a', b', c' to bc, ca, ab. Denote new three points on perpendiculars little inside the abc by $a", b", c"$. We get new 10 acute triangles using $a, b, c, a', b', c', a", b", c"$ instead of abc, abc', bca', cab' with the property: $a'b'c'$ is isocseles with symmetric neighbor 3 triangles. Then we continue the same procedure.

Any triangle of the above 20-acute-triangulation on a round sphere satisfies the assumption of Lemma. In the case of the above 24-acute-triangulation the triangle abc satisfies the assumption of Lemma. In the case of the above 30-acute-triangulation the triangle abc satisfies the assumption of Lemma. Then we get $20 + 6m$-acute-triangulation, $24 + 6m$-acute-triangulation and $40 + 6m$-acute-triangulation on the standard sphere.

At last we show the non-existence of 22-acute-triangulation of round sphere. When the number of triangles is equal to 22, it holds that the number of edges is equal to 22. From the fact that the degree ≥ 5, it follows that the number of vertices is less than 14. We get that the number of vertices is equal to 13 by Euler's formula. It holds that there must be 12 vertices with degree 5 and one vertex with degree 6. It is impossible.

Theorem B. *For any even number n which is ≥ 14 and excluded 18, 24, there is an acute triangulation of the icosahedron with n triangles. Furthermore for any odd number n, there are not any acute triangulation of the icosahedron with n triangles.*

Remark. In the case of $n = 18, 24$ we don't know whether there is an acute triangulation of the icosahedron with n triangles or not.

Proof of Theorem B. To prove this, we need construct 20-acute-triangulation, 30-acute-triangulation and 34-acute-triangulation satisfying the condition of Lemma. The icosahedron has a natural 20-acute-triangulation. A 14-acute-triangulation is constructed by Theorem 4.

Firstly we construct a 30-acute-triangulation. Take the 5 triangle abc, acd, ade, aef, afb of icosahedron and denote the middle points of ab, ad, ae, de by b', d', e', m (See Figure 5.). Denote by $b"$ the point which is moved b' to a on the edge ab with $d(b', b") = \epsilon$. Denote by $d"$ (resp. $e"$) the point which is moved d' (resp. e') to a on the edge ad (resp. ae) with $d(d', d") = \frac{\epsilon}{2}$ (resp. $d(e', e") = \frac{\epsilon}{2}$). We get new ten triangles

$$bcb", bfb", cd"b", fe"b", cdd", fee", dmd", eme", b"d"e", d"me"$$

and new one vertex m on boundary $bcdefb$. We will check that all these triangles are acute. Note that $\angle b'd'e' = \angle b'e'd' = \frac{\pi}{2}$ and $\angle d'b'e' = \frac{\pi}{3}$. If positive number ϵ is small, then $\angle d"b"e" < \frac{\pi}{2}$. From the definition of $b", d", e"$ it follows that $\angle b'd'e' = \angle b'e'd' < \frac{\pi}{2}$. Hence the triangle $b"d"e"$ is acute. Note that the triangle $cb'd'$ is equilateral and all angles are $\frac{\pi}{3}$. If positive number ϵ is small, then $\angle cb"d" < \frac{\pi}{2}$, $\angle cd"b" < \frac{\pi}{2}$ and $\angle b"cd" < \frac{\pi}{2}$. Hence the triangle $cb"d"$ is acute. From the same discussion it follows that the triangle $fb"e"$ is acute. It is trivial that the rest triangles are acute. Next, take the other 5 triangles which is the opposite side of de. By the same subdivision we get 10 acute triangles (including only one new vertex m on de) instesad of 5 triangles. Hence we get 30-acute-triangulation on the icosahedron.

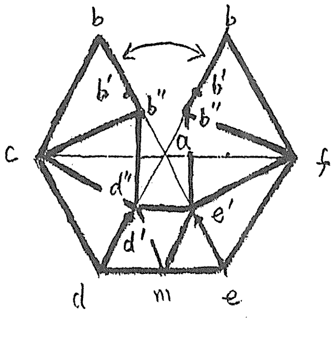

Figure 5.

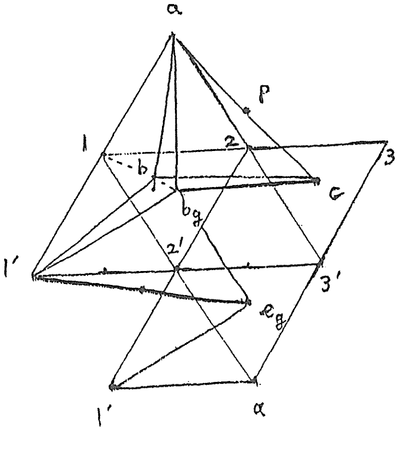

Figure 6.

Now we construct the 16-acute-triangulation as follows (See Figure 7.).

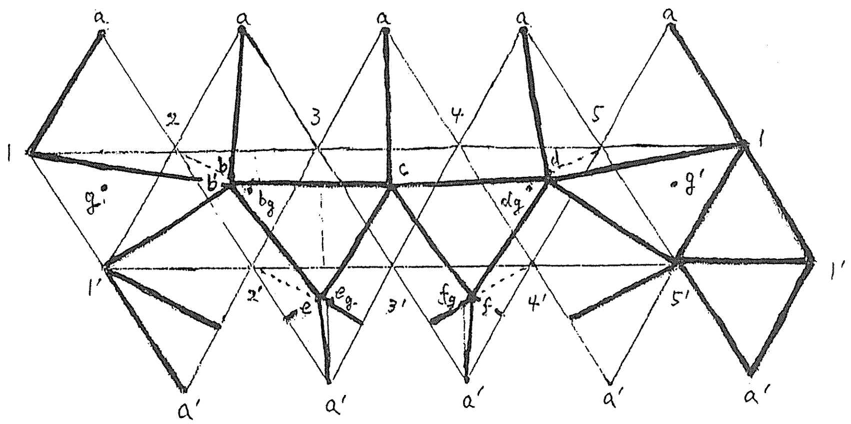

Figure 7.

Let a, a' be two vertices of the icosahedron such that they are antipodal points each other. Take 5 triangles around a (resp. a') and denote their vertices without a (resp. a') by $1, 2, 3, 4, 5$ (resp. $1', 2', 3', 4', 5'$), where we may assume that there are 10 triangles

$$11'2, 22'3, 33'4, 44'5, 55'1, 1'22', 2'33', 3'44', 4'55', 5'11'.$$

Denote the gravities of triangle $11'2, 22'3, 33'4, 44'5, 55'1, a'2'3'$ and $a'3'4'$ by g, b_g, c, d_g, g', e_g and f_g. Take a point b (resp. d) on the segment $2b_g$ (resp. $5d_g$) enough close to b_g (resp. d_g). It holds that $\angle abc < \angle ab_gc = \frac{\pi}{2}$ and $\angle adc < \angle ad_gc = \frac{\pi}{2}$. These are derived as follows: it holds that $angle\tilde{2}\tilde{b}_g\tilde{p} > \frac{\pi}{2}$, where $\tilde{*}$ denotes the point on the Euclidean plane on which the unfolded icosahedron is drawn as Figure 6, corresponding to the point $*$ on the icosahedron and $\tilde{p}$ is the middle point of the segment $\tilde{1}'\tilde{e}_g$ in Figure 6. Indeed the inner product the vector from $\tilde{p}$ to $\tilde{b}_g$ and the vector from $\tilde{2}$ to $\tilde{b}_g$ is negative (i.e. $(-1, -\frac{2\sqrt{3}}{3}) \cdot (1, -\frac{\sqrt{3}}{3}) = -\frac{1}{3} < 0$). It holds that $\angle 1'be_g < \angle 1'b_ge_g = \frac{\pi}{2}$ and $\angle 5'df_g < \angle 5'd_gf_g = \frac{\pi}{2}$. These are shown as the above, too. Then we can take a point b' (resp. d') on the ray from a to b (resp. ad) in the triangle $2b_gg$ (resp. $5d_gg'$) very enough close to b (resp. d) as $\angle 1'b'e_g < \frac{\pi}{2}$ (resp. $\angle 5'd'f_g < \frac{\pi}{2}$). Note that $\angle b'e_g1' < \angle b_ge_g1' = \frac{\pi}{2}$ and $\angle d'f_g5' < \angle d_gf_g5' = \frac{\pi}{2}$. Then we can take a point e (resp. f) on the segment $2'e_g$ (resp. $5'f_g$) enough close to e_g (resp. f_g) as $\angle b'e1' < \frac{\pi}{2}$ (resp. $\angle d'f5' < \frac{\pi}{2}$). Note that we can take e, f as the length of ee_g is equal to the length of ff_g. Take 16 triangles

$$a1b', ab'c, acd', ad'1, a'1'e, a'ef, a'f5', a'5'1', 11'b', 1'b'e, b'ec, ecf, cfd', fd'5', d'5'1, 5'11'.$$

We will check that all these triangles are acute. It is easy that all angles around a, a' are acute. Indeed, $\angle 1ab' < \angle 1ab_g = \frac{\pi}{2}$, $\angle 1ad' < \angle 1ad_g = \frac{\pi}{2}$, $\angle b'ac < \angle 2ac = \frac{\pi}{2}$, $\angle d'ac < \angle 5ac = \frac{\pi}{2}$, $\angle 1'a'e < \angle 1'a'e_g = \frac{\pi}{2}$, $\angle 5'a'f < \angle 5'a'f_g = \frac{\pi}{2}$, $\angle ea'f = \frac{\pi}{3} + \epsilon$, $\angle 1'a'5' = \frac{\pi}{3}$. Let's discuss for other angles of triangle around a, a'. Note that the segment $1b'$ is contained in the tiangles $11'2, 21'2', 22'3$. It holds $\angle a1b' = \angle a12 + \angle 21b' < \angle a12 + \angle 212' = \frac{\pi}{2}$. Note that $\angle ab_g1 < \angle 1b_gg = \frac{\pi}{2}$. Then we get $\angle ab'1 < \frac{\pi}{2}$ for b' is very close to b_g. It is trivial that $\angle ab'c < \angle abc$. It have been still checked that $\angle abc < \frac{\pi}{2}$. Then, $\angle ab'c < \frac{\pi}{2}$. It holds that $\angle acb' < \frac{\pi}{2}$, for b' is contained in the triangle $2b_gg$. The angles $\angle acd, \angle adc, \angle ad1, \angle a1d$ are the same as above. It is trivial that $\angle a'1'5' = \angle a'5'1' = \frac{\pi}{3}$, $\angle a'1'e < \angle a'1'2' = \frac{\pi}{3}$, $\angle a'5'f < \angle a'5'4' = \frac{\pi}{3}$, $\angle a'e1' < \angle a'e_g1' = \frac{\pi}{3}$, $\angle a'f5' < \angle a'f_g5' = \frac{\pi}{3}$, $\angle a'ef < \angle a'e_gf = \frac{\pi}{3}$, $\angle a'fe < \angle a'f_ge = \frac{\pi}{3}$.

Let's consider the triangle $11'b'$. It holds that $\angle 11'b' < \angle 11'b_g = \frac{\pi}{2}$. It is trivial that $\angle 1b'1' < \frac{\pi}{2}$ and $\angle b'11' < \frac{\pi}{2}$. Next let's consider the triangle $1'b'e$. It holds that $\angle b'1'e = \angle b'1'2' + \angle 2'1'e < \angle 21'2' + \angle 2'1'e_g = \frac{\pi}{2}$ and $\angle 1'eb' < \angle 1'eb_g = \frac{\pi}{2}$. It have been still checked that $\angle 1'b'e < \frac{\pi}{2}$. The triangle $b'ec$ is enough close to the equilateral triangle b_ge_gc whose angles are equal to $\frac{\pi}{3}$. Then $\angle b'ec, \angle ecb', \angle cb'e$ are almost equal to $\frac{\pi}{3}$. For the triangle cef, $\angle fce$ is close to $\angle f_gce_g = \frac{\pi}{3}$. It holds that $\angle cef < \angle ce_gf_g = \frac{\pi}{2}$. Indeed, note that $e_gf_g \subset eg$. The triangles $cfd', d'f5'$ are acute triangles from the same discussions as above. The triangle $15'1'$ is an equilateral triangle whose angles are equal to $\frac{\pi}{3}$. Hence we proved that the above all triangles are acute.

At last we must check that these acute triangulations satisfy the assumption of Lemma. It is trivial that the 20-acute-triangulation satisfies it. In the case of 30-acute-triangulation, the isosceles $md"e"$ satisfies the condition of Lemma. In the case of 16-acute triangulation, we can take the point b', d' as the isocseles $15'1'$ satisfies the condition of Lemma. Hence we get $14 + 6m, 30 + 6m, 16 + 6m$-acute-triangulations of the icosahedron for any positive integer m.

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