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SOME GEOMETRIC PROPERTIES OF p-HARMONIC MAPS

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1. Introduction

In this note we consider geometric properties of p-harmonic maps and p-harmonic functions. Let (M, g) and (N, h) be connected Riemannian manifolds and $p \geq 2$. We suppose from a technical reason that (N, h) is isometrically imbedded in a Euclidean space $\mathbb{R}^m$ and denote by $H^{1,p}(M, N)$ the space

$$\{u \in H^{1,p}(M, \mathbb{R}^m) \mid u(x) \in N \, a.e. \}.$$

A $H^{1,p}$-map u from M into N is called a (weakly) p-harmonic map if it is a weak solution of the following equation

$$\mathrm{Tr}\, \nabla(|\mathrm{d}u|^{p-2}\, \mathrm{d}u) \;=\; 0\,,$$

where "Tr" denotes the trace. It is the Euler-Lagrange equation of the p-energy functional

$$\mathrm{E}_p(u) = \int_M |\mathrm{d}u|^p.$$

This definition of p-harmonic maps does not depend on the embedding on N in a Euclidean space and coinside with that of harmonic maps when $p = 2$.

Because we are now interested only in "geometric" properties, p-harmonic maps are assumed to be of class C^1 or C^2 in the sequel. The following are typical examples of p-harmonic maps.

EXAMPLE 1. (totally geodesic maps) Every totally geodesic map $u : M \to N$ is a p-harmonic map for any p. Indeed if $\nabla \mathrm{d}u = 0$, then we have

$$\begin{aligned} \mathrm{Tr}\nabla(|\mathrm{d}u|^{p-2}\mathrm{d}u) &= |\mathrm{d}u|^{p-2}\mathrm{Tr}(\nabla \mathrm{d}u) + \langle \nabla|\mathrm{d}u|^{p-2}, \mathrm{d}u\rangle \\ &= 0 + (p-2)|\mathrm{d}u|^{p-4}\langle\langle \nabla \mathrm{d}u, \mathrm{d}u\rangle, \mathrm{d}u\rangle \\ &= 0. \end{aligned}$$

If $v : N \to L$ is an isometric totally geodesic immersion, then the identity

$$\begin{aligned} \mathrm{Tr}\nabla(|\mathrm{d}(v \circ u)|^{p-2}\mathrm{d}(v \circ u)) &= \mathrm{Tr}\nabla(|\mathrm{d}u|^{p-2}\mathrm{d}(v \circ u)) \\ &= \mathrm{d}v \left(\mathrm{Tr}\nabla(|\mathrm{d}u|^{p-2}du)\right) \end{aligned}$$

holds for every map $u : M \to N$. Hence when N is embedded by i in L as a totally geodesic submanifold, a map $u : M \to N$ is p-harmonic if and only if $i \circ u : M \to L$ is p-harmonic.

EXAMPLE 2. (p-harmonic functions) Consider the case $M = \mathbb{R}^n \setminus \{O\}$ and $N = \mathbb{R}$. Then the following functions which depend only on $r = |x|$ are p-harmonic:

$$u(r) = \begin{cases} r^{(p-n)/(p-1)} & (n \neq p) \\ \\ \log r & (n = p). \end{cases}$$

This is shown for example from the formula

$$\begin{aligned} &\mathrm{Tr}\nabla(|\nabla u|^{p-2}\nabla u) \\ &= \frac{\partial}{\partial r}(|\nabla u|^{p-2}\frac{\partial u}{\partial r}) + \frac{n-1}{r}|\nabla u|^{p-2}\frac{\partial u}{\partial r} + \mathrm{Tr}_S\nabla_S(|\nabla u|^{p-2}\nabla_S u), \end{aligned}$$

where Tr_S, ∇_S denote the trace and differentiation in the direction of spheres of radius r respectively.

EXAMPLE 3. (equator maps) For n, m with $n \geq m+1$, consider the unit ball B^n in $\mathbb{R}^n = \mathbb{R}^{m+1} \times \mathbb{R}^{n-m-1}$ and unit sphere S^m in $\mathbb{R}^{m+1}$. Let $M = B^n \setminus (\{O\} \times \mathbb{R}^{n-m-1})$ and $N = S^m$. Then a map $u : M \to N$ is defined by the equation

$$u(y, z) = \frac{y}{|y|} \quad (y \in \mathbb{R}^{m+1},\ z \in \mathbb{R}^{n-m-1}).$$

If $n < m + 1$, then $u : B^n \setminus \{O\} \to S^m \subset \mathbb{R}^{m+1}$ is defined by

$$u(y) = \left(\frac{y}{|y|}, O\right) \in S^{n-1} \times \{O\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n+1}$$

These maps are p-harmonic for any p. More strongly some of them are in fact p-energy minimizing for their boundary data ([1]).

Remark. If u is a p-harmonic function, then Cu and $u + C$ are p-harmonic function for any constant C. But $u + v$ is not necessarily p-harmonic even if u and v are p-harmonic functions. Also coordinate functions of a p-harmonic map to a Euclidean space are not necessarily p-harmonic functions.

2. Existence of p-harmonic maps from the spheres.

In this section we treat the existence problem.

THEOREM 1. ([8]) *Let N be a compact simply connected Riemannian manifold isometrically embedded in a Euclidean space. Then for any C^1-map $u : S^n \to N$ with $n \geq 2$, there exist a finite number of n-harmonic maps $u_1, u_2, \ldots, u_k : S^n \to N$ which satisfy the following:*

(1) $[u] = [u_1] + [u_2] + \cdots + [u_k]$.
(2) $\inf_{v \in [u]} \mathrm{E}_n(v) = \mathrm{E}_n(u_1) + \cdots + \mathrm{E}_n(u_k)$.
(3) *u_j is a minimizer of E_n in $[u_j]$ for $j = 1, 2, , \ldots k$.*

In the above [ ] denotes the free homotopy class of maps from the sphere. If N is not simply connected, the same results holds up to the action of $\pi_1(N)$ on $\pi_n(N)$. This result is shown by a bubbling argument and a generalization of the work in [9], [11] for harmonic maps from S^2. The case $\pi_n(N) = \{0\}$ with arbitrary M was first proved by a different method in an unpublilshed paper [6], and can be deduced from [14] too. As an application of this theorem, we can give an alternative proof of a result concerning manifolds with strongly p-th moment stable stochastic dynamical systems.

THEOREM 2. ([4], [8]) *If a compact manifold N admit a strongly p-th moment stable stochastic dynamical system, then $\pi_k(N) = 0$ for $k = 1, 2, \dots, p$. In particular for $p \geq \dim N/2$, a p-th moment stable stochastic dynamical system can only exist on homotopy spheres.*

¿From the previous theorem we can show that the elements of p-dimensional homotopy groups are represented by p-harmonic maps. The proof of the following theorem is almost the same as that for $\pi_2(N)$ in [9].

THEOREM 3. ([8]) *Let N be a compact simply connected Riemannian manifold isometriccaly embedded in a Euclidean space. Then for every $n \geq 2$ there exist finite number of n-harmonic maps $f_1, f_2, \dots$ such that*

(1) $[f_1], [f_2], \dots \neq 0$.
(2) $[f_1], [f_2], \dots$ *generate* $\pi_n(N)$.
(3) $\mathrm{E}_n(f_1) = \inf\{\mathrm{E}_n(f) \mid [f] \neq 0\}$.
$\mathrm{E}_n(f_j) = \inf\{\mathrm{E}_n(f) \mid [f] \notin \langle [f_1], [f_2], \dots, [f_{j-1}] \rangle\}$

where $\langle a, b, c, \dots \rangle$ denotes the subgroup generated by elements $a, b, c, \dots$

3. Convex functions and p-harmonic maps.

In this section, we consider geometric properties of p-harmonic maps to manifolds which have convex functions. The next result on harmonic maps is well known.

THEOREM 4. ([5]) *If $\varphi : M \to N$ is a harmonic map and $f : N \to \mathbb{R}$ is a C^2 convex function, then $f \circ \varphi$ is a subharmonic function on M. If in addition M is compact and f is strictly convex, then φ is constant.*

This is derived by the identity

$$\mathrm{Tr}\nabla(\mathrm{d}(f \circ \varphi)) = \mathrm{Tr}(\nabla \mathrm{d} f)(\mathrm{d}\varphi, \mathrm{d}\varphi)$$

which is easily verified from the composition law. For p-harmonic maps with $p \geq 2$, the following lemma is useful.

LEMMA. ([7]) *If $\varphi : M \to N$ is a p-harmonic map and $f : N \to \mathbb{R}$ is a convex function (both of class C^2), then we have the following identity:*

$$\mathrm{Tr}\nabla(|\mathrm{d}\varphi|^{p-2}\mathrm{d}(f \circ \varphi)) = |\mathrm{d}\varphi|^{p-2}\mathrm{Tr}(\nabla \mathrm{d} f)(\mathrm{d}\varphi, \mathrm{d}\varphi).$$

This is also a consequence of simple computation including the composition law. If φ is assumed to be only of class C^1, then we get an identity "in a weak form". In the rest of this section, we consider the applications of the above identity. The first is a generalization of the result of W.B Gordon which was obtained in [2] by calculating the first variation directly.

THEOREM 5. ([2], [7]) *Let $\varphi : M \to N$ be a p-harmonic map of class C^1. If M is compact and there exists a strictly convex function f of class C^2 on N, then φ is constant.*

Proof. If ϕ is of class C^2, then we have only to integrate both sides of the identity in the above lemma. When ϕ is only of class C^1, we use the identity "in a weak form".

THEOREM 6.([7]) *Let M, N be Riemannian manifolds. Suppose that M is complete and noncompact. If N has a strictly convex function f of class C^2 such that the uniform norm $|df|$ is bounded. Then every p-harmonic map $\varphi : M \to N$ of class C^1 with*

$$\int_M |d\varphi|^{p-1} < \infty$$

is a constant map.

Proof. Fix a point x of M and take a function η on M which satisfies the property

$$\begin{cases} 0 \leq \eta \leq 1, & \\ |d\eta| \leq C/R, & \\ \eta \equiv 1 & \text{on } B_R, \\ \eta \equiv 0 & \text{on } M \setminus B_{2R}. \end{cases}$$

In the above, B_R and C denote the ball of radius R centered at x and a constant which does not depend on R respectively. Making use of Lemma we get the inequlaity

$$\begin{aligned} \int_{B_R} |d\varphi|^{p-2}\mathrm{Tr}(\nabla d\varphi)(d\varphi, d\varphi)\eta &\leq \int_{B_{2R}} \mathrm{Tr}\nabla \left(|d\varphi|^{p-2} d(f \circ \varphi)\right) \eta \\ &\leq \frac{C}{R} \int_M |d\varphi|^{p-1}. \end{aligned}$$

Letting $R \to \infty$, we obtain the desired result. □

In fact we have only to assume that the growth of $p-1$ energy of φ on the spheres of radius R is $o(R)$. Relating results including the case of p-subharmonic functions are found in [7].

In [12] J.H.Sampson proved a form of maximum principle for harmonic maps. Let us say, following [12], the second fundamental form ∇di of a hypersurface S in N embedded by a map i is definite at a point $y_0 \in N$ iff $\nabla di(X, X)$ is nonzero and points in a given direction for any tangent vector $X \in T_{y_0}S$. The concave side of a hypersurface is the side pointed at by $\nabla di(X, X)$.

THEOREM 7. ([12]) *Let $\varphi : M \to N$ be a nonconstant harmonic map Assume that S is a hypersurface of N with definite second fundamental form at a point $y_0 = \varphi(x_0)$. Then no neighborhood of x_0 is mapped entirely in the concave side of S in N.*

Proof. ([3]) There exists a strictly convex function f defined in a neighborhood V of y_0 in N which depends only on the distance from S. In addition we can assume that $f < 0$ on the concave side of S, and $f^{-1}(0) \cap V = S$ ([3] p.23). ¿From the theorem of W.B. Gordon, the composition $f \circ \varphi$ is a subharmonic function. In addition it is nonpositive in a neighborhood of x_0 and achieve the maximum in the interior. Hence it is constant from the strong maximum principle, and we obtain

$$\mathrm{Tr}(\nabla \mathrm{d} f)(\mathrm{d}\varphi, \mathrm{d}\varphi) = \triangle(f \circ \varphi) = 0.$$

Because f is assumed to be strictly convex, the map φ must be a constant. □

Though we cannot completely generalize his result for the case of p-harmonic maps with $p \geq 2$, the next weak property can be proved.

THEOREM 8. *Let $\varphi : M \to N$ be a nonconstant p-harmonic map such that no point in a neighborhood of $x_0 \in M$ is mapped to $\varphi(x_0)$. Assume that S is a hypersurface of N with definite second fundamental form at a point $y_0 = \varphi(x_0)$. Then no neighborhood of x_0 is mapped entirely in the concave side of S in N.*

Proof. Take a strictly convex function f with the same property as in the previous theorem. To prove by contradiction, we assume that φ maps a neighborhood U of x_0 in M to the concave side of S. Then the inequality

$$f(\varphi(x)) = 0 \leq f(\varphi(x_0))$$

holds for all x in U. From Lemma the following one also holds:

$$\mathrm{Tr}\nabla(|\mathrm{d}\varphi|^{p-2}\mathrm{d}(f \circ \varphi)) \geq 0.$$

We want to claim that

$$\sup_{\partial U}(f \circ \varphi) = \sup_{U}(f \circ \varphi) = 0.$$

For this purpose suppose that

$$a = \sup_{\partial U}(f \circ \varphi) < \sup_{U}(f \circ \varphi) = 0.$$

Then there exist a constand $c > 0$ and a subdomain U' in U such that

$$g = f \circ \varphi - a - c > 0$$

in U' and $g = 0$ on $\partial U'$. Let us consider a function ψ which is equal to g in U' and vanishes outside U'. Using an previous inequality we obtain

$$\int_U |\mathrm{d}\varphi|^{p-2}\langle \mathrm{d}(f \circ \varphi), \mathrm{d}\psi\rangle \leq 0.$$

From the definition of ψ,

$$\int_{U'} |\mathrm{d}\varphi|^{p-2}|\mathrm{d}(f\circ\varphi)|^2 \le 0$$

which implies $|\mathrm{d}\varphi| = 0$ or $|\mathrm{d}(f\circ\varphi)| = 0$ in U'. In both cases we get $\mathrm{d}(f\circ\varphi) = 0$, and the function g is constant in U'. This contradicts the definition of g and the equality

$$\sup_{\partial U}(f\circ\varphi) = \sup_{U}(f\circ\varphi) = 0$$

holds.

Hence there exists some point $x_1 \in \partial U$ with $\varphi(x_1) \in S \setminus x_0$. The hypersurface S is locally a graph of some convex function defined on a neighorhood of x_0 in the tangent plane at x_0. We can take a hypersurface S' with definite second fundamental form whose intersection with S is only the point x_0 and the mapping φ maps U entirely in the concave side of S'. Repeating the same reasoning for this hypersurface S', we get a contradiction. □

4. Inequality between $|\nabla|\mathrm{d}\varphi||$ and $|\nabla \mathrm{d}\varphi|$

For harmonic maps the following inequality is known.

THEOREM 9. ([10], [13]) *Let $\varphi : M^n \to N$ be a harmonic map. Then on the set $\{x \mid \mathrm{d}\varphi(x) \ne 0\}$ we have the inequality*

$$\frac{n}{n-1}|\nabla|\mathrm{d}\varphi||^2 \le |\nabla \mathrm{d}\varphi|^2.$$

For a tensor T the inequality

$$|\nabla|T|| \le |\nabla T|$$

holds generally. The above theorem states that it can be improved in the case $T = \nabla \mathrm{d}\varphi$ for some harmonic map φ. Such kind of inequalities appear in many geometric prpblems. The case of harmonic map was proved in [13] with a different constant, and is used in the regularity problem of harmonic maps to spheres. Later [10] proved the above form and improved some of regularity results in [13]. It is not known whether this type of inequality is true for p-harmonic maps or not. But at least we can prove an inequlaity for p-harmonic functions.

THEOREM 10. *Let u be a p-harmonic function on a Riemannian manifold. Then the following inequality holds on the set $\{x \mid |\nabla u|(x) \ne 0\}$:*

$$\min\left\{2,\ 1+\frac{(p-1)^2}{n-1}\right\}|\nabla|\nabla u||^2 \le |\nabla\nabla u|^2.$$

Proof. For x with $\nabla u(x) \ne 0$, let us take a normal coordinate around x such that

$$u_1(x) = |\nabla u|(x),\ u_2(x) = \cdots = u_n(x) = 0.$$

Then we have at x

$$\nabla_j|\nabla u| = u_{1j}, \ |\nabla|\nabla u||^2 = \sum_j u_{1j}^2.$$

On the other hand, since u is p-harmonic

$$|\nabla u|^{p-2}\triangle u + \left\langle \nabla|\nabla u|^{p-2}, \nabla u\right\rangle = 0.$$

Consequently we get

$$\begin{aligned}
&|\nabla u|^{2(p-2)}|\nabla\nabla u|^2 - |\nabla u|^{2(p-2)}|\nabla|\nabla u||^2 \\
&= |\nabla u|^{2(p-2)}\sum_{i,j} u_{i,j}^2 - |\nabla u|^{2(p-2)}\sum_j u_{1j}^2 \\
&\geq |\nabla u|^{2(p-2)}\sum_{i\geq 2} u_{i1}^2 + |\nabla u|^{2(p-2)}\sum_{i\geq 2} u_{ii}^2 \\
&\geq |\nabla u|^{2(p-2)}\sum_{i\geq 2} u_{i1}^2 + \frac{1}{n-1}|\nabla u|^{2(p-2)}\left(\sum_{i\geq 2} u_{ii}\right)^2.
\end{aligned}$$

The second term on the right hand side is equal to

$$\begin{aligned}
&\frac{1}{n-1}\left(|\nabla u|^{p-2}u_{11} + \left\langle \nabla|\nabla u|^{p-2}, \nabla u\right\rangle\right)^2 \\
&= \frac{1}{n-1}\left(|\nabla u|^{p-2}u_{11} + (p-2)|\nabla u|^{p-3}u_{11}u_1\right)^2 \\
&= \frac{(p-1)^2}{n-1}|\nabla u|^{2(p-2)}u_{11}^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&|\nabla u|^{2(p-2)}|\nabla\nabla u|^2 - |\nabla u|^{2(p-2)}|\nabla|\nabla u||^2 \\
&\geq |\nabla u|^{2(p-2)}\sum_{i\geq 2} u_{i1}^2 + \frac{(p-1)^2}{n-1}|\nabla u|^{2(p-2)}u_{11}^2 \\
&\geq \min\left\{1, \ \frac{(p-1)^2}{n-1}\right\}|\nabla u|^{2(p-2)}|\nabla|\nabla u||^2
\end{aligned}$$

which implies the desired inequlaity. □

Remark. For the equator map $\varphi : B^n\backslash(\{O\}\times\mathbb{R}^{n-m-1}) \to S^m$ defined in Example 3,

$$\sqrt{2}\,|\nabla|\mathrm{d}\varphi|| = |\nabla\mathrm{d}\varphi|.$$

Remark. We can also show the next inequality for a p-harmonic function u with a similar calculation:

$$\left(1 + \min\left\{\frac{1}{(p-1)^2}, \ \frac{1}{(n-1)}\right\}\right)|\nabla|\nabla u|^{p-1}|^2 \leq \left|\nabla\left(|\nabla u|^{p-2}\nabla u\right)\right|^2.$$

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