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MAXIMUM PRINCIPLES AND ITS APPLICATIONS TO SUBMANIFOLDS

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ABSTRACT. In this paper, we study n-dimensional complete and connected submanifolds in an $(n+p)$-dimensional Euclidean space E^{n+p}. The classification of n-dimensional complete and connected submanifolds in an $(n+p)$-dimensional Euclidean space E^{n+p} is given under some conditions on submanifolds.

1. INTRODUCTION

The purpose of this paper is to study n-dimensional connected submanifolds in an $(n+p)$-dimensional Euclidean space E^{n+p}. In 1900, Liebmann proved that compact surfaces with constant Gaussian curvature in E^3 and a compact surfaces with constant mean curvature and with nonnegative Gaussian curvature in E^3 are the standard spheres. In 1951, Hopf [8] proved the following theorem.

Theorem 1.1 (H. Hopf). *Let M be a compact surface with constant mean curvature $|H|$ and with genus zero in E^3, then M is a standard sphere.*

The theorem of Hopf was extended to complete surfaces in E^3 by Klotz-Osserman [9] as following.

Theorem 1.2 (T. Klotz and R. Osserman). *Let M be a complete and connected surface with constant mean curvature $|H|$ in E^3. If the Gaussian curvature G of M is nonnegative, then M is a plane E^2 in E^3, a sphere $S^2(c)$ in E^3 or a cylinder $S^1(c) \times E^1$ in E^3.*

From the equation of Gauss, we know that the Gaussian curvature G of a surfaces in E^3 is nonnegative if and only if $\langle h \rangle^2 \leq n^2|H|^2/(n-1)$, where $n = 2$ and $\langle h \rangle$ is the length of the second fundamental form of a surfaces in E^3 and $|H|$ is the mean curvature of a sufaces in E^3. The result due to Klotz-Osserman is extended to higher codimensions by Shen in [14] as following.

Theorem 1.3 (B. Y. Shen). *Let M be a complete and connected surface with parallel mean curvature vector H in an $(2+p)$-dimensional Euclidean space E^{2+p}. If the second fundamental form h of M satisfies*

$$\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1} \quad (n = 2),$$

then M is a plane E^2 in E^{2+p}, a sphere $S^2(c)$ in E^{2+p}, a cylinder $S^1(c) \times E^1$ in E^{2+p} or a product surface with circles $S^1(c_1) \times S^1(c_2)$ in E^{2+p}.

For connected hypersurfaces in an $(n+1)$-dimensional Euclidean space E^{n+1}, in [12], Nomizu and Smyth proved the following :

Theorem 1.4 (K. Nomizu and B. Smyth).
(1) Let M be a compact hypersurface with constant mean curvature $|H|$ in an $(n+1)$-dimensional Euclidean space E^{n+1}. If the sectional curvatures of M are nonnegative, then M is a hypersphere $S^n(c)$ in E^{n+1}.
(2) Let M be a complete and connected hypersurface with constant mean curvature $|H|$ and with constant scalar curvature r in an $(n+1)$-dimensional Euclidean space E^{n+1}. If the sectional curvatures of M are nonnegative, then M is a hyperplane E^n in E^{n+1}, a hypersphere $S^n(c)$ in E^{n+1} or the generalized cylinder $S^{n-k}(c) \times E^k \, (1 \leq k \leq n-1)$ in E^{n+1}.

In the case of complete and connected hypersurfaces in E^{n+1}, Nomizu and Smyth assumed three conditions, that is, constant scalar curvature and constant mean curvature and nonnegative sectional curvature, but the conditions of the theorem of Nomizu and Smyth are too strong. In [5] and [6], Cheng and Yau improved the theorem of Nomizu and Smyth as following.

Theorem 1.5 (S. Y. Cheng and S. T. Yau).
(1) Let M be a complete and connected hypersuface with constant scalar curvature r in an $(n+1)$-dimensional Euclidean space E^{n+1}. If the sectional curvatures of M are nonnegative, then M is a hyperplane E^n in E^{n+1}, a hypersphere $S^n(c)$ in E^{n+1} or the generalized cylinder $S^{n-k}(c) \times E^k \, (1 \leq k \leq n-1)$ in E^{n+1}.
(2) Let M be a complete and connected hypersurface with constant mean curvature $|H|$ in an $(n+1)$-dimensional Euclidean space E^{n+1}. If the sectional curvatures of M are nonnegative, then M is a hyperplane E^n in E^{n+1}, a hypersphere $S^n(c)$ in E^{n+1} or the generalized cylinder $S^{n-k}(c) \times E^k \, (1 \leq k \leq n-1)$ in E^{n+1}.

We can expect that complete and connected hypersufaces with constant mean curvature $|H|$ and with constant scalar curvature r in an $(n+1)$-dimensional Euclidean space E^{n+1} are the hyperplane E^n in E^{n+1}, the hypersphere $S^n(c)$ in E^{n+1} or the generalized cylinder $S^{n-k}(c) \times E^k \, (1 \leq k \leq n-1)$. But it is known that the above conjecture holds when the dimension of hypersurfaces in E^{n+1} is 3.

Remark 1.1. *It is known that compact hypersurfaces with constant mean curvature and with constant scalar curvature in an $(n+1)$-dimensional Euclidean space E^{n+1} are the hypersphere $S^n(c)$ in E^{n+1}.*

On the other hand, Cheng and Nonaka [3] studied submanifolds with higher dimensions and higher codimensions in a Euclidean space. They extended the theorem of Klotz and Osserman to higher dimensions and higher codimensions.

Theorem 1.6 (Q. M. Cheng and K. Nonaka). *Let M be an n-dimensional complete and connected submanifold with parallel mean curvature vector H in an $(n+p)$-dimensional*

Euclidean space $E^{n+p}\,(n \geq 3)$. *If the second fundamental form* h *of* M *satisfies*

$$\langle h\rangle^2 \leq \frac{n^2|H|^2}{n-1}\,, \tag{1.1}$$

then M *is a totally geodesic Euclidean space* E^n *in* E^{n+p}, *a totally umbilical sphere* $S^n(c)$ *in* E^{n+p} *or the generalized cylinder* $S^{n-1}(c) \times E^1$ *in* E^{n+p}.

Remark 1.2. *In* [14], *Shen intended to prove the theorem 1.6 by making use of the result of Motomiya* [11]. *But, since the result of Motomiya is wrong (see the section 2), the proof of Shen about the theorem 1.6 is not valid. A proof of the theorem 1.6 was given by Cheng and Nonaka* [3] *(see the section 3).*

2. The Maximum Principle

In this section, we shall mention the maximum principles which play on an important role in the study of differential geometry on Riemannian manifolds. First of all, we state the well known theorem which is called Hopf's maximum principle as the following.

Hopf's maximum principle :

Let M be an n-dimensional connected Riemannian manifold. If a C^2-function f satisfies $\Delta f \geq 0\,(resp.\,\Delta f \leq 0)$ on M and has a maximum (resp. a minimum) on M, then f is a constant function, where Δ denotes the Laplacian on M.

When the Riemannian manifolds are compact, Hopf's maximum principle is often used in the following form.

Hopf's maximum principle :

Let M be an n-dimensional compact Riemannian manifold. If a C^2-function f on M satisfies $\Delta f \geq 0$ or $\Delta f \leq 0$ on M, then M is a constant function.

When the Riemannian manifolds are complete and connected, Omori and Yau proved a very important theorem which is called the genaralized maximum principle. In [13], Omori proved the following.

Omori's maximum principle :

Let M be an n-dimensional complete and connected Riemannian manifold with the sectional curvatures bounded from below. If a C^2-function f is bounded from above, then for all $\varepsilon > 0$, there exists a point $x \in M$ such that

$$\begin{cases} \sup f - \varepsilon < f(x)\,, \\ ||\mathrm{grad}\, f(x)|| < \varepsilon\,, \\ \max\{\sum_{i,j=1}^n X^i X^j \nabla_i \nabla_j f(x) \,|\, X \in T_x(M),\ |X| = 1\} < \varepsilon\,, \end{cases}$$

where ∇ denotes the Riemannian connection on M and $X = \sum_{i=1}^n X^i \partial/\partial x^i$ for the natural frame $\partial/\partial x^i$ $(i = 1, 2, \cdots, n)$.

In this paper, we shall make use of the following convention on the ranges of indices :

$$1 \leq i,\ j,\ k, \cdots \leq n, n+1 \leq \alpha,\ \beta,\ \gamma, \cdots \leq n+p.$$

In [15], Yau generalized Omori's maximum principle.

The generalized maximum principle :

Let M be an n-dimensional complete and connected Riemannian manifold with Ricci curvature bounded from below. If a C^2-function f is bounded from above, then for all $\varepsilon > 0$, there exists a point $x \in M$ such that

$$\sup f - \varepsilon < f(x)\,, \tag{2.1}$$

$$||\mathrm{grad}\, f(x)|| < \varepsilon\,, \tag{2.2}$$

$$\Delta f(x) < \varepsilon\,. \tag{2.3}$$

In [11], Motomiya intended to improve the generalized maximum principle due to Omori and Yau as the following.

Motomiya's wrong result :

Let M be an n-dimensional complete and connected Riemannian manifold with Ricci curvature bounded from below. If a C^2-function f is bounded from above and has no maximum, then for all $\varepsilon > 0$, there exists a point $x \in M$ such that

$$\sup f - \varepsilon < f(x) < \sup f - \frac{\varepsilon}{2}\,, \tag{2.4}$$

$$||\mathrm{grad}\, f(x)|| < \varepsilon\,, \tag{2.5}$$

$$\Delta f(x) < \varepsilon\,. \tag{2.6}$$

But this result is wrong. In fact, Cheng and Wu gave the following counter example in [4].

Counter example of result of Motomiya : Let $M = E^2$, $f(x,y) = -\exp(cx)$ $(c \geq 2)$. Obviously, M is a 2-dimensional complete and connected Riemannian manifold with Ricci curvature (=0) bounded from below and a smooth function $f(< 0)$ is bounded from above and f has no maximum on M. So, if for $\varepsilon > 0$, there exists a point $q = (x, y) \in M$ such that (2.4)-(2.6) hold, then from (2.5), we see that

$$||\mathrm{grad}\, f(q)|| = c\exp(cx) < \varepsilon\,.$$

Hence, we have

$$-\frac{\varepsilon}{c} < -\exp(cx) = f(q)\,. \tag{2.7}$$

On the other hand, since $\sup f = 0$, from (2.4), we have

$$-\varepsilon < f(q) < -\frac{\varepsilon}{2}\,. \tag{2.8}$$

Taking $c \geq 2$, we find that (2.7) and (2.8) are contradictory.

The Motomiya's wrong result was used by Shen in [14] in order to prove the theorem 1.6 in the section 1. Hence, Shen's proof is not valid. A proof of the theorem 1.6 was given by Cheng and Nonaka [3] by making use of the generalized maximum principle due to Omori and Yau (see the section 3).

3. Proofs of The Theorems

In this section, we shall prove the theorem 1.4, 1.5 and 1.6. We denote by h the second fundamental form of M and choose an orthonormal frame field $e_1, e_2, \cdots, e_{n+p}$ on E^{n+p}, restricted to M, so that $e_1, e_2, \cdots, e_n$ are tangent to M and $h_{ji}{}^{n+1} = h_{ji} = \rho_j \delta_{ji}$. Let $\omega_1, \omega_2, \cdots, \omega_{n+p}$ be dual coframe field on E^{n+p}, that is, $\omega_i(e_j) = \delta_{ij}$ and (ω_{ji}) denotes the Riemannian connection form of M and $(\omega_{\alpha\beta})$ is the connection form in the normal bundle $T^{\perp}(M)$ of M. We can prove that the second fundamental form h of M can be given by

$$h = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} h_{ij}{}^{\alpha} \omega_i \otimes \omega_j e_\alpha .$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} h_{ii}{}^{\alpha} e_\alpha .$$

We denote by $K_{ijk\ell}$ the components of the curvature tensor of M and by $K_{\alpha\beta k\ell}$ the components of the normal curvature tensor of M, then the equation of Gauss (3.1) and the equation of Ricci (3.2) are given by

$$K_{ijk\ell} = c(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h_{ik}{}^{\alpha} h_{j\ell}{}^{\alpha} - h_{i\ell}{}^{\alpha} h_{jk}{}^{\alpha}) , \tag{3.1}$$

$$K_{\alpha\beta k\ell} = \sum_{i=1}^{n} (h_{ik}{}^{\alpha} h_{i\ell}{}^{\beta} - h_{i\ell}{}^{\alpha} h_{ik}{}^{\beta}) . \tag{3.2}$$

From the equation of Gauss (3.1), we have

$$r = n(n-1)c + n^2|H|^2 - \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}{}^{\alpha})^2 ,$$

where r denotes the scalar curvature of M.

Defining $h_{ijk}{}^{\alpha}$ by

$$\sum_{k=1}^{n} h_{ijk}{}^{\alpha} \omega_k = dh_{ij}{}^{\alpha} + \sum_{k=1}^{n} h_{jk}{}^{\alpha} \omega_{ik} + \sum_{k=1}^{n} h_{ik}{}^{\alpha} \omega_{jk} , \tag{3.3}$$

then we have the equation of Codazzi as the following.

$$h_{ijk}{}^{\alpha} - h_{ikj}{}^{\alpha} = 0 . \tag{3.4}$$

Morover, by taking exterior differentiation of (3.3) and defining $h_{ijk\ell}{}^{\alpha}$ by

$$\sum_{\ell=1}^{n} h_{ijk\ell}{}^{\alpha} \omega_\ell = dh_{ijk}{}^{\alpha} + \sum_{\ell=1}^{n} h_{\ell jk}{}^{\alpha} \omega_{i\ell} + \sum_{\ell=1}^{n} h_{i\ell k}{}^{\alpha} \omega_{j\ell} + \sum_{\ell=1}^{n} h_{ij\ell}{}^{\alpha} \omega_{k\ell} , \tag{3.5}$$

then we have

$$h_{ijk\ell}{}^{\alpha} - h_{ij\ell k}{}^{\alpha} = \sum_{t=1}^{n} h_{it}{}^{\alpha} K_{tjk\ell} + \sum_{t=1}^{n} h_{tj}{}^{\alpha} K_{tik\ell} - \sum_{\beta=n+1}^{n+p} h_{ij}{}^{\beta} K_{\alpha\beta k\ell}\,. \tag{3.6}$$

The Laplacian $\Delta h_{ij}{}^{\alpha}$ of the second fundamental form h of M is defined by

$$\Delta h_{ij}{}^{\alpha} = \sum_{k=1}^{n} h_{ijkk}{}^{\alpha}\,, \tag{3.7}$$

then, from the equation of Gauss (3.1) and the equation of Codazzi (3.4) and (3.6), we have

$$\Delta h_{ij}{}^{\alpha} = \sum_{k=1}^{n} h_{kkij}{}^{\alpha} + \sum_{k=1}^{n} \left(\sum_{t=1}^{n} h_{kt}{}^{\alpha} K_{tijk} + \sum_{t=1}^{n} h_{ti}{}^{\alpha} K_{tkjk} - \sum_{\beta=n+1}^{n+p} h_{ki}{}^{\beta} K_{\alpha\beta jk} \right). \tag{3.8}$$

Proof of Theorem 1.4. (1). From (3.8), by a direct computation, we have

$$\frac{1}{2}\Delta \langle h \rangle^2 = \sum_{i,j,k=1}^{n} (h_{ijk})^2 + \sum_{i<j} (\rho_i - \rho_j)^2 K_{ij}\,, \tag{3.9}$$

where $\langle h \rangle$ denotes the length of the second fundamental form h of M and $K_{ij} = \rho_i \rho_j$ $(i \neq j)$ is the sectional curvatures of M for the plane section spanned by e_i and e_j. Since the sectional curvatures of M are nonnegative and M is compact, by Hopf's maximum principle, we have

$$\sum_{i,j,k=1}^{n} (h_{ijk})^2 = 0\,, \quad \sum_{i,j=1}^{n} (\rho_i - \rho_j)^2 K_{ij} = 0\,. \tag{3.10}$$

Hence, M is isoparametric and the number of distinct principal curvatures are at most two. Therefore, by a theorem due to Cartan [1], M is a totally umbilical sphere.

(2). From the equation of Gauss, we see that $\langle h \rangle^2$ is constant. Then we have $\Delta \langle h \rangle^2 = 0$. Hence, (3.10) holds. We find that M is isoparametric and the number of distinct principal curvatures are at most two. Therefore, (2) holds.

Proof of Theorem 1.5. (1). In [6], S. Y. Cheng and S. T. Yau introduced a differential operater $\Box$ which is defined by

$$\Box f = \sum_{i,j=1}^{n} (n|H|\delta_{ij} - h_{ij}) \nabla_i \nabla_j f \tag{3.11}$$

for any C^2-function f on M, where $|H|$ denotes the mean curvature of M. We can prove that it is self-adjoint.

When M is compact, we consider $\Box(n|H|)$. Since the scalar curvature r of M is constant, by a direct computation,

$$\Box(n|H|) = \sum_{i,j,k=1}^{n} (h_{ijk})^2 - n^2 \|\mathrm{grad}\, H\|^2 + \sum_{i<j} (\rho_i - \rho_j)^2 K_{ij}\,, \tag{3.12}$$

where $K_{ij} = \rho_i\rho_j \ (i \neq j)$ is the sectional curveture of M for the plane section spanned by e_i and e_j. From the equation of Gauss, we can prove

$$\sum_{i,j,k=1}^{n} (h_{ijk})^2 - n^2 \|\mathrm{grad}\, H\|^2 \geq 0\,. \tag{3.13}$$

Hence, from Stokes formula, we have

$$\int_M \Box(n|H|) = 0\,. \tag{3.14}$$

Then we have $\Box(n|H|) = 0$. Now, Since M is compact, there exists a point $x \in M$ such that the scalar curvature $r > 0$ at x. Then, the scalar curvature $r > 0$ on M since r is constant. Hence, we see that $\Box$ is elliptic. We infer that $|H|$ is constant. Therefore, from the theorem 1.4, we see that M is a totally umbilical sphere $S^n(c)$.

In the following, we consider a complete and connected hypersurface M in E^{n+1}. Since the scalar curvature $r = \sum_{i\neq j} K_{ij}$ is nonnegative constant, $r = 0$ on M if and only if $K_{ij} = 0$ for all distinct i and j. In this case that the scalar curvature r of M is zero, (1) holds from the theorem due to Hartman and Nirenberg [7]. Hence, we may asuume that the scalar curvature r of M is positive constant.

Let ξ be a unit normal vector field on M. Now, M is convex because the sectional curvatures of M are nonnegative. Then, since the Gauss image of a complete convex hypersurface M in E^{n+1} lies in a closed hemisphere, there exists a unit vector $\widetilde{X}$ on E^{n+1} such that $\widetilde{g}(\xi, \widetilde{X}) \geq 0$ on M(for see [17]), where $\widetilde{g}$ is a Euclidean metric of E^{n+1}. By a direct computation, we have

$$\Box\widetilde{g}(\xi, \widetilde{X}) = -\sum_{k,\ell=1}^{n} (n|H|\delta_{k\ell} - h_{k\ell}) \sum_{i=1}^{n} (h_{ki}h_{i\ell})\widetilde{g}(\xi, \widetilde{X})\,. \tag{3.15}$$

Since we see that $(n|H|\delta_{k\ell} - h_{k\ell})\sum_{i=1}^{n}(h_{ki}h_{i\ell}) > 0$ as above, we find that $\Box\widetilde{g}(\xi, \widetilde{X}) \leq 0$ on M and $\widetilde{g}(\xi, \widetilde{X}) \geq 0$ on M. Since the differential operater $\Box$ is elliptic, we inter that $\widetilde{g}(\xi, \widetilde{X})$ is constant. Hence, $\widetilde{g}(\xi, \widetilde{X}) = 0$ on M or $\widetilde{g}(\xi, \widetilde{X}) > 0$ on M. Therefore we conclude that (1) holds from this assertion (see [6] for details).

(2). Let ξ be a unit normal vector field on M. Then we have the well known formula

$$\Delta\xi = -\langle h\rangle^2 \xi\,, \tag{3.16}$$

where $\langle h\rangle$ denotes the length of the second fundamental form h of M. Since the sectional curvatures of M are nonnegative, there exists a unit vector $\widetilde{X}$ on E^{n+1} such that $\widetilde{g}(\xi, \widetilde{X}) \geq 0$ on M as the proof of (1). Then, from (3.16), we can prove that $\widetilde{g}(\xi, \widetilde{X})$ has to be identically zero unless $\langle h\rangle^2$ tends to zero. If the mean curvature $|H|$ of M is nonzero constant, $\langle h\rangle^2$ is bounded from below by a positive constant. Hence, either $|H| = 0$ on M or $\widetilde{g}(\xi, \widetilde{X}) = 0$ on M. Then (2) holds from this assertion (see [5] for details).

Proof of Theorem 1.6. Since the mean curvature vector H of M is parallel, that is, $\omega_{\alpha\beta} = 0$, the mean curvature $|H|$ of M is constant. we now consider the case $|H| = 0$ and case $|H| \neq 0$ separately.

Case(i) : $|H| = 0$ on M

From the assumption of theorem 1.6, we have

$$\langle h\rangle^2 \leq \frac{n^2|H|^2}{n-1} = 0\,,$$

that is, M is an n-dimensional totally geodesic Euclidean space E^n in E^{n+1}. so the theorem holds.

Case(ii) : $|H| \neq 0$ on M

Let $e_1, e_2, \cdots, e_{n+p}$ be an orthonormal frame field on E^{n+p} such that $H = |H|e_{n+1}$. We set

$$|T|^2 = \sum_{\alpha=n+2}^{n+p} \operatorname{trace} A_\alpha^2\,, \tag{3.17}$$

where $\operatorname{trace} A_\alpha = \sum_{i=1}^n h_{ii}{}^\alpha$. Then we find that the nonnegative smooth function $|T|^2$ is defined globally on M. From the following two algebraic lemmas and the fundamental equations about submanifolds, we have

$$\frac{1}{2}\Delta|T|^2 \geq \frac{n-3}{2}|T|^4\,, \tag{3.18}$$

where Δ denotes the Laplacian on M (for see [3]).

Lemma 3.1 (B. Y. Chen and M. Okumura [2]). *Let $a_1, a_2, \cdots, a_n, b$ be $n+1$ real number $(n > 1)$ satisfies the following inequality,*

$$\left(\sum_{i=1}^n a_i\right)^2 \geq (n-1)\sum_{i=1}^n a_i^2 + b \quad (resp. >)\,.$$

Then we have

$$2a_ia_j \geq \frac{b}{n-1} \quad (resp. >)$$

for all distinct i and j.

Lemma 3.2 (A. M. Li and J. M. Li [10]). *Let $A_1, A_2, \cdots, A_p$ be symmetric $(n \times n)$ - matrices. Then we have*

$$\sum_{\alpha,\beta=1}^p \{N(A_\alpha A_\beta - A_\beta A_\alpha) + S_{\alpha\beta}^2\} \leq \left(\sum_{\alpha=1}^p N(A_\alpha)\right)^2$$

and equality holds if and only if one of the following conditions holds :

(i) $A_1 = A_2 = \cdots = A_p = 0$

(ii) only two of $A_1, A_2, \cdots, A_p$ are different from zero. Moreover assuming $A_1 \neq 0$, $A_2 \neq 0$, then $S_1 = S_2$ and there exists an $(n \times n)$-matrix T such that

$${}^tTA_1T = \sqrt{\frac{S_1}{2}}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad {}^tTA_2T = \sqrt{\frac{S_1}{2}}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $S_\alpha = N(A_\alpha) = \operatorname{trace}({}^tA_\alpha A_\alpha)$, $S_{\alpha\beta} = \operatorname{trace}(A_\alpha A_\beta)$.

Condition (1.1) implies that $|T|^2$ is bounded from above by $n^2|H|^2/(n-1)$ and by the lemma 3.1, we can prove that the sectional curvatures of M are nonnegative (for see [2]). Hence, we can apply the generalized maximum principle due to Omori and Yau to function $|T|^2$. Then there exists a sequence $\{x_k\} \subset M$ such that

$$\lim_{k\to\infty} |T|^2(x_k) = \sup |T|^2\,, \tag{3.19}$$

$$\limsup_{k\to\infty} \Delta |T|^2(x_k) \le 0\,. \tag{3.20}$$

From (3.18), (3.19) and (3.20), we see that

$$\limsup_{k\to\infty} \Delta |T|^2(x_k) = (n-3)\big(\sup |T|^2\big)^2 = 0\,. \tag{3.21}$$

Hence, if $n \ge 4$, then we have $|T|^2 = 0$ on M. In case of $n = 3$, from (3.21), we can prove that $|T|^2 = 0$ on M by the lemma 3.1 and the lemma 3.2 (for see [3]). From which, we denote N_1 the normal subbundle spanned by $e_{n+2}, e_{n+3}, \cdots, e_{n+p}$ of the normal bundle $T^{\perp}(M)$ of M, then M is geodesic with respect to N_1. Since the mean curvature vector H of M in E^{n+p} is parallel, we see that N_1 is parallel. Hence, from the theorem 1 in [16], we conclude that M lies $(n+1)$-dimensional totally geodesic Euclidean space E^{n+1}. Then we find that the mean curvature $|H'|(=|H|)$ of M in E^{n+1} is constant and the sectional curvatures of M in E^{n+1} are nonnegative. The theorem 1.6 can be obtain from the theorem 1.5 in section 1.

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