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COMPARISON THEOREMS FOR MANIFOLDS WITH RADIAL CURVATURE BOUNDED BELOW

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1. Introduction

This is the first draft of our recent work [11] on the Bishop-Gromov volume comparison theorem and the Toponogov comparison theorem for manifolds with the radial curvature bounded below. The details will be published some where else. Curvature and topology of Riemannian manifolds is one of the mai n stream in differential geometry. The Toponogov triangle comparison theorem and the Bishop-Gromov v olume comparison theorem for concentric metric balls on a complete Riemannia n n-manifold M plays an important role for the investigation of curvatur e and topology of Riemannian manifolds. The volumes of concentric metric bal ls on M are usually compared with those on the complete simply connected s pace forms of constant sectional curvature. It is the purpose of this articl e to establish the Bishop-Gromov volume comparison theorem between M and t he model spaces, where our model spaces do not have metrics of constant sect ional curvature but their curvature may change sign. Furthermore we want to establish the Toponogov triangle comparison theorem for such manifolds. We also discuss the maximal diameter theorems due to Toponog ov [17] and Cheng [8] for a certain class of manifolds as stated below.

Here we discuss connected, complete and smooth Riemannian n-manifolds with out boundary. Geodesics are parametrized by arclength unless otherwise is s tated.

For the construction of our model manifold, we first choose a consta nt $0 < \ell \leq \infty$ and a smooth function $K : [0, \ell) \to \mathbf{R}$ which are associated with the model manifold M^* in such a way that the metric is rotationaly symmetric with respect to p^*. A complete Riemannian n-manifold M^* with the base point at $p^* \in M^*$ is said to have *the radial sectional curvature* $K : [0, \ell) \to \mathbf{R}$ at p^* if and only if the following are satisfied:

1. The tangential cut locus $C_{p^*} \subset M^*_{p^*}$ at p^* is the sphere $\mathbf{S}^{n-1}(\ell)$ with radius ℓ if $\ell < \infty$ and $C_{p^*} = \emptyset$ if $\ell = \infty$
2. Along every geodesic $\gamma^* : [0, \ell) \to M^*$ emanating form $p^* \in M^*$ the sectional curvature satisfies

$$K_{M^*}(\dot\gamma^*(t), X) = K(t),\ \forall t \in (0, \ell],\ \forall X \in M^*_{\gamma(t)},\ X \perp \dot\gamma(t), \tag{1.1}$$

3. If $\ell < \infty$, then $\gamma^*(\ell)$ is the first conjugate point to p^* along γ

When M^* is simply connected and $\ell < \infty$, (1) implies that every geodesic γ^* emanating from p^* has its first conjugate point to p^* at length ℓ with the multiplicity λ being independe nt of the choice of initial direction and $\lambda = 1, 3, 7$ or $\lambda = n-1$. Thus the first conjugate locus to p^* coincides with $C(p^*)$. If $\ell = \infty$, then M^* is diffeomorphic to $\mathbf{R}^n$.

When M^* is not simply connected, $\ell < \infty$ is automatically satisfied and the first conjugate point to p^* along every geodesic emanating from p^* appears at length 2ℓ (which is p^* itself) with the multiplici ty $\lambda = n-1$. The cut locus $C(p^*)$ to p^* is a compact hypersurface diffeomorphic to the quotient space of a fixed point free $\mathbf{Z}_2$ action of $\mathbf{S}^{n-1}$. Here the action sends $u \in \mathbf{S}^{n-1}$ to the vector $-\dot\gamma_u(2\ell)$, and $\gamma_u : [0, \ell] \to M$ is the geodesic with $u = \dot\gamma_u(0)$.

When (1) and (3) are satisfied, $\lambda = n-1$ and the cut locus $C(p^*)$ to p^* coincides with the first conjugate locus which is a single point, say, q^*. Thus $C(q^*) = \{p^*\}$ and hence M^* has the radial sectional curvature $K^{-1}(t) := K(\ell - t)$ at q^*. We then see that M^* is diffeomorphic to $\mathbf{S}^n$. The details will be reffered to Nakagawa-Shiohama [14] and Besse [3].

We say that *a complete Riemannian n-manifold M has the radial Ricci curvature at* $p \in M$ *bounded below by* $(n-1)K : [0, \ell] \to \mathbf{R}$ if and only if there exists the model M^* with the radial sectional curvature K at p^* such that

$$\mathrm{Ric_M}(\dot\gamma(\mathrm{t})) \geq (\mathrm{n}-1)\mathrm{K}(\mathrm{t}), \quad \forall \mathrm{t} \in [0, \beta]$$

for every minimizing geodesic $\gamma : [0, \beta] \to M$ with $\gamma(0) = p$.

We say that *the radial sectional curvature at* $p \in M$ *of a complete Riemannian n-manifold M is bounded below by* $K : [0, \ell) \to \mathbf{R}$ if and only if there exists the model M^* with the radial sectional curvature K at p^* such that

$$K_M(\dot\gamma(t), X) \geq K(t),\ \forall t \in [0, \beta],\ \forall X \in M_{\gamma(t)},\ \langle X, \dot\gamma(t)\rangle = 0$$

for every minimizing geodesic $\gamma : [0, \beta] \to M$ with $\gamma(0) = p$.

Recently we have proved in [11] the Toponogov comparison theorem for manifolds with radial sectional curvature bounded below by $K : [0, \ell) \to \mathbf{R}$. Here $\ell \leq \infty$. We first note that M is not assumed to be compact.

Theorem 1.1. Let M be a complete Riemannian n-manifold with base point at p. Assume that the radial sectional curvature of M at p is bounded below by $K : [0, \ell) \to \mathbf{R}$. Here $\ell \leq \infty$ and K is the radial sectional curvature of the model M^* with base point at p^*. For every geodesic triangle $\Delta = \Delta(p\,x\,y) \subset M$ there exists the correspon ding triangle $\Delta^* = \Delta(p^* x^* y^*) \subset M^*$ such that

$$d(p, x) = d(p^*, x^*),\ d(p, y) = d(p^*, y^*),\ d(x, y) = d(x^*, y^*) \tag{1.2}$$

and such that

$$\angle(p\,x\,y) \geq \angle(p^* x^* y^*), \quad \angle(p\,y\,x) \geq \angle(p^* y^* x^*), \quad \angle(x\,p\,y) \geq \angle(x^* p^* y^*) \tag{1.3}$$

The equality case in the above result is also discussed. Namely we have the

Corrolary to Theorem 1.1 If $\angle(p\,x\,y) = \angle(p^*x^*y^*)$, then there exists a piece of totally geodesic surface bounded by $\Delta(p\,x\,y)$ which is isometric to the interior of the corresponding triangle $\Delta^{\cdot}(p^*x^*y^*) \subset M^*$. The same is true for other angles in (1.3).

Remark 1. The angle comparisons at the vertices x, y have already been established in [13] for the case of K being constant. We have first proved the angle comparison at the base point p. It should be noted that the equality case has not been discussed before.

Remark 2. When K is a positive constant, the existence of a geodesic triangle $\Delta(p\,x\,y) \subset M$ with its circumference $L(\Delta(p\,x\,y)) = 2\ell$ implies M being isometric to the standard n-sphere $\mathbf{S}^n(K)$ of constant curvature K, (and $\ell = \pi/\sqrt{K}$). However this is not our case. Let M have the radial sectional curvature bounded below by $K : [0, \ell) \to \mathbf{R}$ and $\ell < \infty$. Then the corresponding model M^* may admit a geodesic triangle $\Delta^*(p^*x^*y^*)$ whose circumference is 2ℓ and has the property that the edge x^*y^* does not pass through the point q^*, where $q^* \in M^*$ is the antipodal point $d(p^*, q^*) = \ell$. This means that the existence of such a geodesic triangle $\Delta(p\,x\,y)$ with $L(\Delta)) = \ell$ will not imply $\mathrm{diam(M)} = \ell$.

The following problem seems to be interesting.

Problem 1.1. Let the radial sectional curvature at p of M be bounded below by $K : [0, \ell) \to \mathbf{R}$ for $\ell < \infty$. Then is M isometric to M^* if there exists a geodesic triangle $\Delta(p\,x\,y) \subset M$ such that $L(\Delta(p\,x\,y)) = \ell$?

We next discuss the Bishop-Gromov volume comparison theorem for manifolds wi th bounded radial Ricci curvature from below. Here we do not assume the compactness of M.

Theorem 1.2. Let M be a connected, complete Riemannian n-manifold with a base point at $p \in M$. Assume that the radial Ricci curvature of M at p is bounded below by $(n-1)K : [0, \ell) \to \mathbf{R}$. Then the function

$$\varphi(t) := \frac{\mathrm{vol(B(p,t))}}{\mathrm{vol(B(p^*,t))}}, \quad \forall t \in (0, \ell) \tag{1.4}$$

is monotone non-increasing in $t \in [0, \ell]$. Moreover $\varphi(t)$ is constant if and only if M is isometric to M^*.

We next discuss the maximal diameter theorem corresponding to the above resu lts. Assume that $\ell < \infty$ and that the radial Ricci curvature at $p \in M$ of M is bounded below by $(n-1)K : [0, \ell] \to \mathbf{R}$. The first conjugate point to $p \in M$ along every geodesic $\gamma : [0, \ell] \to M$ with $\gamma(0) = p$ appears in $(0, \ell]$ and hence the tangen tial cut locus $C_p \subset M_p$ to p is contained entirely in the closed ℓ-ball centered at o. The maximal domain $U_p \subset M_p$ on which $\exp_p$ is an embedding has the property that

$$\partial U_p = C_p, \qquad U_p \subset B(o, \ell)$$

Setting $\delta(p) := \sup\{ d(p,x) : x \in M\}$ and $\delta(p^*) := \sup\{ d(p^*,x^*) : x^* \in M^*\} = d(p^*,q^*)$, we state the maximal diameter theorem in our situation as follows.

Theorem 1.3. Let M be a comapct Riemannian n-manifold with base point at p. Assume that the radial Ricci curvature at p is bounded below by $(n-1)K : [0,\ell] \to \mathbf{R}$. We then have

$$\delta(p) \leq \delta(p^*) \tag{1.5}$$

Moreover, assume that there exists a point $q \in M$ such that

$$\delta(p) = d(p,q) = \delta(p^*) = \ell \tag{1.6}$$

and that the radial Ricci curvature of M at q is bounded below by $(n-1)K^{-1}$. Then $C(p) = \{q\}$ and $C(q) = \{p\}$ and M is isometric to M^*.

The following result is the maximal diameter theorem corresponding to the Toponogov theorem.

Theorem 1.4. Assume that the radial sectional curvature at p of a compact Riemannian n-man ifold M is bounded below by $K : [0,\ell] \to \mathbf{R}$. Here K is the radial sectional curvature of a model M^*. If the diameter diam(M) of M satisfies

$$\mathrm{diam(M)} = \ell$$

then M is isometric to M^*.

By relaxing the above result, we may obtain a new sphere theorem which contain the manifolds with $K_M \geq 1$ and diam(M) being sufficiently close to π. The following problem seems to be very interesting.

Problem 1.2. Given a model manifold M^* with the radial sectional curvature $K : [0,\ell] \to \mathbf{R}$ at p^*, does there exist an $\varepsilon = \varepsilon(n,\ell,K)$ with the following properties ? Let M be a complete n-manifold with base point at p such that

1. the radial Ricci curvature at p is bounded below by $(n-1)K$
2. there is a point $q \in M$ with $\delta(p) = d(p,q)$ such that the radial Ricci curvature at q is bounded below by $(n-1)K^{-1}$

If

$$\max\{ \delta(q^*) - \delta(q), \delta(p^*) - \delta(p)\} < \varepsilon, \tag{1.7}$$

then M is diffeomorphic to $\mathbf{S}^n$.

Here we only prove Theorems 1.2 and 1.3. The basic tools in Riemannain geome try are referred to Besse [3], Cheeger-Ebin [4], Bishop-Crittenden [6] and Sakai [7].

2. Definitions and Notations

Let M be a complete Riemannian n-manifold with a base point at $p \in M$. Let $U_p \subset M_p$ be the largest domain containing the origi n on which the exponential map $\exp_p|_{U_p} : U_p \to M$ is an embedding. Then the boundary ∂U_p of U_p (if is is nonempty) is then the tangential cut locus C_p to p and

$$C(p) := \exp_p(C_p)$$

is the cut locus to p. Let $\mathbf{S}_p \subset M_p$ be the unit hypersphere centered at the origin. For a vector $u \in \mathbf{S}_p$ we denote by $\gamma_u : [0, \infty) \to M$ the unit speed geodesic tangent to u. Let $\rho : \mathbf{S}_p \to \mathbf{R}^+$ be defined by

$$\rho(u)u \in C_p \qquad \text{or} \qquad \gamma_u(\rho(u)) \in C(p) \tag{2.1}$$

For a domain $\Omega \subset \mathbf{S}_p$ and for a positive number a, we define $\Omega_a \subset M_p$ by

$$\Omega_a := \{tu \in M_p \ : \ u \in \Omega, \ 0 < t \leq a\} \tag{2.2}$$

If $\Theta = (\theta^1, \cdots, \theta^{n-1}) \in \Omega$ is a local coordinates on Ω, then the metric of M is expressed in terms of the geodesic polar coordinates as

$$ds^2 = dt^2 + \sum_{\alpha,\beta=1}^{n-1} h_{\alpha\beta}(\Theta, t) d\theta^\alpha d\theta^\beta$$

If we set

$$\rho(\Omega) := \inf_{\Omega} \rho, \tag{2.3}$$

then the volume of $\exp_p(\Omega_a)$ for $a \in (0, \rho(\Omega)]$ is expressed as

$$\mathrm{vol}(\exp_p(\Omega_a)) = \int_\Omega \int_0^a \sqrt{\det(h_{\alpha\beta})}\, dt \wedge d\Theta$$

For each $t \in (0, \rho(\Omega)]$ we set

$$V_\Omega(t) := \mathrm{vol}(\exp_p(\Omega_t)) \tag{2.4}$$

From now on $M_p, M^*_{p^*}$ and $\mathbf{R}^n$ are naturally identified and hence $\mathbf{S}_p = \mathbf{S}_{p^*} = \mathbf{S}^{n-1} \subset \mathbf{R}^n$, where there is no confusion.

The metric of a model M^* with the radial sectional curvature at a base point p^* is expressed, by using the geodesic polar coordinates arou nd p^* as follows.

$$(ds^*)^2 = dt^2 + f^2(t)\, d\Theta^2$$

where $d\Theta^2$ is the canonical metric on the standard unit $n-1$ sphere $\mathbf{S}^{n-1}$ and $f : [0, \ell) \to \mathbf{R}$ satisfies the Jacobi equation

$$f'' + Kf = 0, \quad f(0) = 0, \quad f'(0) = 1$$

Moreover if $\ell < \infty$, then

$$\lim_{t \to \ell} f'(t) = -1, \quad \lim_{t \to \ell} f(t) = 0$$

The above condition for the radial sectional curvature implies that if $\ell < \infty$, then C_{p^*} is the standard ℓ-sphere and $U_{p^*} = B(p^*, \ell) \subset M_{p^*}$ and $\gamma^*_u : [0, \ell] \to M^*$ for every $u \in \mathbf{S}^{n-1}$ is minimizing and $\gamma_u^*(\ell)$ is the first conjugate point to p^* along γ_u^* with its multiplicity $n-1$. Therefore $C(p^*)$ consists of a single point

$$C(p^*) = \{q^*\} \tag{2.5}$$

We now want to discuss the curvature assumptions for manifolds. We say that *the radial Ricci curvature of M at p is bounded below by K* if for every $u \in \mathbf{S}^{n-1}$ and for every $t \in (0, \ell)$,

$$\mathrm{Ric_M}(\dot\gamma_u(t)) \geq (n-1)K(t) \tag{2.6}$$

Note that the above inequality implies that

$$\rho(u) \leq \ell, \qquad \forall u \in \mathbf{S}^{n-1}$$

Therefore we have

$$U_p \subset U_{p^*} = B(o, \ell)$$

We denote by $\nabla_D X$ the covariant differential of a vector field X along γ with respect to the arclength parameter $D = d/dt$ of γ. Let Y^* be a Jacobi field along $\gamma^* : [0, \ell] \to M^*$ such that $Y^*(0) = 0$, $\nabla_D Y^*(0) = v \neq 0$ for $\langle v, \dot\gamma^*(0)\rangle = 0$. Then the metric property implies that $E^*(t) := Y^*(t)/||Y^*(t)||$ is unit parallel field along γ^* generated by v and

$$Y^*(t) = f(t)E^*(t), \quad t \in [0, \ell] \tag{2.7}$$

Example 2.1. Let $A, B \gg 1$ be fixed numbers. Let M^* with base point at p^* have the radial sectional curvature $K : [0, \ell] \to [1, \infty)$ such that

$$K = A, \quad \text{near } 0, \quad K = B, \quad \text{near } \ell \tag{2.8}$$

$$K = 1, \quad \text{outside a neighborhood of } \{0\} \cup \{\ell\} \tag{2.9}$$

We then observe that the diameter of M^* is close to π and $\mathrm{vol(M^*)} \ll 1$.

Example 2.2. A hypersurface M^* of revolution in $\mathbf{R}^{n+1}$ obtained by rotating a proper curve around the x^{n+1}-axis and the porifile curve intersecting orthogonally with the rotation axis exactly at two points p^* and q^*. Each point of the intersection is the base point of this model. M^* has the radial sectional curvature $K : [0, \ell] \to \mathbf{R}$ at p^*, where $\ell < \infty$.

3. Outline of the Proofs of Theorems

For an arbitrary fixed $u \in \mathbf{S}^{n-1}$ we choose a local coordinates $\Theta = (\theta^1, \cdots, \theta^{n-1})$ on a domain Ω containing u such that

$$\langle \frac{\partial}{\partial\theta^i}(u), \frac{\partial}{\partial\theta^j}(u)\rangle = \delta_{ij} \tag{3.1}$$

We denote by Y_i^*, $\quad i = 1, \cdots, n-1$ the Jacobi field along γ_u^* such that

$$\langle Y_i^*, \dot\gamma_u^*\rangle = 0, \; Y_i^*(0) = 0, \; \dot Y_i^*(0) = \frac{\partial}{\partial\theta^i}(u) \tag{3.2}$$

Clearly $Y_i^*/||Y_u^*||$ is parallel along γ_u^* and $\langle Y_i^*, Y_j^*\rangle = 0$ for all $i \neq j$. Therefore we observe that

$$Y_i^*(t) = f(t)E_i^*, \qquad 0 \le t < \ell, \tag{3.3}$$

where E_i^* for every $i = 1, \cdots, n-1$ is the unit parallel field along γ_u^* such that

$$\frac{\partial}{\partial\theta^i}(u) = E_i^*(0)$$

Let Y_i for $i = 1, \cdots, n-1$ be the corresponding Jacobi field along $\gamma_u : [0, \rho(u)] \to M$ such that

$$Y_i(0) = 0, \quad Y_i'(0) = \frac{\partial}{\partial\theta^i}(u) \tag{3.4}$$

We then observe that the function $\varphi_\Omega : [0, \infty) \to [0, 1]$ defined by

$$\varphi_\Omega(t) := \frac{V_\Omega(t)}{V_\Omega^*(t)}, \qquad 0 \le t \le \rho(\Omega) \tag{3.5}$$

and also

$$\varphi_\Omega(t) := \frac{V_\Omega(\rho(\Omega))}{V_\Omega^*(t)}, \qquad \rho(\Omega) \le t < \ell \tag{3.6}$$

is continuous monotone non-increasing in $[0, \infty)$ and differentiable on $(0, \rho(\Omega)) \cup (\rho(\Omega), \infty)$ and

$$\lim_{t\uparrow\rho(\Omega)} \varphi_\Omega'(t) \ge \lim_{t\downarrow\rho(\Omega)} p_\Omega'(t)$$

Moreover we observe that $\varphi_\Omega(0) = 1$. Here $\varphi_\Omega(\ell) = 1$ holds if and only if

$$\rho(\Omega) = \ell,\ Y_i(t) = f(t)E_i(t),\ K_M(Y_i(t), \dot\gamma_u(t)) = K(t) \tag{3.7}$$

for all $i = 1, \cdots, n-1$ and for all $t \in [0, \ell)$. Therefore in this case $\exp_p(\Omega_\ell)$ is isometric to $\exp_{p^*}(\Omega_\ell)$. This proves Theorem 1.1.

We are now in position to prove Theorem 1.4. The following Lemma is usefull for the proof of Theorem 1.4.

Lemma 3.1. In addition to the assumptions as in Theorem 1.4, if $q \in M$ satisfies $d(q,p) = \delta(p) = \delta(p^*)$ and if the radial Ricci curvature at q is bounded below by $(n-1)K^{-1} := (n-1)K(\ell - t)$, we then have

$$C(p) = \{q\}, \qquad C(q) = \{p\} \tag{3.8}$$

Proof of Lemma 3.1.

The crucial point is to show that

$$d(p, x) + d(x, q) = \delta(p), \quad \forall x \in M$$

Once the above relation has been established, we see that every geodesic segmment emanating from p (or q) with length ℓ reaches to q (or p). Thus we have $C(q) = \{p\}$ and $C(p) = \{q\}$.

Suppose that there exists a point $x \in M$ such that

$$d(p,x) + d(x,q) > \delta(p)$$

This inequality is equivalent to state that $M \setminus B(p,t) \cup B(q, \ell - t)$ for every $t \in (0,\ell)$ has non-empty i nterior, and hence

$$\mathrm{vol(B(p,t))} + \mathrm{vol(B(q,\ell - t))} < \mathrm{vol(M)}$$

Let $\varphi_p, \varphi_q : [0,\ell] \to [0,1]$ be defined by

$$\varphi_p(t) := \frac{\mathrm{vol(B(p,t))}}{\mathrm{vol(B(p^*,t))}}, \quad \varphi_q(t) := \frac{\mathrm{vol(B(q,t))}}{\mathrm{vol(B(q^*,t))}} \tag{3.9}$$

There exists a number $\alpha \in (0,\ell)$ such that

$$\mathrm{vol(B(p^*,\alpha))} = \mathrm{vol(B(q^*,\ell - \alpha))} = \frac{1}{2}\mathrm{vol(M^*)}$$

Theorem 1.2 then implies that

$$\varphi_p(\alpha) = \frac{\mathrm{vol(B(p,\alpha))}}{\mathrm{vol(M^*)}/2} \geq \varphi_p(\ell) = \frac{\mathrm{vol(M)}}{\mathrm{vol(M^*)}} \tag{3.10}$$

$$\varphi_q(\ell - \alpha) = \frac{\mathrm{vol(B(q,\ell - \alpha))}}{\mathrm{vol(M^*)}/2} \geq \varphi_q(\ell) = \frac{\mathrm{vol(M)}}{vol(M^*)} \tag{3.11}$$

Therefore we obtain

$$\mathrm{vol(B(p,\alpha))} \geq \frac{1}{2}\mathrm{vol(M)}, \qquad \mathrm{vol(B(q,\ell - \alpha))} \geq \frac{1}{2}\mathrm{vol(M)}$$

This is a contradiction.

□

Outline of the Proof of Theorem 1.2 For the proof of Theorem 1.2 the following Lemmas are useful. Let M be a compact n-manifold with base point at $p \in M$. Let the radial sectional curvatu re at p be bounded below by $K : [0,\ell] \to \mathbf{R}$. Let $x, y \neq p$ be such that

$$d(p,x) + d(x,y) + d(y,p) < 2\ell$$

and $\beta, \gamma : [0,1] \to M$ abe minimizing geodesics with $\beta(0) = \gamma(0) = p$, $\beta(1) = y$ and $\gamma(1) = x$. Take a minimizing geodesic $\alpha : [0,1] \to M$ such that $\alpha(0) = x$, $\alpha(1) = y$. Choose a partition $0 = t_0 < t_1 \cdots < t_k = 1$ of $[0,1]$ and minimizing geodesics $\gamma_0^{\pm}, \cdots, \gamma_k^{\pm} : [0,1] \to M$ such that for each $i = 1, \cdots, k-1$, the geodesic triangle $\Delta_i := \Delta_i(\gamma_{i-1}^{+}\, \gamma_i^{-}\, \alpha|[t_{i-1}, t_i])$ forms a narrow triangle and

$$\gamma_0^{-} := \gamma, \qquad \gamma_k^{+} := \beta$$

Here by a narrow triangle we mean that there exists a constant $c > 0$ such that $d(\gamma_{i-1}^{+}(s), \gamma_i^{-}(s)) < c|t_i - t_{i-1}|$ for all $s \in [0,1]$. From the lemma on limit angles (see Toponogov [17]) we observe that

$$\angle(pxy) \geq \angle(-\dot{\gamma}_0^{+}(1), \dot{\alpha}(0)), \quad \angle(pyx) \geq \angle(-\dot{\gamma}_k^{-}(1), -\dot{\alpha}(1))$$

We next take a sufficiently small $\varepsilon > 0$ and consider a rotationally symmetr ic metric $ds^*_\varepsilon \cdot ds^*_\varepsilon$ defined on the ℓ-ball

$$(ds^*_\varepsilon)^2 = dt^2 + f^2_\varepsilon(t)d\Theta^2, \quad 0 < t < \ell$$

where $f_\varepsilon(t)$ is the solution of the Jacobi equation

$$y'' + (K - \varepsilon)y = 0, \quad y(0) = 0, \; y'(0) = 1 \tag{3.12}$$

Once such a small ε has been fexed, we choose a partition of $[0, 1]$ such that for each Δ_i the Berger comparison theorem applies to both of the edges γ^+_{i-1} and γ^-_i. Clearly we have the correspond ing geodesic triangle Δ^*_i as well as $\Delta^*(p\,x\,\alpha(t_i))$. Therefore the angle comparison holds for the angles opposite these two edges. Further, the Alexandrov convexity holds for angles at x^* and y^*. Namely, if $\theta(t)$ is the angle at x^* of the triangle $\Delta^*(p\,x\,\alpha(t))$, then $\theta(0) = \angle(-\dot\gamma^+_0(1), \dot\alpha(0))$ and θ is monotone inincreasi ng. The same is true for the angles at y^*. Thus we have proved the following

Lemma 3.2. Assume that the circumference of $\Delta = \Delta(p\,x\,y)$ is less than 2ℓ. If there exists for a geodesic triangle $\Delta = \Delta(p\,x\,y)$ the corresponding triangle $\Delta^* = \Delta(p^*\,x^*\,y^*)$ on M^* such that

$$d(p^*, x^*) = d(p, x), \; d(p^*, y^*) = d(p, y), \; d(x^*, y^*) = d(x, y)$$

then

$$\angle(p\,x\,y) \geq \angle(p^*\,x^*\,y^*), \quad \angle(p\,y\,x) \geq \angle(p^*\,y^*\,x^*) \tag{3.13}$$

Here equality holds if and only if there exists a totally geodsic surface bounded by $\gamma^+_0[0, 1] \cup \gamma^-_k[0, 1] \cup \alpha[0, 1]$ which is isometric to $\Delta(p^*\,x^*\,y^*)$.

Lemma 3.3. Let $x, y \in M$ be distinct from p and $\alpha : [0, 1] \to M$ a minimizing geodesic such that $\alpha(0) = x, \alpha(1) = y$ and such that

$$d(p, \alpha(t)) < \ell, \quad \forall t \in [0, 1]$$

Then there wxists the corresponding geodesic triangle $\Delta^* = \Delta(p^*\,x^*\,y^*)$ on M^*. Moreover, the circumference $L(\Delta)$ of Δ does not exceed 2ℓ. If $L(\Delta) = 2\ell$, then there exists a totally geodesic surface bounded by Δ which is isometric to the corresponding triangle Δ^*.

Proof of Lemma Suppose that Δ does not admit the corresponding geodesic triangle on M^*. We first observe that the set of all such pair of points $\{x, y\}$ that has no corresponding triangle on M^* forms an open set in $M \times M$. Thus we may assume without loss of generality that

$$x \notin C(p)$$

Then there exists a number $t^* \in (0, 1)$ such that $\Delta(p\,x\,\alpha(t))$ for every $t \in (0, t^*)$ has its corresponding triangle $\Delta(p^*\,x^*\,\alpha^*(t))$ in M^* and such that $\Delta(p\,x\,\alpha(t'))$ for all $t' > t^*$ sufficiently close to t^* does not have its corresponding triangle on M^*. Because $z^* := \alpha(t^*) \neq q^*$, the gradient vector field of the distance function to p^* at z^* is the unit vector tangent to the minimizing goedesic joining p^* to z^*. Also the gradient vector field to the distance function to x at $z := \alpha(t^*)$ is $\dot\alpha(t^*)/||\dot\alpha(t^*)||$. It follows from what is supposed that the gradient vector field to the distance function to x^* at z^* is not a unit vector. Therefore we see that z^* belongs to the cut locus to x^*.

If $z^* \in C(p^*)$ and if $\Theta(x^*) \neq \Theta(z^*)$, then $d(x^*, \cdot)$ is non-critical at z^*. This fact means that if the distance function to x^* is restricted on the metric sphere $\mathcal{S} := \{w^* \in M^*; d(x^*, w^*) = d(p^*, x^*)\}$, then there is a small $\eta > 0$ such that $\mathcal{S} \cap B(z^*, \eta)$ contains a point atwhich $d(x^*, *)$ can take any value sufficiently close to $d(x^*, z^*)$. Therefore we find a corresponding geodesic triangle $\Delta^*(p\, x\, \alpha(t))$ for all $t \in (t^* - \delta, t^* + \delta)$. This is a contradiction.

The above argument implies that $\Theta(x^*) = -\Theta(z^*)$ and hence $d(x^*, *)|_{\mathcal{S}}$ takes maximum at z^*. The corresponding triangle $\Delta(p^*\, x^*\, z^*)$ exists and has the property that the angle at p^* is π. Then the Toponogov theorem recently improved by Machigashira [12] then implies that for every $t \in (0, 1)$

$$\angle(x\, p\, y) = \angle(x^*\, p^*\, z^*) = \pi$$

and

$$\angle(x\, p\, \alpha(t)) \geq \angle(x^*\, p^*\, \alpha^*(t)),\ \angle(z\, p\, \alpha(t)) \geq \angle(z^*\, p^*\, \alpha^*(t))$$
$$\angle(x^*\, p^*\, \alpha^*(t)) + \angle(z^*\, p^*\, \alpha^*(t)) = \pi$$
$$\angle(x\, p\, \alpha(t)) + \angle(z\, p\, \alpha(t)) = \angle(x\, p\, y) = \pi$$

In particular, there exists a totally geodesic surface bounded by $\Delta(p\, x\, z)$ which is isometric to the corresponding $\Delta(p^*\, x^*\, z^*)$.

On the other hand, we find a minimizing geodesic $\alpha_1 : [0, 1] \to M$ with $\alpha_1(0) = x, \alpha_1(1) = z$ and $\dot{\alpha}_1(0) \neq \dot{\alpha}(0)$. Thus we see that $z \in C(x)$ and hence z cannot be an interior point of α. Therefore we conclude $t^* = 1$.

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