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ON ANAND CONJECTURE

YuXin Dong

Institute of Mathematics, Fudan University,
Shanghai 200433 , P.R. China

1. Introduction

Unitons are harmonic maps $\varphi : S^2 \to U(N)$, which are important object in mathematics and physics. The construction of unitons was investigated from different perspectives by Uhlenbeck, Valli, Segal, Wood, Guest-Ohnita, Gu-Hu, Anand, and Burstall-Guest. For recent progress in this area see [12], [2], [11], [5] and their references.

In [1,2], Anand showed that unitons are equivalent to holomorphic 'uniton bundles', with energy corresponding to the bundles' second Chern class. Using monad representation, he obtained a simple formula for the unitons and proved that 2-untions have normalized energy at least four. This bound is sharp. In a personal communication with the author, Anand proposed the following conjecture:

Conjecture (Finite Gap): m-unitons have energy at least m^2.

The purpose of this note is to give a positive answer to this conjecture. It is known that harmonic maps of simply connected Riemann surfaces into $U(N)$ can be lifted into holomorphic maps into the based loop group $\Omega U(N)$, which are called extended harmonic maps. Unitons have algebraic extended harmonic maps $\Phi : S^2 \to \Omega_{\mathrm{alg}} U(N)$ (see [20] for details). Our idea is to deform a given extended harmonic map into a simpler one, which allows us to confirm the above conjecture of Anand.

2. Harmonic maps and extended harmonic maps

Let φ : $R^2 \to U(N)$ be a smooth map. Define $A = \frac{1}{2}\varphi^{-1}d\varphi$, and decompose it as follows:

$$A = A_z dz + A_{\overline{z}} d\overline{z} \quad \text{with } (A_z)^* = -A_{\overline{z}},$$

where $z = x + iy \in R^2$. Then A satisfies the Maurer-Cartan equation:

$$\overline{\partial} A_z - \partial A_{\overline{z}} + 2[A_{\overline{z}}, A_z] = 0. \tag{1}$$

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In the case of maps into a matrix group, with the standard (left-invariant) metric, the energy takes the form

$$E(\varphi) = \frac{1}{2}\int_{R^2}(|\varphi^{-1}\frac{\partial\varphi}{\partial x}|^2 + |\varphi^{-1}\frac{\partial\varphi}{\partial y}|^2)dx\wedge dy$$

$$= -4i\int trA_z A_{\overline{z}}dzd\overline{z}. \tag{2}$$

It is known that φ is harmonic if and only if A is co-closed; i.e.,

$$d^*A = \frac{\partial}{\partial\overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}} = 0. \tag{3}$$

Untions are harmonic maps from S^2 to $U(N)$. From [19], we know that harmonic maps from $R^2 \to U(N)$ extend to S^2 iff they have finite energy, and that such maps are always smooth. In the following, we will use this fact and work in terms of coordinates x and y on R^2.

The Lax pair of the harmonic maps is ([20]):

$$\begin{cases} \overline{\partial}\Phi = (1-\lambda)\Phi A_{\overline{z}}, \\ \partial\Phi = (1-\lambda^{-1})\Phi A_z, \end{cases} \tag{4}$$

whose integrability conditions are (1) and (3). A nondegenerated $N\times N$ matrix solution $\Phi(\lambda) := \Phi_\lambda$ to (4) is called extended solution or extended harmonic map. It can be normalized so that

$$\Phi_1 = I, \qquad \Phi_\lambda\Phi^*_{\sigma(\lambda)} = I, \tag{5}$$

where $\sigma(\lambda) = (\overline{\lambda})^{-1}$. Moreover, we have $\Phi_{-1} = Q\varphi$ for some constant $Q\in U(N)$.

Let $\Omega U(N) = \{\gamma : S^1 \to U(N) | \gamma \text{ is smooth}, \gamma(1) = I\}$ be the group of smooth based loops in $U(N)$. By (5), we can regard an extended harmonic map as a smooth map into $\Omega U(N)$. Denote by $\Omega_{\mathrm{alg}}U(N)$ the subgroup of $\Omega U(N)$ consisting of algebraic loops. One of the main results of Uhlenbeck [20] concerning harmonic maps $\varphi : S^2 \to U(N)$ is :

Theorem 2.1[20]. *Let $\varphi : S^2 \to U(N)$ be harmonic. Then there exists an algebraic extended harmonic map $\Phi : M \to \Omega_{\mathrm{alg}}U_n$ of φ.*

It is easy to see that the algebraic extended harmonic map Φ can be expressed as the following type after multiplication on the left by a suitable constant loop $\gamma\in\Omega U(N)$:

$$\Phi = \sum_{\alpha=0}^{m}\lambda^\alpha T_\alpha. \tag{6}$$

According to [20], we call the harmonic map $\varphi : S^2 \to U(N)$ and the extended harmonic map $\Phi : S^2 \to \Omega U(N)$ a uniton and an extended uniton respectively. We say that φ or Φ has finite uniton number m. The minimal such number m is called the minimal uniton number of φ or Φ. It was shown in [20] that an extended uniton Φ can be normalized further so that

$$V_0(\Phi) := Span\{Im_{z\in S^2}T_0(z)\} = C^N, \tag{7}$$

and if an extended uniton Φ satisfies the condition (7) then m is just the minimal uniton number. An extended uniton Φ with the property (7) will be called the standard extended uniton of φ.

lemma 2.2. *Let $\Phi_\lambda = \sum_{\alpha=0}^{m} \lambda^\alpha T_\alpha : S^2 \to \Omega U(N)$ be an extended uniton. Then $ImT_0 : S^2 \to G_k(C^N)$ is a holomorphic map , where $rank(ImT_0) = k$. Furthermore, Φ_λ is standard if and only if ImT_0 is a linearly full holomorphic curve in $G_k(C^N)$. Similarly, ImT_m is an anti-holomorphic map.*

Proof. We have two unitary connections on the trivial bundle $\underline{C}^N$: $d_A = d + A$ and the trivial connection d. They determine their corresponding Koszul-Malgrange holomorphic structures on $\underline{C}^N$ by [15]. From (4), we have

$$\overline{\partial}T_0 = T_0A_{\overline{z}}$$

which means that T_0 is a global holomorphic section of $Hom[(\underline{C}^N, d_A) , (\underline{C}^N, d)]$. Thus $\underline{Im}T_0$ is a holomorphic subbundle of $(\underline{C}^N, d)$, i.e., ImT_0 is a holomorphic map to $G_k(C^N)$. Φ_λ is standard if and only if it satisfies (7), which means that ImT_0 is a linearly full holomorphic map. By (4), we have also

$$\partial T_m = T_mA_z$$

We can prove similarly that ImT_m is an anti-holomorphic map. □

We introduce various fundamental loop groups as follows:

$$\begin{aligned} \Lambda GL(N,C) =& \{\gamma : S^1 \to GL(N,C) | \gamma \text{ is smooth}\}, \\ \Lambda^+GL(N,C) =& \{\gamma \in \Lambda GL(N,C) | \gamma \text{ extends continuously to} \\ & \text{a holomorphic map } D^+ \to GL(N,C)\} \end{aligned}$$

where $D^+ = \{\lambda \mid |\lambda| < 1\}$. We recall the Iwasawa decomposition:

Theorem 2.3[18]. *Each γ in $\Lambda GL(N,C)$ can be uniquely decomposed as*

$$\gamma = \gamma_u\gamma_+,$$

where $\gamma_u \in \Omega U(N)$ and $\gamma_+ \in \Lambda^+GL(N,C)$.

By the theorem, we have an identification

$$\Omega U(N) \cong \Lambda GL(N,C)/\Lambda^+GL(N,C) \tag{8}$$

and thus, a natural action of the complex group $\Lambda GL(N,C)$ on the complex manifold $\Omega U(N)$ ([13]): if $\gamma \in \Lambda GL(N,C)$ and $\delta \in \Omega U(N)$, then

$$\gamma \cdot \delta = (\gamma\delta)_u. \tag{9}$$

Let us choose a maximal torus T of $U(N)$ and a fundamental Weyl chamber W_0 in the Lie algebra t of T. Denote by I' the intersection of the integer lattice $I = (2\pi)^{-1}\exp^{-1}(e)\cap t$ with W_0 . Then we know that I' ($\ni \xi$) parametrizes all the conjugacy classes of homomorphisms $\gamma_\xi : S^1 \to U(N)$. By [18], we have the following Bruhat decomposition :

$$\Omega_{a\lg}U(N) = \sqcup_{\xi\in I'}\Lambda^+_{a\lg}GL(N,C)\cdot\gamma_\xi, \tag{10}$$

where $\Lambda^+_{a\lg}GL(N,C) = \Lambda_{a\lg}GL(N,C)\cap\Lambda^+GL(N,C)$.

There is a beautiful Morse theoretic explanation of these decompositions. Let $H : \Omega U(N) \to R$ be the usual energy functional on paths, $H(\gamma) = \int_{S^1}|\gamma'|^2$. This is a Morse-Bott function. The critical manifolds are the conjugacy classes of homomorphisms. We write

$$\Omega_\xi = \{g\gamma_\xi g^{-1}|\ g \in U(N)\} \tag{11}$$

for the conjugacy class of the homomorphism γ_ξ.Denote by U_ξ the unstable manifold of Ω_ξ with respect to the flow of $-gradH$ (i.e., U_ξ is the stable manifold of Ω_ξ with respect to the flow of $gradH$). Then we have ([17])

$$U_\xi = \Lambda^+_{a\lg}GL(N,C)\cdot\gamma_\xi. \tag{12}$$

In [17], the energy functional of loops H is viewed as a Hamiltonian function associated to a symplectic action of the group S^1on $\Omega U(N)$, which is given by

$$\mu\cdot\gamma(\lambda) = \gamma(\mu\lambda)\gamma(u)^{-1},\quad \mu\in S^1, \gamma\in\Omega U(N). \tag{13}$$

It was observed by Terng (see sect.7 of [20]) that this action preserves the extended harmonic equation (4). We have the following action of C^* on $\Omega_{\rm alg}U(N)$:

$$\mu\cdot\gamma(\lambda) = \gamma(\mu\lambda)\Lambda^+GL(N,C),\quad \mu\in C^*, \gamma\in\Omega_{\rm alg}U(N), \tag{14}$$

which is the complexification of the S^1-action. It turns out that the flow line of $gradH$ starting at a point $\gamma\in\Omega_{\rm alg}U(N)$ is given by the action of subsemigroup $(0,1]$ of C^*.

For an extended untion $\Phi : S^2 \to \Omega_{\rm alg}U(N)$, we cannot conclude that $\Phi(S^2) \subseteq U_\xi$ for some $\xi \in I'$ in general. However, we have

proposition 2.4[5]. *Let $\Phi : S^2 \to \Omega_{\rm alg}U(N)$ be an extended uniton. Then there exists some $\xi\in I'$, and some discrete subset D of S^2 , such that $\Phi(S^2 - D)\subseteq U_\xi$.*

3. Deformations of S^1- invariant unitons

In the classification theory of unitons in [5], a fundamental role is played by the following special extended unitons:

Definition 3.1. An S^1-invarinat extended uniton is an extended uniton Φ such that $Im\Phi \subseteq \Omega_\xi$, for some $\xi \in I'$; or equivalently Φ is invariant under the S^1-action (13). In this case, Φ_{-1} is also called S^1-invariant uniton.

By [20], we know that Φ is an S^1-invarinat extended uniton into $\Omega U(N)$ if and only if it can be expressed as

$$\Phi = \sum_{\alpha=0}^{k} \Pi_\alpha \lambda^\alpha \tag{15}$$

where $f_\Phi := (\Pi_0, \Pi_1, ..., \Pi_k)$ is a super-horizontal holomorphic map into a complex flag manifold $F = F(n_0, n_1, ..., n_k)$ with $n_\alpha = rank\Pi_\alpha$ $(\alpha = 0, 1, ..., k)$ and $N = n_0 + n_1 + \cdots + n_k$. We say that f_Φ is standard if the extended untion Φ is standard.

In the following, we will investigate the deformation of the S^1-invariant harmonic map. We know that $F(n_0, ..., n_k)$ can be identified with the complex homogeneous space $U(N)/U(n_0) \times ... \times U(n_k)$. A point B of F may be viewed as a $(k+1)-$ tuple of orthogonal subspaces $B_0 \oplus ... \oplus B_k$ where $\dim(B_i) = n_i$, or , equivalently, as a flag

$$0 \subset V_0 = B_0 \subset V_1 = B_0 + B_1 \subset ... \subset V_k = B_0 + B_1 + ... + B_k = C^N.$$

Let $\{V_0, ..., V_k\}$ denote the flag corresponding $(B_0, ..., B_k)$. An element of $F(n_0, \cdots, n_k)$ is written as $B = (B_0, B_1, ..., B_k)$ or $\{V_0, V_1, ..., V_k\}$. Set $J = \{0, 2, ..., 2[\frac{k}{2}]\}$ and $J^c = \{0, 1, ..., k\} \setminus J$. Define the projection

$$\pi : F(n_0, n_1, ..., n_k) \ni (B_0, B_1, ..., B_k) \to \bigoplus_{j \in J^c} B_j \in Gr_t(C^N) \tag{16}$$

where $t = \sum_{j \in J^c} n_j$. It is known that π is the so called canonical twistor fibration(see [6] or [9]).

Define the following tautological complex vector bundles for $B \in F(n_0, n_1, ..., n_k)$: $(V_0)_B = B_0, (V_1)_B = B_0 \oplus B_1, ..., (V_k)_B = \bigoplus_{i=0}^k B_i$. A map $f : S^2 \to F(n_0, n_1, ..., n_k)$ is called super-horizontal holomorphic map if

$$\begin{cases} \partial'' C^\infty(f^{-1}V_i) \subset C^\infty(f^{-1}V_i), \\ \partial' C^\infty(f^{-1}V_i) \subset C^\infty(f^{-1}V_{i+1}), \quad i = 0, 1, ..., k, \end{cases} \tag{17}$$

where we set $V_{k+1} = C^N$.

Hence a uniton $\varphi : S^2 \to Gr_t(C^N)$ is S^1-invariant if and only if there exists a super-horizontal holomorphic map $f_\Phi : S^2 \to F$ such that $\varphi = \pi \circ f_\Phi$. The group

$GL(N,C)$ acts on $F(n_0,n_1,...,n_k)$ in the natural way : for $A \in GL(N,C)$ and $V_0 \subset V_1 \subset \cdots \subset V_k$, we have $AV_0 \subset AV_1 \subset \cdots \subset AV_k$. It is easy to see that this action preserves the super- horizontality and holomorphicity conditions. Thus it induces an action on the S^1−invariant untions.

Let $G = U(N)$ and $g = u(N)$ denote the unitary group and its Lie algebra. Then we can regard $F = F(n_0,n_1,...,n_k)$ as an "Ad-orbit" of $U(N)$ as follows: If we let $(\widehat{B}_0,\widehat{B}_1,...,\widehat{B}_k)$ be a fixed element of $F(n_0,n_1,...,n_k)$ and set $\xi = \sqrt{-1}\sum_{i=0}^k \pi_{W_i}$ where $W_i = \bigoplus_{\alpha=0}^i \widehat{B}_\alpha$ for $i = 0,1,...k$, then we have $F(n_0,n_1,...,n_k) \cong Ad(G)\xi$. Here π_V denotes the Hermitian projection in C^N onto a complex subspace V.

Fix an element $L \in CP^{N-1}$ and put $Q = \sqrt{-1}\pi_L \in u(N)$. For a $X = \sqrt{-1}\sum_{i=0}^k \pi_{V_i} \in Ad(G)\xi$, where $(V_0, V_0^\perp \cap V_1,...,V_{k-1}\cap V_k^\perp) \in F(n_0,n_1,...,n_k)$, we define the height function $h^Q : Ad(G)\xi \to R$ by

$$h^Q(X) = (X,Q). \tag{18}$$

Here $(,)$ is an $Ad(G)$−invariant inner product on g. Then it is known that h^Q is a Morse-Bott function. Let $gradh^Q$ be the gradient of h^Q with respect ot the Kähler metric. The following fact is due to Frankel [16] : the flow of $-gradh^Q$ is given by the natural action of $\{\exp\sqrt{-1}tQ\}$. A point $(V_0, V_0^\perp\cap V_1,....,V_{k-1}^\perp\cap V_k)$ is a critical point if and only if $[\sqrt{-1}\sum_{i=0}^k \pi_{V_i}, \sqrt{-1}\pi_L] = 0$, i.e., the line L is contained in one of V_0, $V_0^\perp\cap V_1,...,V_{k-1}^\perp \cap V_k$.

lemma 3.1. *There are $k+1$ connected no-degenerate critical manifolds of h^Q :*

$$F_0 = \{\{V_0,V_1,...,V_k\} \in F(n_0,n_1,...,n_k)|L \subset V_0\} \cong F(n_0-1,n_1,...,n_k),$$
$$F_1 = \{\{V_0,V_1,...,V_k\} \in F(n_0,n_1,...,n_k)|L \subset V_0^\perp\cap V_1\} \cong F(n_0,n_1-1,...,n_k),$$
$$\cdots\cdots$$
$$F_k = \{\{V_0,V_1,...,V_k\} \in F(n_0,n_1,...,n_k)|L \subset V_{k-1}^\perp\cap V_k\} \cong F(n_0,n_1,...,n_k-1).$$

We set $G_Q = \{A \in GL(N,C)|A(L) = L\}$. It is known that the stable manifold for a non-degenerate critical manifold N is given by $S^Q(N) = G_QX$ for $X \in N$. In this case, we have

lemma 3.2. *The corresponding stable manifolds are*

$$S^Q(F_0) = F_0,$$
$$S^Q(F_1) = \{\{V_0,V_1,...,V_k\} \in F(n_0,n_1,...,n_k)|V_0\cap L = \{0\}, L\subset V_1\},$$
$$\cdots\cdots$$
$$S^Q(F_k) = \{\{V_0,V_1,...,V_k\} \in F(n_0,n_1,...,n_k)|V_{k-1}\cap L = \{0\}, L\subset V_k\}.$$

Let $\varphi : S^2 \to Gr_t(C^N)$ be a S^1−invariant harmonic map, and $f_\Phi : S^2 \to F$ be a super-horizontal holomorphic map corresponding to φ. If $f_\Phi(S^2) \subset S^Q(F_k)$, then $\{(\exp\sqrt{-1}tQ)\circ f_\Phi\}$ provides a continuous deformation to a super-horizontal holomorphic map into $F_k(\cong F(n_0, n_1, ..., n_k-1))$. We shall show that there exists some $L\in CP^{N-1}$ such that $f_\Phi(S^2)\subset S^Q(F_k)$ if $n_k\geq 2$.

Set $Y=\{L\in CP^{N-1}|f_\Phi(z)\notin S^Q(F_k)$ for some $z\in S^2\}$. Then we have $Y=\{L\in CP^{N-1}|L\subset V_{k-1}(z)$ for some $z\in S^2\}$. It suffices to show that Y cannot be equal to CP^{N-1}.

Fix $z\in S^2$, we set $Y_z=\{L\in CP^{N-1}|L\subset V_{k-1}(z)\}$. It is easy to see that

$$\dim_C Y_z\leq\sum_{i=0}^{k-1}n_i-1. \tag{19}$$

Since $\dim_C S^2=1$, we have

$$\dim_C Y\leq\sum_{i=0}^{k-1}n_i. \tag{20}$$

It follows that, if $n_k\geq 2$, the space Y cannot be equal to CP^{N-1}. It suffices to choose $L\in CP^{n-1}\backslash Y$. Hence , by induction, we obtain a map whose image lies in $F(n_0, n_1, ..., n_{k-1}, 1)$. By repeating the argument with $f^{\ *}=\{V_k^\perp, ..., V_0^\perp, C^n\}\in F(1, n_{k-1}, ..., n_0)$, we can similarly deform $f_{\ \Phi}$ into $\{\{V_0, V_1, ..., V_k\}\in F(n_0, n_1, ..., n_{k-1}, 1)|\ C^{n_0-1}\subseteq V_0\}\cong F(1, n_1, ..., n_{k-1}, 1)$. Thus we obtain the following proposition.

proposition 3.3. *If $n\geq 3$, then any super-horizontal holomorphic map f_Φ into $F(n_0,\ n_1, ..., n_k)$ can be deformed continuously through super-horizontal holomorphic maps to a super-horizontal holomorphic map $f_{\widetilde\Phi}$ into $F(1, n_1, ..., n_{k-1}, 1)$. Furthermore, if f_Φ is standard , then $f_{\widetilde\Phi}$ is also standard in its corresponding space.*

Proof. We only need to prove the second part of the proposition. Assume that f_Φ is standard, i.e., the corresponding extended uniton $\Phi=\sum_{\alpha=0}^k\Pi_\alpha\lambda^\alpha$ satisfies the following condition

$$V_0(\Phi)=Span\{Im_{z\in S^2}\Pi_0(z)\}=C^N$$

Let $\Phi^t=\sum_{\alpha=0}^k\Pi_\alpha^t\lambda^\alpha$ be the continuous deformation obtained in the above. It is clear that $\dim V_0(\Phi^t)=const.$ So, $V_0(\Phi^t)=C^N$. In particular, we have

$$V_0(\Phi^\infty)=C^N.$$

The above deformation shows that $f_{\Phi^\infty}(S^2)\subset\{\{V_0, V_1, ..., V_k\}\in F(n_0, n_1, ..., n_k)|\ C^{n_0-1}\subseteq V_0, C^{n_k-1}\subseteq V_{k-1}^\perp\cap V_k\}\overset{i}{\cong}F(1, n_1, ..., n_{k-1}, 1)$, where i denotes the natural isomorphism. Let $f_{\widetilde\Phi}=i\circ f_{\Phi^\infty}: S^2\to F(1, n_1, ..., n_{k-1}, 1)$. We see that

$$V_0(\widetilde\Phi)=C^{\widetilde N}$$

where $\widetilde{N} = N - (n_0 + n_k - 2)$. Hence $\widetilde{\Phi}$ is a standard extended $U(\widetilde{N})$−uniton, i.e., $f_{\widetilde{\Phi}}$ is standard. □

We know from [8] that any harmonic map $S^2 \to CP^{N-1}$ corresponds to a super-horizontal holomorphic map into $F(n_0, 1, n_2)$ with $n_0 + 1 + n_2 = N$. Hence we obtain:

Corollary 3.4[13]. *If $N \geq 3$, then any harmonic map $\varphi : S^2 \to CP^{N-1}$ can be deformed continuously through harmonic maps to a harmonic map $\psi : S^2 \to CP^2$.*

4. Anand's conjecture

We now give an application of the above deformations for unitons.

Theorem 4.1. *Let $\varphi : S^2 \to U(N)$ be a harmonic map with minimal uniton number m. Then*

$$E(\varphi) \geq 4\pi m^2. \tag{21}$$

Furthermore, the equlity holds if and only if φ can be deformed continuously to a harmonic map ψ which is one of the following two cases :

(i) $\psi : S^2 \to CP^1(\subset U(2) \subset U(N))$ is a totally unramified holomorphic map ;

(ii) $\psi : S^2 \to Gr_2(C^3)$ is a harmonic map with minimal uniton number $m = 2$ and $\psi^\perp : S^2 \to CP^2(\subset U(3) \subset U(N))$ is a totally unramified, not±holomorphic , full harmonic map.

Proof. Let $\Phi = \sum_{\alpha=0}^{m} T_\alpha \lambda^\alpha$ be the standard extended harmonic map of φ. By Proposition 2.3 we can express Φ on $M - D$ as follows:

$$\Phi = A\gamma_\xi B \tag{22}$$

where $A, B : M - D \to \Lambda^+_{\text{alg}} GL_n(C)$ and γ_ξ is the geodesic given by

$$\gamma_\xi(\lambda) = diag(\lambda^m, ..., \lambda^m, ..., \lambda, ..., \lambda, 1, ..., 1).$$

It is easy to see that

$$T_0 = A(0)\gamma_\xi(0, ..., 1, ..., 1)B(0). \tag{23}$$

on $M - D$. Now we deform the extended harmonic map Φ by the gradient flow of the Morse-Bott function $H : \Omega G \to R$. From the description of this flow in Sec.2, we have $\widehat{\Phi} = \lim_{t\to+\infty} \Phi^t$, where Φ^t is the extended harmonic map $e^{-t} \cdot \Phi$. The points in D are the possible bubble points in the deformation. If bubbling occurs, then $E(\varphi) > E(\widehat{\Phi}_{-1})$. So, in general , we have $E(\varphi) \geq E(\widehat{\Phi}_{-1})$. Since $Im\Psi \subseteq \Omega_\xi$,the extended uniton Ψ is an S^1−invariant uniton. By [20] $\widehat{\Phi}$ has the following expression:

$$\widehat{\Phi} = \sum_{\alpha=0}^{m} \Pi_\alpha \lambda^\alpha \tag{24}$$

where $(\Pi_0, \Pi_1, ..., \Pi_m)$ is a super-horizontal holomorphci map into $F(n_0, n_1, ..., n_m)$ with $n_\alpha = rank(\Pi_\alpha)$ $(\alpha = 0, 1, ..., m)$. From the definition of the action $e^{-t} \in C^*$ on extended uniton and (22), we have

$$\begin{aligned}\widehat{\Phi} &= A(0) \cdot \gamma_\xi \\ &= A(0)\gamma_\xi C_0.\end{aligned}$$

We see that $\pi_0 = A(0)diag(0, ..., 0, 1, ...1)C_0$. It follows that

$$\begin{aligned}Im\pi_0 &= ImA(0)diag(0, ..., 0, 1, ..., 1) \\ &= ImT_0.\end{aligned}$$

Thus $\widehat{\Phi}$ is a standard extended uniton with minimal uniton number m.

By Proposition 3.3, $\widehat{\Phi}$ can be deformed continuously to an S^1-invariant extended harmonic map Ψ which corresponding a standard super-horizontal holomorphic map into $F(1, n_1, ..., n_{m-1}, 1)$. Hence $E(\widehat{\Phi}_{-1}) = E(\Psi_{-1})$ and the S^1- invariant extended harmonic map Ψ is given by

$$\Psi = \sum_{\alpha=0}^{m} (\pi_\alpha - \pi_{\alpha-1})\lambda^\alpha \tag{25}$$

with $rank(\pi_0) = 1$. From Lemma 2.2, we see that $\psi_0 := Im(\pi_0) : S^2 \to CP^n$ is a full holomorphic map , where $n = 1 + \sum_{i=1}^{m-1} n_i$. The holomorphic map ψ_0 generates the following harmonic sequence (see [4], [22]):

$$\psi_0 \overset{\partial_0}{\to} \psi_1 \overset{\partial_1}{\to} \cdots \overset{\partial_{N-2}}{\to} \psi_n, \tag{26}$$

with $\bigoplus_{j=0}^{n} \underline{\psi}_j = \underline{C}^{n+1}$, where ∂ denotes the fundamental collineation defined by Chern and Wolfson(see [7]). It is easy to see from the section10 of [20] that

$$\underline{\pi}_0 = \underline{\psi}_0, \underline{\pi}_1 = \bigoplus_{j=0}^{i_1} \underline{\psi}_j, \cdots, \underline{\pi}_{m-1} = \bigoplus_{j=0}^{i_{m-1}} \psi_j, \underline{\pi}_m = \underline{C}^{n+1}, \tag{27}$$

with $0 = i_0 < i_1 < ... < i_{m-1} < i_m = n$. It follows from Uhlenbeck's factorization and Valli's formula ([21]) that

$$\begin{aligned}E(\Psi_{-1}) &= 4\pi \sum_{\alpha=0}^{m} \deg \pi_\alpha \\ &= 4\pi \sum_{\alpha=0}^{m-1} \deg(\bigoplus_{j=0}^{i_\alpha} \underline{\psi}_j).\end{aligned} \tag{28}$$

Since $\underline{\pi}_\alpha$ corresponds to the i_α−th osculating curve of the holomorphic curve ψ_0, we have by [4] that

$$\deg(\bigoplus_{j=0}^{i_\alpha} \underline{\psi}_j) = (i_\alpha + 1)(n - i_\alpha) + \frac{n - i_\alpha}{n+1} \sum_{k=0}^{i_\alpha - 1} (k+1) r(\partial_k)$$
$$+ \frac{i_\alpha + 1}{n+1} \sum_{k=i_\alpha}^{n-1} (n-k) r(\partial_k)$$
$$\geq (i_\alpha + 1)(n - i_\alpha), \tag{29}$$

where $r(\partial_k)$ denotes the ramification index of ∂_k. It follows from (28) and (29) that $E(\Psi_{-1}) \geq 4\pi mn$. The minimal uniton number $m \leq n$ by [20], so we obtain $E(\varphi) \geq 4\pi m^2$.

Set $\psi = \Psi_{-1}$. If the equality holds, then we see from (28) and (29) that

$$r(\partial_k) = 0$$

for $k = 0, 1, ..., n-1$ and

$$(i_\alpha + 1)(n - i_\alpha) = n \tag{30}$$

for $\alpha = 0, 1, ..., m-1$. It follows from (30) that either $m = 1$ and $n = 1$, i.e., ψ belongs to the case (i) or $m = 2$, $n = 2$ and

$$\psi = \pi_0 - \pi_1 + \pi_2$$
$$= \pi_0 + \pi_2 - \pi_1,$$

i.e., ψ belongs to the case (ii) . Actually, we see via the Cartan embedding that $\psi = \psi_0$ in the case (i) ; and $\psi = \psi_0 \oplus \psi_2$, $\psi^\perp = \psi_1$ in the case (ii). Conversely, if ψ is one of the cases (i) and (ii), it is easy to verify that the equality in (21) holds. □

Remark 4.1. The above proof shows a little more, namely that if $\varphi : S^2 \to U(N)$ has a standard extended harmonic map $\Phi = \sum_{\alpha=0}^m T_\alpha \lambda^\alpha$ with $\max_{z \in S^2} rank(T_0)$ $= 1$, then $E(\varphi) \geq 4\pi m(N-1)$. We can normalize the energy to be $\widetilde{E}(\varphi) = \frac{1}{4\pi} E(\varphi)$. So, the Anand conjecture states that $m-$ unitons have energy at least m^2.

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References

1. Anand, C.K., *Uniton bundles*, Comm. in Anal. and Geom. 3(3), 371-419(1995).
2. Anand, C.K., *A closed form for Unitons*, J. Math. Soc. Japan 50 (3), 737-751 (1998).
3. Dong, Y.X., Shen, Y.B., *Factorization and uniton numbers for harmonic maps into the unitary group $U(N)$*, Science in China,SerA, 39, 589-597 (1996).
4. Bolton, J., Jesen, G., Rigoli, M. and Woodward, M., *On conformal minimal immersions of S^2 into CP^n*, Math. Ann. 279, 599-620(1988).
5. Burstall, F.E., Guest, M.A., *Harmonic two-spheres in compact symmetric spaces, revisited,* Math. Ann. 309, 541-572(1997).
6. Burstall, F.E., Rawnsley, J.H., *Twistor theory for Riemannian symmetric spaces*, Lectrure Notes in Math. 1424 , Berlin, Heidelberg: Springer, 1990.
7. Chern, S.S., Wolfson, J.G., *Harmonic maps of the two-sphere into a complex Grassmann manifold II*, Ann. of Math. 125, 301-335 (1987).
8. Eells, J., Wood, J.C., *Harmonic maps from surfaces into complex projective spaces*, Adv. in Math. 49, 217-263 (1983).
9. Erdem, S. , Wood, J.C., *On the construction of harmonic maps into a Grassmannian*, J. London Math. Soc. (2) 28 , 161-174 (1983).
10. Gu, C.H., Hu, H.S., *Constructions of unitons via purely algebraic algorithm*, Chin. Ann. of Math., 20B, 1-6 (1999).
11. Gu, C.H., Dong, Y.X., Shen, Y.B., *Constructive factorizations of unitons via singular Darboux transformation*, Science in China (Series A) Vol.43, 149-157(2000).
12. Guest, M.A., *Harmonic maps , loop groups, and integrable systems, London Mathematical Society Student Texts 38 Cambridge University Press*, 1997.
13. Guest, M.A., Ohnita, Y., *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan, Vol. 45 , No. 4, 671-704 (1993).
14. Guest, M.A., Ohnita, Y., *Loop group actions on harmonic maps and their applications, Harmonic maps and Integrable systems edited by A.P. Fordy and J.C. Wood, Vieweg, 273-292 (1994).*
15. Koszul, J.L.,Malgrange, B., *Sur certaines structures fibres complexes*, Arch. Math. 9 , 102-109 (1958).
16. Frankel, T., *Fixed points and torsion on Kähler manifolds*, Ann. of Math. 70, 1-8(1959).
17. Presssley, A.N., *The energy flow on the loop space of a compact Lie group*, J. London Math. Soc. 26, 557-566 (1982).
18. Pressly, A.N., Segal, G.B., *Loop Groups*, Oxford University Press, 1986.
19. Sacks, J. , Uhlenbeck, K., *The existence of minimal immersions of 2-spheres*, Ann. of Math. 113, 1-24 (1981).
20. Uhlenbeck, K., *Harmonic maps into Lie Groups (Classical solutions of the chiral model)*, J. Diff. Geom. 30, 1-50 (1989).
21. Valli, G., *On the energy spectrum of harmonic 2-spheres in unitary groups*, Topology 27 , 129-136 (1988).
22. Wolfson, J.G., *Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifold*, J. Diff. Geom. 27 , 161-178 (1988).
23. Wood, J.C., *Explicit construction and parametrization of harmonic two-spheres in the unitary group*, Proc. London Math. Soc. 59 , 608-624 (1989.