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NOTES ON THE RELATIVE YAMABE INVARIANT

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Dedicated to Professor Katsuhiro Shiohama on his sixtieth birthday

ABSTRACT. The Yamabe invariant [K], [S2] of a closed smooth manifold X is a natural differential-topological invariant which arises from a variational problem for the total scalar curvature of Riemannian metrics on X. The *relative Yamabe invariant* [AB1] of a compact connected smooth manifold W with nonempty boundary is a natural relative version of the Yamabe invariant of X. Hence the relative Yamabe invariant has several fundamental properties analogous to the corresponding ones for the classic Yamabe invariant. In particular, in respect of surgery on X and the interior of W, these two invariants have quite similar properties. In this article, we give those properties.

1. PRELIMINARIES

Let W be a compact connected smooth n-manifold with nonempty boundary $\partial W = M$ (possibly disconnected), and $n = \dim W \geq 3$. We denote by $\mathcal{C}(M)$ and $\mathcal{C}(W)$ the spaces of conformal classes on M and W respectively. Let $C \in \mathcal{C}(M)$ be a conformal class on M and $\overline{C} \in \mathcal{C}(W)$ a conformal class on W. We say that C is the *boundary* of $\overline{C}$ if $\overline{C}|_M = C$. We use the notation $\partial\overline{C} = C$ in this case. Let $\mathcal{R}\mathrm{iem}(W)$ be the space of all Riemannian metrics on W. For each conformal class $C \in \mathcal{C}(M)$, we define the following subspaces of $\mathcal{R}\mathrm{iem}(W)$:

$$\begin{aligned}\mathcal{R}\mathrm{iem}_C(W) &= \{\overline{g} \in \mathcal{R}\mathrm{iem}(W) \mid \partial[\overline{g}] = C\},\\ \mathcal{R}\mathrm{iem}^0_C(W) &= \{\overline{g} \in \mathcal{R}\mathrm{iem}_C(W) \mid H_{\overline{g}} = 0 \quad \text{along } M\},\end{aligned}$$

where $[\overline{g}]$ denotes the conformal class of $\overline{g}$ and $H_{\overline{g}}$ the mean curvature along M with respect to $\overline{g}$. We consider the normalized Einstein-Hilbert functional $I\ :\ \mathcal{R}\mathrm{iem}(W) \longrightarrow \mathbb{R}$ given

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by

$$I(\overline{g}) = \frac{\int_W R_{\overline{g}}\, dv_{\overline{g}}}{\mathrm{Vol}_{\overline{g}}(W)^{(n-2)/n}},$$

where $R_{\overline{g}}$ and $dv_{\overline{g}}$ are respectively the scalar curvature and the volume element of $\overline{g}$. As in the case of closed manifolds, we notice the following (cf. [B]):

Proposition 1.1. ([AB1, Theorem 1.1]) *The set of critical points of I on the space $\mathcal{R}\mathrm{iem}^0_C(W)$ coincides with the set of Einstein metrics $\overline{g}$ on W with $\partial[\overline{g}] = C$ and $H_{\overline{g}} = 0$.*

Remark 1.2. From Claim 3.1 in [AB1], we notice that the set of critical points of I on $\mathcal{R}\mathrm{iem}_C(W)$ is empty for any $C \in \mathcal{C}(M)$. Moreover, the set of critical points of I on the space $\{\overline{g} \in \mathcal{R}\mathrm{iem}_C(W) \mid H_{\overline{g}}$ is constant along $M\}$ is also empty.

Similarly to the case of closed manifolds (see [Au2]), the functional I is not bounded on $\mathcal{R}\mathrm{iem}^0_C(W)$. More precisely, for any W and any $C \in \mathcal{C}(M)$,

$$\inf_{\overline{g} \in \mathcal{R}\mathrm{iem}^0_C(W)} I(\overline{g}) = -\infty , \qquad \sup_{\overline{g} \in \mathcal{R}\mathrm{iem}^0_C(W)} I(\overline{g}) = \infty.$$

For each conformal class $C \in \mathcal{C}(M)$, let denote $\mathcal{C}_C(W)$ the subspace $\{\overline{C} \in \mathcal{C}(W) \mid \partial\overline{C} = C\}$ of $\mathcal{C}(W)$. For each conformal class $\overline{C} \in \mathcal{C}_C(W)$, we consider the subclass $\overline{C}^0 \subset \overline{C}$ defined by

$$\overline{C}^0 = \{\overline{g} \in \overline{C} \mid H_{\overline{g}} = 0 \quad \text{along } M\}.$$

We call $\overline{C}^0$ the *normalized conformal class* of $\overline{C}$. For a fixed $\overline{g} \in \overline{C}$, put $\tilde{g} = u^{\frac{4}{n-2}}\overline{g}$ for $u \in C^\infty_+(W)$, where $C^\infty_+(W)$ denotes the space of all positive smooth functions on W. Then

$$H_{\tilde{g}} = u^{-2/(n-2)}\Big(H_{\overline{g}} + \frac{2(n-1)}{n-2}u^{-1}\frac{\partial u}{\partial \nu}\Big), \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ denotes the normal derivative with respect to the inward unit normal vector field ν along the boundary M. Hence (1.1) implies the following:

Proposition 1.3. ([E2, Section 1])

(1) *$\overline{C}^0$ is nonempty for any $\overline{C} \in \mathcal{C}(W)$.*

(2) *If $\overline{g} \in \overline{C}^0$, then*

$$\overline{C}^0 = [\overline{g}]^0 = \left\{u^{4/(n-2)}\overline{g} \in \overline{C} \mid u \in C^\infty_+(W), \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{along } M\right\}.$$

The *relative Yamabe constant* $Y_{\overline{C}}(W, M; C)$ of the conformal class $\overline{C} \in \mathcal{C}_C(W)$ (or the pair $(\overline{C}, C)$) is defined by

$$Y_{\overline{C}}(W, M; C) = \inf_{\overline{g} \in \overline{C}^0} I(\overline{g}) = \inf_{\overline{g} \in \overline{C}^0} \frac{\int_W R_{\overline{g}}\, dv_{\overline{g}}}{\mathrm{Vol}_{\overline{g}}(W)^{(n-2)/n}}$$

(see [Au2], [LP] for the Yamabe constant of a conformal class on a closed manifold). A metric $\breve{g} \in \overline{C}^0$ is called a *relative Yamabe metric* if $I(\breve{g}) = Y_{\overline{C}}(W, M; C)$, that is, $\breve{g}$ is a minimizer of I on $\overline{C}^0$. A similar argument to the Yamabe constant given by Aubin [Au1] (cf. [E2]) shows that

(1.2) $Y_{\overline{C}}(W, M; C) \leq Y_{[\overline{h}]}(S^n_+, S^{n-1}; [h]) = n(n-1) \cdot \mathrm{Vol}_{\overline{h}}(S^n_+)^{2/n}$ for any $(W, M; \overline{C}, C)$,

where S^n_+ denotes the round n-hemisphere with standard metric $\overline{h}$ of constant curvature 1 and $S^{n-1} \subset S^n_+$ the equator with $h = \overline{h}|_{S^{n-1}}$. It should be pointed out that from [E3, Proposition 1.4]

$$\inf_{\tilde{g} \in \overline{C}} I(\tilde{g}) \leq 0 \quad \text{for any } \overline{C} \in \mathcal{C}(W).$$

This observation combined with Remark 1.2, Proposition 1.3-(1) and (1.2) may implies that the minimal boundary condition is suitable for a natural relative version of the classic Yamabe constant.

The *relative Yamabe invariants* of the conformal class $C \in \mathcal{C}(M)$ (or the triple $(W, M; C)$) and the pair (W, M) are respectively defined by

$$Y(W, M; C) = \sup_{\overline{C} \in \mathcal{C}_C(W)} Y_{\overline{C}}(W, M; C) = \sup_{\overline{C} \in \mathcal{C}_C(W)} \inf_{\overline{g} \in \overline{C}^0} \frac{\int_W R_{\overline{g}} \, dv_{\overline{g}}}{\mathrm{Vol}_{\overline{g}}(W)^{(n-2)/n}},$$

$$Y(W, M) = \sup_{C \in \mathcal{C}(M)} Y(W, M; C) = \sup_{C \in \mathcal{C}(M)} \sup_{\overline{C} \in \mathcal{C}_C(W)} \inf_{\overline{g} \in \overline{C}^0} \frac{\int_W R_{\overline{g}} \, dv_{\overline{g}}}{\mathrm{Vol}_{\overline{g}}(W)^{(n-2)/n}},$$

(see [K], [S2], [Le2] for the Yamabe invariant of a closed manifold). The inequality (1.2) guarantees their least upper bound

$$Y(W, M; C), \; Y(W, M) \leq n(n-1) \cdot \mathrm{Vol}_{\overline{h}}(S^n_+)^{2/n},$$

and hence

$$Y(S^n_+, S^{n-1}) = n(n-1) \cdot \mathrm{Vol}_{\overline{h}}(S^n_+)^{2/n}.$$

The conformal invariant $Y(W, M; C)$ of C has a clear geometrical meaning in terms of positive scalar curvature (abbreviated as "psc"). Namely, $Y(W, M; C) > 0$ if and only if any metric $g \in C$ can be extended conformally to a psc-metric $\overline{g}$ on W with the minimal boundary condition $H_{\overline{g}} = 0$. The invariant $Y(W, M)$ is by definition a differential-topological invariant of the pair (W, M), and also $Y(W, M) > 0$ if and only if there exists a psc-metric $\overline{g}$ on W with the minimal boundary condition.

The rest of the paper is organized as follows. In Section 2, we prove that the relative Yamabe constant of $\overline{C} \in \mathcal{C}(W)$ coincides with the conformal invariant of $\overline{C}$ defined by Escobar [E2], and we summarize fundamental properties on relative Yamabe constants and relative Yamabe metrics. In Section 3, we also give fundamental properties on relative Yamabe invariants.

2. Relative Yamabe constants

Let $\overline{C} \in \mathcal{C}(W)$ be a conformal class on W with $\partial\overline{C} = C$, and $\overline{g}$ a metric in the normalized conformal class $\overline{C}^0$. We define the functional $E_{\overline{g}} : L^{1,2}(W) \longrightarrow \mathbb{R}$ by

$$E_{\overline{g}}(f) = \int_W \Big(\frac{4(n-1)}{n-2}|df|^2_{\overline{g}} + R_{\overline{g}} f^2\Big) dv_{\overline{g}} \quad \text{for } f \in L^{1,2}(W),$$

where $L^{1,2}(W)$ denotes the Sobolev space of square-integrable functions on W up to their first weak derivatives. In terms of the functional $E_{\overline{g}}$, the relative Yamabe constant $Y_{\overline{C}}(W, M; C)$ may be written as the following:

Lemma 2.1.

$$\begin{aligned} Y_{\overline{C}}(W,M;C) &= \inf_{\tilde{g} \in \overline{C}^0} I(\tilde{g}) \\ &= \inf_{u \in C^\infty_+(W),\ \frac{\partial u}{\partial \nu}|_M = 0} E_{\overline{g}}(u)/\|u\|^2_{L^{2n/(n-2)}(\overline{g})} \qquad (2.1) \\ &= \inf_{f \in L^{1,2}(W),\ f \not\equiv 0} E_{\overline{g}}(f)/\|f\|^2_{L^{2n/(n-2)}(\overline{g})}. \end{aligned}$$

Here $\|\cdot\|_{L^{2n/(n-2)}(\overline{g})}$ *stands for the* $L^{2n/(n-2)}$*-norm on* W *with respect to* $\overline{g}$.

Remark 2.2. From (2.1), the relative Yamabe constant $Y_{\overline{C}}(W, M; C)$ coincides with the conformal invariant $Q(W) = Q(W, \overline{C})$ of $\overline{C}$ (up to the positive factor $\frac{4(n-1)}{n-2}$) defined by Escobar [E2].

Proof of Lemma ??. Let $\tilde{g} = u^{4/(n-2)}\overline{g}$ be any metric in $\overline{C}^0$, where u is a positive smooth function on W with $\frac{\partial u}{\partial \nu} = 0$ along M. Then we have the formula

$$R_{\tilde{g}} = u^{-(n+2)/(n-2)}\Big(-\frac{4(n-1)}{n-2}\Delta_{\overline{g}} u + R_{\overline{g}} u\Big) \quad \text{on } W, \qquad (2.2)$$

where $\Delta_{\overline{g}} = \overline{g}^{ij}\overline{\nabla}_i\overline{\nabla}_j$ denotes the Laplace-Beltrami operator of $\overline{g}$ and $\overline{\nabla}$ the Levi-Civita connection of $\overline{g}$. The Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ along M implies

$$I(\tilde{g}) = E_{\overline{g}}(u)/\|u\|^2_{L^{2n/(n-2)}(\overline{g})},$$

and hence

$$Y_{\overline{C}}(W,M;C) = \inf_{u \in C^\infty_+(W),\ \frac{\partial u}{\partial \nu}|_M = 0} E_{\overline{g}}(u)/\|u\|^2_{L^{2n/(n-2)}(\overline{g})}.$$

In order to prove the last equality in (2.1), it is enough to prove the following

$$\inf_{u \in C^\infty_+(W),\ \frac{\partial u}{\partial \nu}|_M = 0} E_{\overline{g}}(u)/\|u\|^2_{L^{2n/(n-2)}(\overline{g})} \leq \inf_{f \in L^{1,2}(W),\ f \not\equiv 0} E_{\overline{g}}(f)/\|f\|^2_{L^{2n/(n-2)}(\overline{g})}. \qquad (2.3)$$

We notice that

$$\inf_{u\in C^\infty_+(W),\ \frac{\partial u}{\partial\nu}|_M=0} E_{\overline{g}}(u)/\|u\|^2_{L^{2n/(n-2)}(\overline{g})}$$
$$= \inf\left\{ E_{\overline{g}}(f)/\|f\|^2_{L^{2n/(n-2)}(\overline{g})} \mid f\in L^{1,2}(W),\ f \text{ is smooth near } M \text{ and } \frac{\partial f}{\partial\nu}\Big|_M = 0\right\}.$$

Let r be the distance function to M in W with respect to $\overline{g}$. Let $(x,r)=(x^1,\cdots,x^{n-1},r)$ denotes a Fermi coordinate system on a tubular neighborhood $U_\varepsilon(M)=\{z\in W\mid r(z)<\varepsilon\}$ of M, where ε is a small positive constant. For each smooth function $f\in C^\infty(W)$ with $f\not\equiv 0$, we define a Lipschitz function f_ε by

$$f_\varepsilon(z)=\begin{cases} f(z) & \text{if } z\in W\setminus U_\varepsilon(M)\\ f(x,\varepsilon) & \text{if } z=(x,r)\in U_\varepsilon(M).\end{cases}$$

Note that f_ε is smooth near M and $\frac{\partial f_\varepsilon}{\partial\nu}=0$ along M. Then we have

$$E_{\overline{g}}(f_\varepsilon)=E_{\overline{g}}(f)-\frac{4(n-1)}{n-2}\int_{U_\varepsilon(M)}(\frac{\partial f}{\partial r})^2\,dv_{\overline{g}}\ +\ O(\varepsilon^2)$$
$$\|f_\varepsilon\|^2_{L^{2n/(n-2)}(\overline{g})}=\|f\|^2_{L^{2n/(n-2)}(\overline{g})}\ +\ O(\varepsilon^2),$$

and hence

$$E_{\overline{g}}(f_\varepsilon)/\|f_\varepsilon\|^2_{L^{2n/(n-2)}(\overline{g})}$$
$$=E_{\overline{g}}(f)/\|f\|^2_{L^{2n/(n-2)}(\overline{g})}-\frac{4(n-1)}{(n-2)}\int_{U_\varepsilon(M)}(\frac{\partial f}{\partial r})^2\,dv_{\overline{g}}\Big/\ \|f\|^2_{L^{2n/(n-2)}(\overline{g})}\ +\ O(\varepsilon^2).$$

This implies the inequality (2.3). □

The Yamabe problem on W related to the relative Yamabe constant $Y_{\overline{C}}(W,M;C)$ was solved by Cherrier [C] and Escobar [E2] under some restrictions (cf. [Au2], [LP], [S1]).

Theorem 2.3.

(1) ([C, Section 6]) *There exists a relative Yamabe metric* $\check{g}\in\overline{C}^0$ *provided*

$$Y_{\overline{C}}(W,M;C)<Y(S^n_+,S^{n-1})=n(n-1)\cdot \mathrm{Vol}_{\overline{h}}(S^n_+)^{2/n}.$$

(2) ([E2, Theorem 6.1]) *Assume that* $W=(W^n,\overline{C})$ *satisfies any of the following three condition :*

(i) $n=3,4,$ *or* $5,$

(ii) W *has a nonumbilic point on* $M=\partial W,$

(iii) $M=(M,C=\partial\overline{C})$ *is umbilic, and either* W *is conformally flat or* $n\geq 6$ *and the Weyl curvature tensor of* $\overline{C}$ *does not vanish identically on* M.

Then there exists a relative Yamabe metric $\check{g}\in\overline{C}^0$.

Standard calculation combined with (2.1), (2.2) also gives the following (cf. [Au2], [LP]):

Proposition 2.4.

(1) ([E2, Lemma 1.1]) *For each conformal class* $\overline{C} \in \mathcal{C}(W)$, *there exists a metric* $\overline{g} \in \overline{C}^0$ *whose scalar curvature does not change sign. The sign is uniquely determined by the conformal structure* $\overline{C}$, *and so there are mutually exclusive possibilities :* $\overline{C}$ *admits a metric of* (i) *positive,* (ii) *negative, or* (iii) *identically zero scalar curvature with the minimal boundary condition.*

(2) *Each relative Yamabe metric* $\breve{g} \in \overline{C}^0$ *is a metric of constant scalar curvature* $R_{\breve{g}} = Y_{\overline{C}}(W, M; C) \cdot \mathrm{Vol}_{\breve{g}}(W)^{-2/n}$.

(3) ([E3, Theorem 4.1]) *Let* $\overline{g}$ *be a metric of nonpositive constant scalar curvature on* W *with the minimal boundary condition. If* $\overline{g}'$ *is another metric of constant scalar curvature in* $[\overline{g}]^0$, *then* $\overline{g}' = c\overline{g}$ *for some positive constant* c. *Thus this implies that* $\overline{g}$ *is a relative Yamabe metric.*

(4) (cf. [K, Lemma 1.6]) *Let* $\overline{C} \in \mathcal{C}_C(W)$ *be a conformal class of nonpositive Yamabe constant* $Y_{\overline{C}}(W, M; C)$. *Then, for any* $\overline{g} \in \overline{C}^0$

$$(\min R_{\overline{g}}) \cdot \mathrm{Vol}_{\overline{g}}(W)^{2/n} \leq Y_{\overline{C}}(W, M; C) \leq (\max R_{\overline{g}}) \cdot \mathrm{Vol}_{\overline{g}}(W)^{2/n}.$$

Similar to the classic Yamabe problem, a metric of positive constant scalar curvature with the minimal boundary condition is not always a relative Yamabe metric. The following uniqueness result on Einstein metrics is a generalization of the corresponding one of Obata [O] to manifolds with boundary (cf. [S2, Proposition 1.4]).

Proposition 2.5. ([E1, Theorem 4.1]) *Let* $\hat{g}$ *be an Einstein metric of positive scalar curvature on* W *with totally geodesic boundary. If* $\overline{g}$ *is another metric of constant scalar curvature in* $[\hat{g}]^0$, *then* $\overline{g}$ *is Einstein. Moreover, if* $(W^n, \hat{g})$ *is not conformally equivalent to* $(S^n_+, \overline{h})$, *then* $\overline{g} = c\hat{g}$ *for some positive constant* c. *Hence this implies that* $\hat{g}$ *is a relative Yamabe metric.*

The second variation of I on $\overline{C}$ at a relative Yamabe metric $\breve{g} \in \overline{C}^0$ of positive scalar curvature implies the following estimate, which is a characterization of relative Yamabe metrics in the positive case.

Proposition 2.6. (cf. [Au2, Proposition 5.24]) *For each relative Yamabe metric* $\breve{g}$ *of positive scalar curvature on* W, *the first nonzero eigenvalue* $\nu_1(-\triangle_{\breve{g}})$ *of* $-\triangle_{\breve{g}}$ *for the Neumann boundary condition can be estimated from below :*

$$\nu_1(-\triangle_{\breve{g}}) \geq \frac{R_{\breve{g}}}{n-1}.$$

The characterization (2.1) of relative Yamabe constants also leads to the following Sobolev inequality on W with a relative Yamabe metric $\breve{g}$ of positive scalar curvature:

$$(2.4) \qquad \Big(\int_W |f|^{2n/(n-2)}\, dv_{\breve{g}}\Big)^{(n-2)/n} \leq \frac{1}{c_{[\breve{g}]}} \int_W |df|^2_{\breve{g}}\, dv_{\breve{g}} + \frac{1}{\mathrm{Vol}_{\breve{g}}(W)^{2/n}} \int_W f^2\, dv_{\breve{g}}$$

for $f \in L^{1,2}(W)$, where $c_{[\check{g}]} = \frac{n-2}{4(n-1)} \cdot Y_{[\check{g}]}(W, M; \partial[\check{g}]) > 0$. Using the Moser iteration method on the Sobolev inequality (2.4), we obtain

Proposition 2.7. (cf. [Ak1, Section 2]) *There exists a positive constant $\delta_n > 0$ depending only on $n = \dim W$ such that the following holds :*

$$(1) \qquad \mathrm{Vol}_{\check{g}}(B_r(p)) \geq \delta_n c_{[\check{g}]}^{n/2} r^n \quad \text{for } p \in W \text{ and } r \leq c_{[\check{g}]}^{-1/2} \mathrm{Vol}_{\check{g}}(W)^{1/n},$$

where $B_r(p)$ denotes the geodesic ball of radius r centered at p with respect to $\check{g}$.

$$(2) \qquad \mathrm{diam}_{\check{g}}(W) \leq 2\delta_n^{-1} c_{[\check{g}]}^{-1/2} \mathrm{Vol}_{\check{g}}(W)^{1/n}.$$

3. Relative Yamabe invariants

Let W be a compact smooth n-manifold ($n \geq 3$) with boundary M (possibly $M = \emptyset$). We remark that the relative Yamabe invariant $Y(W, M)$ is nothing but the Yamabe invariant $Y(W)$ of the closed manifold W when M is empty. From the characterization (2.1) of relative Yamabe constants, every technique of analyzing Yamabe invariants is available for $Y(W, M)$ on the interior $\mathrm{Int}(W)$ of W, under suitable modification. Thus the technique developed by Kobayashi [K] implies the following result, which is a fundamental tool for computations of the relative Yamabe invariant.

Theorem 3.1. (cf. [K, Theorem 2]) *Let W_1 and W_2 be compact connected smooth n-manifolds ($n \geq 3$) with boundary M_1 and M_2 respectively. Then*

$$Y(W_1 \sharp W_2, M_1 \amalg M_2) \geq \begin{cases} -(|Y(W_1, M_1)|^{n/2} + |Y(W_2, M_2)|^{n/2})^{2/n} & \\ & \text{if } Y(W_1, M_1), Y(W_2, M_2) \leq 0, \\ \min\{Y(W_1, M_1), Y(W_2, M_2)\} & \text{otherwise.} \end{cases}$$

Here $W_1 \sharp W_2$ denotes the connected sum of W_1 and W_2.

Similar to the classic Yamabe invariant, the minimax definition of relative Yamabe invariants is also rather unwieldy for many purposes. However, when the relative Yamabe invariant of a manifold is nonpositive, the following gives a very useful reinterpretation of the invariant.

Proposition 3.2. (cf. [An], [BCG], [Le1]) *Let W be a compact connected smooth n-manifold with boundary M. Then*

$$\inf_{\overline{g} \in \mathcal{R}\mathrm{iem}^0(W)} \int_W |R_{\overline{g}}|^{n/2} dv_{\overline{g}} = \begin{cases} 0 & \text{if } Y(W, M) \geq 0, \\ |Y(W, M)|^{n/2} & \text{if } Y(W, M) \leq 0. \end{cases}$$

Here $\mathcal{R}\mathrm{iem}^0(W) = \{\overline{g} \in \mathcal{R}\mathrm{iem}(W) \mid H_{\overline{g}} = 0 \quad \text{along } M\}$.

Using Proposition 3.2, we obtain the following surgery theorems corresponding to ones of Petean-Yun [PY] and Petean [P].

Theorem 3.3. (cf. [PY, Theorem 1]) *Let W_1 and W_2 be compact connected smooth n-manifolds with boundary M_1 and M_2 respectively. Suppose that a closed smooth k-manifold S embeds into both* $\mathrm{Int}(W_1)$ *and* $\mathrm{Int}(W_2)$ *with trivial normal bundle. Assume that $k \leq 3$. Let $W^S_{1,2}$ be the manifold obtained by gluing W_1 and W_2 along S. Then*

(i) *If $Y(W_1, M_1) \leq 0$ and $Y(W_2, M_2) \leq 0$, then*
$Y(W^S_{1,2}, M_1 \amalg M_2) \geq -(|Y(W_1, M_1)|^{n/2} + |Y(W_2, M_2)|^{n/2})^{2/n}$,

(ii) *If $Y(W_1, M_1) \leq 0$ and $Y(W_2, M_2) > 0$, then $Y(W^S_{1,2}, M_1 \amalg M_2) \geq Y(W_1, M_1)$.*

Theorem 3.4. (cf. [PY, Corollary 1], [P, Theorem 1]) *Let W be a compact connected smooth manifold of dimension $n \geq 4$ with boundary M. Let $\widehat{W}$ be a manifold obtained from W by performing surgery on* $\mathrm{Int}(W)$ *of codimension q $(1 \leq q \leq n)$. Suppose that $Y(W, M) \leq 0$. Then*

(i) *If $q \geq 3$, then $Y(\widehat{W}, M) \geq Y(W, M)$.*

(ii) *If q is different from $1, 2$ and $n-1$, then $Y(\widehat{W}, M) = Y(W, M)$.*

Remark 3.5. In [AB1], we developed approximation technique which leads to gluing theorems of the boundary connected sum two manifolds along their boundaries for the relative Yamabe invariant. Using the technique, we showed that there are many examples of manifolds with positive and nonpositive relative Yamabe invariants (see also [Ak2]). In particular, we constructed a family of 4-manifolds with strictly negative relative Yamabe invariant. In [AB2], we are studying the conformal cobordism theory of manifolds with positive conformal classes in terms of the relative Yamabe invariant.

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