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COMPLETE SUBMANIFOLDS IN SPHERES*

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Dedicated to Professor Katsuhiro Shiohama on his sixtieth birthday

It is the purpose of this article to discuss complete submanifolds in spheres. Complete submanifolds with parallel mean curvature vector in spheres and complete submanifolds in spheres with constant scalar curvature will be mentioned. A result of complete submanifolds in Euclidean spaces with constant scalar curvature is also announced.

1. SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR

In this section, we shall study submanifolds with parallel mean curvature vector in spheres. First of all, we consider minimal submanifolds. Let M be an n-dimensional compact minimal submanifold in the unit sphere $S^{n+p}(1)$. Then the following Theorem due to Chern, do Carmo and Kobayashi [9] is well-known:

Theorem 1.1. *If M is an n-dimensional compact minimal submanifold in the unit sphere $S^{n+p}(1)$ with $S \le \frac{n}{2-\frac{1}{p}}$, then M is totally geodesic, or $p = 1$ and M is isometric to the Clifford torus, or $p = n = 2$ and M is isometric to the Veronese surface. Where S denotes the squared norm of the second fundamental form.*

Remark 1.1. In [10], Li and Li extended the pinching constant $S \le \frac{n}{2-\frac{1}{p}}$ to $S \le \frac{2n}{3}$ if the codimension p is greater than 1.

In particular, when M is a minimal hypersurface, Cheng and Yang [7] extended the result due to Chern, do Carmo and Kobayashi as the following:

Theorem 1.2. *Let M be a compact minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $S \le n + \frac{n}{3}$, then $S = 0$ and M is totally geodesic or else $S = n$ and M is isometric to the Clifford torus.*

In particular, when $n = 3$, Cheng and Wan [6] obtained the following

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Theorem 1.3. *The totally geodesic sphere, the Clifford torus and the tube of the Veronese surface are the only complete minimal hypersurfaces in $S^4(1)$ with constant scalar curvature.*

As a generalization of the result due to Chern, do Carmo and Kobayashi, Nomizu and Smyth [12] studied hypersurfaces with constant mean curvature in the unit sphere and proved the following:

Theorem 1.4. *Let M be an n-dimensional hypersurface with constant mean curvature in the unit sphere $S^{n+1}(1)$.*

(1) *If M is compact and the sectional curvature of M is nonnegative, then M is isometric to the totally umbilical sphere or the Riemannian product $S^{n-k}(c_1) \times S^k(c_2)$ of spheres.*
(2) *When M is complete and the scalar curvature of M is constant, if the sectional curvature of M is nonnegative, then M is isometric to the totally umbilical sphere or the Riemannian product $S^{n-k}(c_1) \times S^k(c_2)$ of spheres.*

Proof. By a direct computation, we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2 R_{ijij}. \tag{1.1}$$

(1). If M is compact, we have that principal curvatures are constant and the number of distinct principal curvatures is at most two from the Stokes formula because the sectional curvature of M is nonnegative. The classification of isoparametric hypersurfaces due to Cartan yields that Theorem 1.4 holds.

(2). If M is complete and the scalar curvature is constant, we also have that principal curvatures are constant and the number of distinct principal curvatures is at most two from (1.1) because the sectional curvature of M is nonnegative. This finished the proof of Theorem 1.4.

In particular, when $n = 3$, Cheng and Wan [6] completely classified complete hypersurfaces with constant mean curvature and constant scalar curvature in the unit sphere $S^4(1)$, that is, the following is proved:

Theorem 1.5. *Let M be a 3-dimensional complete hypersurface with constant mean curvature in the unit sphere $S^4(1)$. If the scalar curvature is constant, then M is isometric to the totally umbilical sphere, the Riemannian product $S^{n-k}(c_1) \times S^k(c_2)$ of spheres ($k = 1, 2$) or the isoparametric hypersurface with three distinct principal curvature due to Cartan.*

For general cases, submanifolds with parallel mean curvature vector in spheres were researched by many mathematician (for examples, Alencar and do Carmo [1], Cheng and Nakagawa [4], Shiohama and Xu [14], Yano and Ishihara [15], Yau [18] and so on). In particular, Shiohama and Xu [14] proved the following:

Theorem 1.6. *Let M be an n-dimensional compact submanifold with parallel mean curvature vector in the unit sphere $S^{n+p}(1)$. If*

$$S \leq C(n,p,H),$$

then M is isometric to one of the following:
(1) *the totally umbilical sphere;*
(2) *the Riemannian product $S^{n-1}(c_1) \times S^1(c_2)$ of spheres;*
(3) *the Veronese surface.*

where

$$C(n,p,H) = \begin{cases} n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \\ \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\{n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \\ \frac{1}{3}(2n + 5nH^2)\}, \text{ for } p \geq 3 \text{ or } p = 2 \text{ and } H = 0. \end{cases}$$

Where H denotes the mean curvature.

2. Submanifolds with constant scalar curvature

In this section, we shall consider submanifolds in sphere with constant scalar curvature. When the codimension is one, in [8], S.Y. Cheng and Yau proved the following:

Theorem 2.1. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, if $r \geq 1$ and the sectional curvature of M is non-negative, then M is isometric to either the totally umbilical hypersurface or the Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$ $1 \leq k \leq n-1$, where $S^k(c)$ denote the sphere of radius c.*

Proof. By a direct calculation, we have

$$\Box(nH) = \sum_{i,j,k} h_{ijk}^2 - n^2|\text{grad}H|^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2 K_{ij}, \tag{2.1}$$

where K_{ij} is the sectional curvature for the 2-plane spanned by e_i and e_j and λ_i, for $i = 1, 2, \cdots, n$ are the principal curvatures. Since the scalar curvature is constant, we have from the Gauss equation, we have

$$2n^2H\nabla_k H = 2\sum_i \lambda_i h_{iik}.$$

Hence, we have, from $r \geq 1$,

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2|\text{grad}H|^2. \tag{2.2}$$

From Stokes formula, we obtain that the mean curvature H is constant since the sectional curvature is non-negative. Therefore, from the Theorem 1.4, we know that Theorem 2.1 is true.

Remark 2.1. The differential operator $\Box$ is defined by, for any differentiable function f defined on M,

$$\Box f = \sum_{i,j=1}^{n} (nH - h_{ij})\nabla_i \nabla_j f,$$

where H and h_{ij} denote the mean curvature and the components of the second fundamental form of M, respectively.

By making use of the similar method which was used by Nakagawa and the author in [4] and the above differential operator $\Box$ introduced by S.Y. Cheng and Yau, Li [11] proved the following:

Theorem 2.2. *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, if $r \geq 1$ and*

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}, \tag{2.3}$$

then M is isometric to either the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$, where S is the squared norm of the second fundamental form of M.

Proof. Since (2.3) holds, we can prove that

$$\sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \geq 0.$$

Hence, from the same assertions as in proof of Theorem 2.1, we can infer that the mean curvature H is constant and M is an isoparametric hypersurface with at most two distinct principal curvatures. Hence, Theorem 2.2 is proved.

Remark 2.2. We should notice that the condition $r \geq 1$ plays an essential role in the proofs of the above Theorem 2.1 and Theorem 2.2.

On the other hand, we consider the following example:

Example 2.1. For any $0 < c < 1$, by considering the standard immersions $S^{n-1}(c) \subset \mathbf{R}^n$, $S^1(\sqrt{1-c^2}) \subset \mathbf{R}^2$ and taking the Riemannian product immersion $S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow \mathbf{R}^2 \times \mathbf{R}^n$, we obtain a compact hypersurface $S^1(\sqrt{1-c^2})$ $\times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r > 1 - \frac{2}{n}$.

The Example 2.1 shows that not all Riemannian products $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ appear in these results in Theorem 2.1 and Theorem 2.2. Since the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has only two distinct principal curvatures and its scalar curvature $n(n-1)r$ is constant and satisfies $r > 1 - \frac{2}{n}$. Hence, we would like to ask the following:

Problem 2.1. *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If M is of only two distinct principal curvatures one of which is simple, then, does $r > 1 - \frac{2}{n}$ hold?*

In the following Theorem 2.3 we answer this Problem 2.1, affirmatively, and prove that if $r \neq \frac{n-2}{n-1}$, then $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ is the only complete hypersurfaces in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures one of which is simple such that $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$.

Theorem 2.3. *Let M be an n-dimensional complete hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures one of which is simple. Then $r > 1 - \frac{2}{n}$ and, when $r \neq \frac{n-2}{n-1}$, if $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

Furthermore, we consider the case that $r = \frac{n-2}{n-1}$ holds, we infer the following:

Theorem 2.4.

(1) *Let M be a complete hypersurface with two distinct principal curvatures one of which is simple in $S^{n+1}(1)$. If $r = \frac{n-2}{n-1}$, then $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$ and if $S = n$ holds on M, then M is isometric to the Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$;*

(2) *There are no complete hypersurfaces with two distinct principal curvatures one of which is simple in $S^{n+1}(1)$ such that $r = \frac{n-2}{n-1}$ and $S > (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$.*

Remark 2.3. The Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$ is a complete minimal hypersurface in $S^{n+1}(1)$ such that $r = \frac{n-2}{n-1}$ and $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$.

From Theorem 2.3, Theorem 2.4, we have

Corollary 2.1. *Let M be an n-dimensional ($n > 3$) complete locally conformally flat hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. Then $r > 1 - \frac{2}{n}$ and, when $r \neq \frac{n-2}{n-1}$, if $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

Proof. Since M is a locally conformally flat hypersurface, we know that M is of at most two distinct principal curvatures and one of them is of at least $n-1$ multiplicities. If at some point p, these principal curvatures are equal with each other, then $S = nH^2$ at this point p. From the Gauss equation, we have $S = nH^2 = n(r-1) \geq 0$. Since $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, we have $n(r-1) \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$. This is impossible. Hence M is of two distinct principal curvatures and one of them is of $n-1$ multiplicities. From the Theorem 2.3, we know that M is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$. That is, Corollary 2.1 is proved.

Corollary 2.2. *There are no complete locally conformally flat n-dimensional ($n > 3$) hypersurfaces in $S^{n+1}(1)$ such that $r = \frac{n-2}{n-1}$ and $S > (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$.*

Proof. By making use of Theorem 2.4 and the same argument in Corollary 2.1, we know that Corollary 2.2 is true.

From Theorem 2.3, Theorem 2.4 and Example 2.1, it is interesting to generalize these results in Theorem 2.1 and Theorem 2.2 to the case $r > 1 - \frac{2}{n}$. That is, it is interesting to prove the following:

Problem 2.2. *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to either the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$.*

When $r = \frac{n-2}{n-1}$, we answer this Problem 2.2, affirmatively, in the following Theorem 2.5. For the general case, we can not answer it yet.

Theorem 2.5. *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ ($r = \frac{n-2}{n-1}$) in $S^{n+1}(1)$. If $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is isometric to the Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$.*

Proof. Since $r = \frac{n-2}{n-1}$, we know $n(r-1)+2 = r$. From the Gauss equation, we have

$$\begin{aligned} 0 &\leq n^2H^2 \\ &= S + n(n-1)(r-1) \\ &\leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} + n(n-1)(r-1) \\ &= (n-1)\frac{r}{n-2} + \frac{n-2}{r} + n(n-1)(r-1) \\ &= 1 + (n-1) - n = 0. \end{aligned}$$

Hence, we infer $H = 0$ on M, that is, M is minimal and $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$. Therefore, the assertion in Theorem 2.5 is true from the Theorem 1.1.

In order to prove Theorem 2.3 and Theorem 2.4, we consider n-dimensional hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. From now on, we assume that M is an n-dimensional complete hypersurface with constant scalar curvature and with two distinct principal curvatures in $S^{n+1}(1)$ and one of these two distinct principal curvatures is simple, that is, we assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda \quad \text{and} \quad \lambda_n = \mu.$$

The following Lemma 2.1 and Lemma 2.2 hold.

Lemma 2.1. *If M is of two distinct principal curvatures and λ is the principal curvature of $n-1$ multiplicities, we obtain the following:*

(1) *When $r-1 \geq 0$, $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds if and only if $w^{-n} = \lambda^2 - r + 1 \geq \frac{2r}{n-2}$.*

(2) *When $1-\frac{2}{n} < r < 1$ and $r \neq \frac{n-2}{n-1}$, $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds if and only if one of the following conditions holds*

(a). $w^{-n} = \lambda^2 - r + 1 \geq \frac{2r}{n-2}$ and $\lambda^2 \geq 1-r$

(b). $w^{-n} = \lambda^2 - r + 1 \leq \frac{2r}{n-2}$ and $\lambda^2 < 1-r$.

Lemma 2.2. *Let M be of two distinct principal curvatures and let λ denote the principal curvature of $n-1$ multiplicities. If $r \leq 1-\frac{2}{n}$, then $\frac{dw(s)}{ds}$ is a monotone increasing function of s.*

Proof of Theorem 2.3. From the assumption of Theorem 2.3, we know that if w is constant, then λ is constant because of $w^{-n} = \lambda^2 - r + 1$. Hence, the Theorem 2.3. is true. Next, we shall prove that w must be constant. If w is not constant, we shall obtain a contradiction. In fact, since $w^{-n} = \lambda^2 - r + 1$ satisfies the equation

$$\frac{d^2w}{ds^2} - w(\frac{n-2}{2}\frac{1}{w^n} - r) = 0, \tag{2.4}$$

we know that $w(s)$ is a function defined in $(-\infty, +\infty)$ because M is complete and the integral curve of principal vector field corresponding to μ is a geodesic. From Lemma 2.1, we conclude the following:

(1) if $r-1 \geq 0$, then $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds if and only if $w^{-n} = \lambda^2 - r + 1 \geq \frac{2r}{n-2}$.

(2) if $1-\frac{2}{n} < r < 1$ and $r \neq \frac{n-2}{n-1}$, $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds if and only if one of the following conditions holds

(a). $w^{-n} = \lambda^2 - r + 1 \geq \frac{2r}{n-2}$ and $\lambda^2 \geq 1-r$

(b). $w^{-n} = \lambda^2 - r + 1 \leq \frac{2r}{n-2}$ and $\lambda^2 < 1-r$.

From the equation (2.4), we obtain

(1) if $r \leq 1-\frac{2}{n}$, then from Lemma 2.2, we know $\frac{dw}{ds}$ is an increasing function of s

(2) if $r-1 \geq 0$, then

$$\frac{d^2w}{ds^2} \geq 0;$$

(3) if $1-\frac{2}{n} < r < 1$ and $r \neq \frac{n-2}{n-1}$, then one of the following inequalities:

$$\frac{d^2w}{ds^2} \geq 0$$

and

$$\frac{d^2w}{ds^2} \leq 0$$

is satisfied.

Hence, either $\frac{dw}{ds}$ is an increasing function of s or $\frac{dw}{ds}$ is a decreasing function of s if w is not constant. Hence, we infer that it is impossible from the following Proposition 2.1. The proof of Theorem 2.3. is completed.

Proposition 2.1. *Let M be an n-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures in $S^{n+1}(1)$. Let λ denote the principal curvature of $n-1$ multiplicities. Then $w(s)$ is constant if $\frac{dw(s)}{ds}$ is a monotone function of s.*

Proof of Theorem 2.4. (1). Since $r = \frac{n-2}{n-1}$, we have

$$\begin{aligned} S &= (n-1)\lambda^2 + \mu^2 \\ &= (n-1)\lambda^2 + (\frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda)^2 \\ &= \frac{n^2}{4}\lambda^2 + \frac{n^2(r-1)^2}{4\lambda^2} - \frac{n(n-2)}{2}(r-1) \\ &= \frac{n^2}{4}(\lambda - \frac{1}{(n-1)\lambda})^2 + n \\ &\geq n. \end{aligned}$$

And $S = n$ at some point if and only if $\lambda - \frac{1}{(n-1)\lambda} = 0$, that is, $\lambda^2 = \frac{1}{n-1}$ at this point. From the Gauss equation, we know that M is minimal at this point. Hence, if $S = n$ on M holds, then M is a minimal hypersurface. From the theorem 1.1, we know that M is isometric to the Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$.

(2). Since $r = \frac{n-2}{n-1}$, by making use of the same proof as one in Theorem 2.3, we know that $S > (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds if and only if one of the following conditions holds

(a). $w^{-n} = \lambda^2 - r + 1 > \frac{2r}{n-2}$ and $\lambda^2 > 1 - r$

(b). $w^{-n} = \lambda^2 - r + 1 < \frac{2r}{n-2}$ and $\lambda^2 < 1 - r$.

and $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$ holds if and only if $\lambda^2 = 1 - r$ holds. Since $S > (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds and λ^2 is a continuous function of s, we conclude that $\lambda^2 \neq 1 - r$. Hence, we infer that either $\frac{dw}{ds}$ is an increasing function of s or $\frac{dw}{ds}$ is a decreasing function of s if w is not constant. Hence, we infer that $w(s)$ is constant from the above Proposition 2.1. From (2.4) we have $\lambda^2 = \frac{1}{n-1}$. Hence, $S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2} = n$. This is impossible. The proof of Theorem 2.4 is completed.

For submanifolds with higher codimensions, we obtained in [2]:

Theorem 2.6. *Let M^n be an n-dimensional $(n > 2)$ compact submanifold with constant scalar curvature $n(n-1)r$ satisfying $r > 1$ in the unit sphere $S^{n+p}(1)$. If*

$$S \leq \alpha(n, r),$$

then M^n is isometric to the totally umbilical sphere $S^n(r)$ or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c = \frac{n-2}{nr}$, where S is the squared norm of the second fundamental form of M^n and

$$\alpha(n,r) = \begin{cases} (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}, & \text{for } p \leq 2, \\ n(r-1) + \frac{3n-5+(n^2-n-1)(r-1)}{2n^2r^2} \\ -\frac{(n-2)\sqrt{4+2(3n-1)(r-1)+(n^2+2n-2)(r-1)^2}}{2n^2r^2}, & \text{for } p \geq 3. \end{cases}$$

3. Submanifolds in Euclidean spaces

In [5], Cheng and Nonaka proved that a complete n-dimensional submanifold with parallel mean curvature vector in a Euclidean space $\mathbf{R}^{n+p}$ is isometric to the totally umbilical sphere, or the totally geodesic Euclidean space $\mathbf{R}^n$ or the generalized cylinder $\mathbf{R}^1 \times S^{n-1}$ if $S \leq \frac{n^2H^2}{n-1}$ is satisfied. Where S and H denote the squared norm of the second fundamental form and the mean curvature, respectively. In [2], the author studied submanifolds with constant scalar curvature in Euclidean spaces and successfully proved the following:

Theorem 3.1. *The totally umbilical sphere, the totally geodesic Euclidean space $\mathbf{R}^n$ and the generalized cylinder $\mathbf{R}^1 \times S^{n-1}$ are the only complete n-dimensional submanifolds with constant scalar curvature $n(n-1)r$ in a Euclidean space $\mathbf{R}^{n+p}$, which satisfy $S \leq \frac{n(n-1)r}{n-2}$. Where S denotes the squared norm of the second fundamental form.*

Remark 3.1. The investigation of hypersurfaces with constant scalar curvature in Euclidean spaces was been done very well, but the study of submanifolds with higher codimensions and with constant scalar curvature in Euclidean spaces is not done almost. Hence, the result is very important for the forward research of submanifolds with constant scalar curvature.

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