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TOTAL CURVATURE FOR OPEN SUBMANIFOLDS OF EUCLIDEAN SPACES

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ABSTRACT. The classical Cohn-Vossen inequality states that for any Riemannian 2-manifold the difference between $2\pi\chi(M)$ and the total curvature $\int_M KdA$ is always nonnegative. For complete open surfaces in E^3 this curvature defect can be interpreted in terms of the length of the curve "at infinity". The goal of this paper is to investigate higher dimensional analogues for open submanifolds of euclidean space with cone-like ends. This is based on the extrinsic Gauss-Bonnet formula for compact submanifolds with boundary and its extension "to infinity". It turns out that the curvature defect can be positive, zero, or negative, depending on the shape of the ends "at infinity". Furthermore we study the variational problem for the total curvature of hypersurfaces where the ends are not fixed. It turns out that for open hypersurfaces with cone-like ends the total curvature is stationary if and only if each end has vanishing Gauss-Kronecker curvature in the sphere "at infinity". For this case of stationary total curvature we prove a result on the quantization of the total curvature.

1. Introduction. The total curvature of Riemannian manifolds and submanifolds has been a field of active research during the last 150 years. For compact manifolds the so-called *Gauss-Bonnet theorem* is a milestone in differential geometry, both in an extrinsic and an intrinsic version. It states that a certain curvature quantity of the interior of a compact manifold plus another curvature quantity of the boundary (including a discussion of angles if there are any) equals the Euler characteristic, up to a constant depending only on the dimension. The intrinsic version for even-dimensional manifolds is nowadays often called the *Gauss-Bonnet-Chern theorem.* The extrinsic version is closely related with the Hopf index theorem, with the mapping degree of the Gauss map and with the study of critical points of height functions.

In the non-compact case Cohn-Vossen investigated the total curvature of a complete open 2-manifold. In this case the boundary term is missing, and therefore in general the same equality between the total curvature and the Euler characteristic cannot hold. However, the missing boundary term is always nonnegative, leading to the so-called *Cohn-Vossen inequality.*

For higher-dimensional open manifolds, this missing boundary term is much less understood, neither extrinsically nor intrinsically. In any case one has to assume that the manifold is of finite topology and that the curvature is globally absolutely integrable.

For hypersurfaces or submanifolds of euclidean space $\mathbb{E}^{n+1}$ an extrinsic version was investigated by P.Wintgen by means of the set of *limit directions*. By definition this set is the part of the unit sphere $S^n \subset \mathbb{E}^{n+1}$ which appears as the compactification of M "at infinity". If the submanifold behaves "asymptotically cone-like" at the ends (in a sense to be specified below), then the ordinary Gauss-Bonnet theorem implies the following result:
Theorem 1. *If $M^n \subset \mathbb{E}^{m+1}$ is a complete submanifold with finitely many cone-like ends, then the difference between the Euler characteristic and the total curvature can be explicitly expressed as a sum of the even higher total mean curvatures of the set "at infinity", denoted by $M_\infty \subset S^m$:*

$$c_m\chi(M) - TC_n(M) = \sum_{0\leq 2i\leq n-1} \frac{c_m}{c_{m-n+2i}c_{n-1-2i}} TC_{2i}(M_\infty)$$

Here c_k denotes the volume of the k-dimensional unit sphere, and TC_k denotes the (non-normalized) k^{th} extrinsic *total curvature* in $\mathbb{E}^{m+1}$ or in S^m, respectively.
This expression allows a further discussion of the validity of the Cohn-Vossen inequality. It turns out that there is a simple 4-dimensional example in euclidean 5-space where this inequality does not hold. Remarkably enough, for this example the total curvature is *stationary* in the class of all submanifolds with cone-like ends. In more generality the variation of the total curvature leads to the following:
Theorem 2. *Let $M^n \subset \mathbb{E}^{n+1}$ be a complete open hypersurface with finitely cone-like ends, n even. Then the gradient of the total curvature functional is the Gauss-Kronecker curvature of the hypersurface "at infinity".*

This raises the question for a classification of compact hypersurfaces in the standard unit sphere with vanishing Gauss-Kronecker curvature. One can also ask for the possible values of the total curvature in the stationary case. There is a partial result as follows:
Theorem 3. *Let $M^4 \subset \mathbb{E}^5$ be a complete open hypersurface with finitely many cone-like ends and with stationary total curvature. Assume that for each end the rank of the shape operator in the sphere "at infinity" is constant. Then the normalized total curvature takes values in the integers:*

$$\frac{3}{4\pi^2}\int_M K dV \in \mathbb{Z}.$$

Details and proofs will appear elsewhere.

2. The Cohn-Vossen inequality. For a compact oriented (and connected) Riemannian 2-manifold (M, g) with boundary ∂M the classical Gauss-Bonnet theorem states the equation

$$2\pi\chi(M) - \int_M K dA = \int_{\partial M} \kappa(s) ds$$

where κ denotes the geodesic curvature on the oriented boundary.

In the case of non-compact 2-manifolds things are a little bit more complicated. First of all one should assume that (M, g) is complete because for non-complete metrics one cannot expect general results on the total curvature. Secondly, the quantities $\chi(M)$ and $\int_M K dA$ need not be finite numbers. If we assume that M is of finite topological type then M is homeomorphic to a closed surface $\widetilde{M}$ with a finite number $p_1, \ldots, p_k$ of points removed (called *ends*), $k \geq 1$. Finally, one has to assume that the Gaussian curvature K is absolutely integrable over M, that is, $\int_M |K| dA < \infty$. Then the following holds:
Theorem. (Cohn-Vossen) *If (M, g) is a complete Riemannian 2-manifold of finite topological type and with absolutely integrable Gaussian curvature K then the inequality*

$$2\pi\chi(M) - \int_M K dA \geq 0$$

holds. In particular we have $\int_M K dA \leq 2\pi$ if M is non-compact.

There are more subtle versions for the case that M is not of finite topological type (then we can formally set $\chi(M) = -\infty$) and that $\int_M K dA$ attains a value in the extended real numbers $[-\infty, +\infty]$. Here the statement is that the Cohn-Vossen inequality still holds. Furthermore, there are a number of additional conditions under which the Gauss-Bonnet equality $2\pi\chi(M) - \int_M K dA = 0$ continues to hold in the non-compact case.

From the Gauss-Bonnet formula it seems to be obvious that the curvature defect $2\pi\chi(M) - \int_M K dA$ can be calculated or controlled by the geodesic curvature of the boundary curves in an exhaustion

$$M_1 \subset M_2 \subset M_3 \subset \cdots \subset M$$

of the given surface M by compact surfaces M_i with boundary. However, it took a surprisingly long time until this curvature defect was well understood.

P.Wintgen suggested that the curvature defect of a complete and properly immersed surface in 3-space is the length of the set M_∞ of so-called *limit directions* $\lim_{n\to\infty} \frac{f(x_n)}{||f(x_n)||}$. He conjectured that one can always assign a finite length to this set it the total curvature is finite. Unfortunately, this is not true in general, not even if the norm of the second fundamental form is square integrable, a stronger assumption.

3. The extrinsic Gauss-Bonnet theorem. For investigating higher dimensional analogues of the classical Gauss-Bonnet formula for 2-manifolds, one can look at the integrated extrinsic curvature of a compact hypersurface. Here a suitable type of curvature is the *Gauss-Kronecker curvature* (defined as the determinant of the shape operator) $K = K_n$ where n is the dimension of the manifold. In the even-dimensional case this curvature is independent of the unit normal, in the odd-dimensional case its sign depends on the unit normal. It is well known that K is intrinsic if n is even.

NOTATIONS: In the sequel K denotes the Gauss-Kronecker curvature. The constant c_n denotes the volume of the standard unit sphere. These constants can be expressed in terms of the Gamma function as follows: $c_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$. The symbol dV denotes the volume element of a submanifold, sometimes in the form dV_M for specifying the manifold on which it is defined.

The following theorem is classical.

Theorem (Gauss-Bonnet-Hopf). *Let $M^n \subset \mathbb{E}^{n+1}$ be an embedded compact hypersurface such that M is the boundary of its interior $M_{int} \subset \mathbb{E}^{n+1}$, and let K denote the Gauss-Kronecker curvature of M with respect to the inner normal (pointing to M_{int}). Then the following hold:*

(i) $\int_M K dV_M = c_n \cdot \chi(M_{int})$

(ii) *If n is even then $\chi(M) = 2\chi(M_{int})$ and, consequently, $\int_M K dV_M = \frac{c_n}{2} \cdot \chi(M)$. Moreover, this equality holds for arbitrary immersions $f\colon M \to \mathbb{E}^{n+1}$ of a compact orientable n-manifold, even if M is not the boundary of some $(n+1)$-manifold.*

The essential difference between even and odd dimensions is that (ii) holds independently of the nature (or even the existence) of an interior M_{int}. As a matter of fact, for odd dimensions the total curvature does depend on the choice of M_{int}, i.e., on the choice of the embedding.

In the case of submanifolds of higher codimension one has to regard the so-called *Lipschitz-Killing curvature* which is defined as the determinant of the shape operator A_e in direction of a specific unit normal e

$$\langle A_e(X), Y\rangle = \langle \nabla_X Y, e\rangle.$$

Therefore integrating the curvature requires the space of all unit normals of all points (the total space of the unit normal bundle of an embedding or immersion)

$$\perp^1 (M) = \{(p,e) \mid p \in M, ||e|| = 1, e \perp T_pM\}.$$

For a submanifold $M^n \subset \mathbb{E}^{n+1}$, $\perp^1 (M)$ can be regarded as a submanifold of the tangent bundle of $\mathbb{E}^{n+1}$, or as a submanifold of $M \times \mathbb{E}^{n+1}$. This space $\perp^1 (M)$ carries a canonical orientation (induced by the outer normal) which is compatible with the orientation of the ambient space, and it caries a so-called *canonical volume form* dV_{can} which is induced from this orientation. Locally we have $dV_{can} = dV_M \wedge dV_{S^{n-m}}$.

NOTATIONS: In the sequel K or K_n denotes the Lipschitz-Killing curvature where n indicates the dimension of the manifold where it is defined. More precisely, we use the symbol $K(e)$ or $K_n(e)$ for the Lipschitz-Killing curvature in direction of a unit normal e.

Theorem. *Let $M^n \subset \mathbb{E}^{m+1}$ be an embedded compact submanifold without boundary (or an immersion of M), and let K denote the Lipschitz-Killing curvature, defined on the unit normal bundle $\perp^1 (M)$. Then the Gauss-Bonnet formula holds in the following form:*

$$\int_{\perp^1(M)} K dV_{can} = c_m \cdot \chi(M)$$

Moreover, if M is even we have $\chi(\perp^1(M)) = 2\chi(M)$.

In order to extend the extrinsic Gauss-Bonnet theorem to compact submanifolds with boundary, one has to find an appropriate analogue for the right hand boundary term in the classical formula

$$2\pi\chi(M) - \int_M K dA = \int_{\partial M} \kappa(s)ds.$$

In any case we have to distinguish between *inner points* $p \in M \setminus \partial M$ and *boundary points* $p \in \partial M$. In the interior the curvature will be defined as above, i.e., at a point p we consider the curvature

$$K(p) = \int_{e\in\perp_p^1} K_n(e)dV_{S^{m-n}}$$

and the total Lipschitz-Killing curvature

$$\int_{p\in M} K(p)dV_M = \int_{\perp^1} K_n dV_{can}.$$

At the boundary it is quite natural to consider only the *outer unit normals* and to integrate only over the set

$$\perp_+^1(\partial M) = \{(p,e) \mid p \in \partial M, ||e|| = 1, p \perp T_p\partial M, \langle e, \nu_{out}\rangle \geq 0\}$$

where ν_{out} denotes the specific *outer unit normal vector* which is tangent to M, which is perpendicular to ∂M and which points away from M.

Definition. (unit normal space, total curvature) For a compact submanifold $M^n \subset \mathbb{E}^{m+1}$ with boundary ∂M we define the *unit normal space* N^1 by

$$N^1 = \perp^1(M) \cup \perp_+^1(\partial M).$$

It carries a canonical volume form dV_{can} as in the case without boundary. Then the *total curvature* of M is the sum of the total curvatures of the two parts from $\perp^1(M \setminus \partial M)$ and from $\perp_+^1(\partial M)$:

$$TC(M, \partial M) = \int_{N^1} K dV_{can} = \int_{e\in\perp^1(M\setminus\partial M)} K_n(e)dV_{can} + \int_{e\in\perp_+^1(\partial M)} K_{n-1}(-e)dV_{can}.$$

Theorem. *For a compact submanifold $M^n \subset \mathbb{E}^{m+1}$ with boundary ∂M (or an immersion of M) the Gauss-Bonnet formula holds as follows:*

$$TC(M, \partial M) = c_m \cdot \chi(M).$$

If m is even then we have $\chi(N^1) = 2\chi(M)$.

Hence the Gauss-Bonnet difference term

$$c_m\chi(M) - \int_{\perp^1(M\setminus\partial M)} K_n dV_{can}$$

can be expressed as the integral over K_{n-1} over the set of outer unit normals at ∂M.

In view of an exhaustion of a noncompact manifold by compact manifolds with boundary, the Gauss-Bonnet defect $c_m\chi(M) - \int_{\perp^1(M\setminus\partial M)} K_n dV_{can}$ is closely related to this "outer curvature" of the "ideal boundary" in the sphere at infinity. For this purpose we first formulate the following theorem for submanifolds in the unit ball which can be regarded as a model for the euclidean space after compactification by a unit sphere at infinity.

Theorem. (Gauss-Bonnet theorem for submanifolds in the closed unit ball) *Let* $(M^n, \partial M^n) \subset (B^{m+1}, S^m)$ *be a compact submanifold which is orthogonal at the boundary, i.e., the outer normal* N *of* M *at each boundary point coincides with the outer normal of* S^m*. Then for the Gauss-Bonnet defect the equation*

$$c_m\chi(M) - \int_{\perp^1(M\setminus\partial M)} KdV_{can} = \sum_{0\le 2i\le n-1} \frac{c_m}{c_{m-n+2i}c_{n-1-2i}} \int_{\perp^1(\partial M)} K_{2i}dV_{can}$$

holds where K_j *denotes the* j^{th} *elementary symmetric function of the shape operator of the embedding* $\partial M \to S^m$*.*

Corollary. (Special cases)

1. *For a compact surface* $(M^2, \partial M^2) \subset (B^3, S^2)$ *of this type we have*

$$4\pi\chi(M) - 2\int_M KdV_M = 2\cdot length(\partial M).$$

2. *For a compact hypersurface* $(M^4, \partial M^4) \subset (B^5, S^4)$ *of this type we have*

$$\frac{8}{3}\pi^2\chi(M) - 2\int_M K_4dV_M = \frac{1}{3}\int_{\partial M}(S-2)dV_{\partial M}$$

where S denotes the scalar curvature of ∂M^4.

4. Limit directions and cone-like ends. It was the idea of P.Wintgen to study the total curvature and total absolute curvature of complete open submanifolds in $\mathbb{E}^{m+1}$ by means of limit directions. A unit vector $e \in S^m$ is called a *limit direction* if there is a sequence $(p_n)_{n\in\mathbb{N}}$ of points in M converging to one particular end such that

$$e = \lim_{n\to\infty} \frac{p_n}{||p_n||}.$$

The set of all limit directions of M is denoted by M_∞. One of Wintgen's results was that the Gauss-Bonnet theorem

$$\int_{\perp^1} KdV_{can} = c_m\chi(M)$$

holds if M is even-dimensional, if K is absolutely integrable and if there are only finitely many limit directions.

Especially, Wintgen's set M_∞ of limit directions in S^m provides an extrinsic analogue of the ideal boundary, provided that M_∞ has a reasonable structure, e.g., as a smooth submanifold of lower dimension.

Definition. (cone-like end) An end E of a complete submanifold $M^n \subset \mathbb{E}^{m+1}$ with associated component M_∞^E in the set of limit directions is said to be *(asymptotically) cone-like* if the following conditions are satisfied:

1. There is a point q such that for sufficiently large R the intersection $E \cap S^m(R;q)$ is an $(n-1)$-dimensional submanifold of the sphere of radius R around q, and
$$\lim_{R\to\infty} \frac{1}{R}(E \cap S^m(R;q)) = M_\infty^E$$
in the C^2-topology. This property is actually independent of the choice of q so we may assume that q is the origin 0.
2. For every ϵ there is a number R_0 such that for each $R > R_0$ the angle between outer unit normal of the submanifold $E \cap B^{m+1}(R;0)$ at any point $p \in E$, $||p|| = R$, and the position vector p is at most ϵ.

Theorem. *For a complete submanifold $M^n \subset \mathbb{E}^{m+1}$ with with finitely many cone-like ends the Gauss-Bonnet defect equals the total outer curvature of $M_\infty \subset S^m$ where one has to sum up over the set of ends separately:*
$$c_m\chi(M) - \int_{\perp^1} KdV_{can} = \sum_{0\leq 2i\leq n-1} \frac{c_m}{c_{m-n+2i}c_{n-1-2i}}\mathbf{K}_{2i}(M_\infty)$$
where $\mathbf{K}_j(M_\infty) = \int_{\perp^1(M_\infty)} K_j dV_{can}$ denotes the total j^{th} curvature of the set M_∞ (for each end separately), regarded as a submanifold of the unit sphere.

Corollary.
1. *If in addition all curvatures K_{2i} of M_∞ are nonnegative then the Cohn-Vossen inequality holds.*
2. *If in addition for each end M_∞^E is totally geodesic in S^m then we have*
$$\chi(M) - \frac{1}{c_m}\int_{\perp^1} KdV = k$$
where k denotes the number of ends.

Corollary. *For a 2-dimensional open surface $M^2 \subset \mathbb{E}^{m+1}$ with cone-like ends the Gauss-Bonnet defect equals the total length of $M_\infty \subset S^m$ (counted with multiplicity, i.e. for each end separately):*
$$c_m\chi(M) - \int_M KdA = \frac{c_m}{2\pi} length(M_\infty) \geq 0.$$
This implies the Cohn-Vossen inequality.
Corollary. *For an open hypersurface $M^4 \subset \mathbb{E}^5$ with cone-like ends the Gauss-Bonnet defect is*
$$\frac{8}{3}\pi^2\chi(M) - 2\int_M K_4 dV_M = \frac{1}{3}\int_{M_\infty}(S-2)dV_{M_\infty}$$
where the integral has to be taken for each end separately.
Corollary. *The Cohn-Vossen inequality does not hold in general for complete open 4-dimensional hypersurfaces in 5-space.*

Indeed, it is sufficient to construct a hypersurface such that the hypersurface at infinity has vanishing Gauss-Kronecker curvature, for instance Cartan's isoparametric hypersurface.

5. The variational problem for the total curvature. For compact hypersurfaces of euclidean space one has Reilly's formula for the variation of the higher mean curvature functionals.

Theorem. *For any compact hypersurface in euclidean space the gradient of the* i^{th} *curvature functional* $\mathbf{K_i} = \int K_i$ *is the function* $-(i+1)K_{i+1}$.

For hypersurfaces of spheres one has the following.

Theorem. *For a hypersurface in the unit n-sphere the gradient of the curvature functional* $\mathbf{K_i} = \int K_i$ *is the function* $-(i+1)K_{i+1} + (n-i)K_{i-1}$.

Using this formula we have the following.

Theorem. *The gradient of the total outer curvature functional (= the right hand side in Theorem 1) of a hypersurface in* S^n *(n even) is the negative Gauss-Kronecker curvature* $-K_{n-1}$ *of this hypersurface.*

Corollary. *The total curvature* $\int_M K_n dV$ *of an even-dimensional open hypersurface* $M \subset \mathbb{E}^{n+1}$ *with cone-like ends is stationary (within the class of such hypersurfaces having cone-like ends) if and only if each component of* M_∞ *has vanishing Gauss-Kronecker curvature in the sphere "at infinity" or, equivalently, if it has one vanishing principal curvature at each point.*

This corollary raises the question what we can say about compact hypersurfaces of even-dimensional spheres with vanishing Gauss-Kronecker curvature.

6. Hypersurfaces of S^{n+1} with vanishing Gauss-Kronecker curvature and the quantization of total curvature. If M^n is a hypersurface of S^{n+1} with vanishing Gauss-Kronecker curvature, then we can observe that the Gauss map G is degenerate. In particular, if the rank of the shape operator is constantly k on M, then the image $G(M)$ is a submanifold of dimension k. It turns out that M then is a tube of radius $\pi/2$ over $G(M)$.

If $n = 3$ one can prove that, if M is compact, then k cannot be 1. Moreover, if $k = 2$, then one can use the classical Gauss-Bonnet theorem to relate the total curvature of M to the Euler characteristic of $G(M)$.

Theorem. *Let* $M^3 \subset S^4$ *be a compact hypersurface with vanishing Gauss-Kronecker curvature. Assume that for each end the rank of the shape operator is constant. Then*

$$\frac{1}{8\pi^2}\int_M (S-2)dV \in \mathbb{Z}.$$

Combining this theorem with Theorem 1 and Theorem 2 we obtain the following.

Theorem. *Let* $M^4 \subset \mathbb{E}^5$ *be a complete open hypersurface with finitely many cone-like ends and with stationary total curvature. Assume that for each end the rank of the shape operator in the sphere "at infinity" is constant. Then the normalized total curvature takes values in the integers:*

$$\frac{3}{4\pi^2}\int_M K dV \in \mathbb{Z}.$$

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