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EXISTENCE OF A FAMILY OF COMPLETE MINIMAL SURFACES OF GENUS ONE WITH ONE END AND FINITE TOTAL CURVATURE

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ABSTRACT. We will report our recent result on existence of a complex one-parameter family of complete minimal surfaces of genus one with one end and finite total curvature. The family connects a minimal surface with total curvature -12π and that with total curvature less than -12π.

1. INTRODUCTION

The purpose of this paper is to report our recent result [8] on existence of a complete minimal surface with finite total curvature in Euclidean space in terms of Weierstrass representation.

Let M be a two-dimensional oriented manifold. If $X\colon M \to \mathbb{R}^3$ is a complete minimal surface with finite total curvature, then we may assume M an open Riemann surface biholomorphic to a compact Riemann surface $\bar{M}$ with finitely many points removed. The genus of $\bar{M}$ is called the *genus* of X and each removed point a *puncture point.* The image of an neighborhood of each puncture point by X is called an *end* of X. The integral $\int_M K\,dA$ for the Gaussian curvature K and the area element dA is called the *total curvature* of X. The total curvature of an oriented complete minimal surface is either $-4\pi m$ for some non-negative integer m or infinite.

We will focus the case where a Riemann surface M is a square torus $\bar{M}$ with one puncture point. By applying Jorge-Meeks's formula ([7, Theorem 4]), the total curvature is less than -4π in this case. We can see existence of the minimal surfaces from examples constructed by applying Weierstrass representation and theory of elliptic functions, for examples, Chen and Gakstatter [3], Bloß [2], Fang [4], and Abi-Khuzam [1].

In [8], we discussed existence of the minimal surfaces in terms of Weierstrass representation without constructing examples by elliptic functions. We will describe the outline of the proof of the following theorem:

Theorem 1.1 ([8]). *There exists a complex one-parameter family of complete minimal surfaces of genus one with one end and total curvature less than -12π and greater than -36π.*

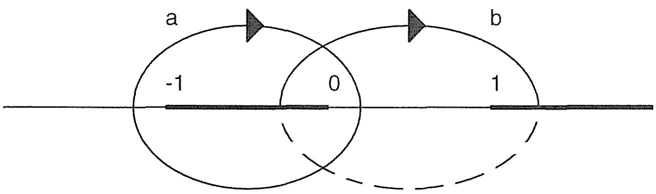

FIGURE 1. A canonical basis $\{a, b\}$.

2. ELLIPTIC CURVES

We will start with preliminaries on the theory of elliptic curves. For more details, see [5] or [6].

Let us denote by $\bar{M}$ the elliptic curve which is the zero locus of a cubic polynomial

$$w^2 = z(z+1)(z-1).$$

The Riemann surface $\bar{M}$ is conformally equivalent to a square torus. We will introduce a representation of the elliptic curve $\bar{M}$ as a two sheeted covering of the sphere $\mathbb{C} \cup \{\infty\}$ branched at -1, 0, 1, and ∞. The covering is obtained by cutting the sphere $\mathbb{C} \cup \{\infty\}$ along two intervals $[-1, 0]$ and $[1, \infty]$ and by pasting two copies of this slit sphere along the slits. We will denote by S_I and S_{II} these two sheets. We will take the branches of w as follows:

$$w = \begin{cases} \sqrt{x(x^2-1)} & x \in [-1, 0] \subset S_I, \\ -\sqrt{x(x^2-1)} & x \in [-1, 0] \subset S_{II}. \end{cases}$$

Let a and b be two oriented one-cycles on $\bar{M}$ indicated in Figure 1, where the parts of the cycles that lie on the sheet S_I are indicated by solid lines and those on the sheet S_{II} by broken lines. The set $\{a, b\}$ of cycles forms a *canonical basis* of the first homology group of $\bar{M}$, that is, the cycle a intersects the cycle b once positively.

We will recall that all holomorphic differentials on an elliptic curve form a one-dimensional complex vector space. Since dz/w is a holomorphic one-form on $\bar{M}$, any holomorphic one-form on $\bar{M}$ is a constant multiple of dz/w. We can see that the integral $\int_a dz/w$ is not equal to 0. Hence, the holomorphic differential

$$\omega := \left(1 \Big/ \int_a \frac{dz}{w}\right) \frac{dz}{w}$$

on $\bar{M}$ becomes the *dual* of a.

For a divisor D on $\bar{M}$, let D_+ and D_- be two nonnegative divisors satisfying $D = D_+ - D_-$. For a meromorphic function f on $\bar{M}$ and a meromorphic one-forms η on $\bar{M}$, we will denote by (f) and (η) the divisors of f and η respectively. we will denote by P the point $(w, z) = (0, 0)$ on $\bar{M}$. Let L be the complex vector space of meromorphic functions on $\bar{M}$ whose divisors are equal to or greater than $-8P$:

$$L = \{f \mid (f) \geq -8P\}.$$

We can see $\dim L = 8$ from the Riemann-Roch theorem. Since

$$(z)_- = 2P, \quad (\omega)_- = 3P, \tag{2.1}$$

the vector space L is spanned by the following basis:

$$1, \frac{1}{z}, w, \frac{1}{z^2}, \frac{w}{z}, \frac{1}{z^3}, \frac{w}{z^2}, \frac{1}{z^4}.$$

3. An example of a minimal surface with total curvature -12π

In this section, we will recall the theory of complete minimal surfaces of genus one with one end and finite total curvature. For more details, see [9] or [10].

We will denote by $\bar{M}$, a, b, P, and ω as in Section 2 and M the open Riemann surface $\bar{M} \setminus \{P\}$. We can see that $\{a, b\}$ is a basis for the first homology group of M, too.

We can see the following holds:

Lemma 3.1. *A pair (g, f) of meromorphic functions on $\bar{M}$ satisfying*

$$\mathrm{Re} \int_\gamma \left(\frac{1}{2}(1-g^2)f\omega, \frac{\sqrt{-1}}{2}(1+g^2)f\omega, gf\omega \right) = (0,0,0) \quad (\gamma = a, b), \tag{3.1}$$

$$-2(g)_- + (f) = -nP, n \in \mathbb{Z}, \, n \geq 2 \tag{3.2}$$

produces a complete minimal surface with finite total curvature $X\colon M \longrightarrow \mathbb{R}^3$ by

$$X(p) = \mathrm{Re} \int^p \left(\frac{1}{2}(1-g^2)f\omega, \frac{\sqrt{-1}}{2}(1+g^2)f\omega, gf\omega \right).$$

The meromorphic function g of a pair (g, f) in Lemma 3.1 is the stereo-graphic projection of the normal Gauss map of the corresponding minimal surface X. Hence, the total curvature of the minimal surface corresponding to (g, f) is equal to $-4\pi \deg g$.

For the convenience, we will denote by Φ the the triplet of integrands in (3.1) corresponding to a pair (g, f):

$$\Phi = \left(\frac{1}{2}(1-g^2)f\omega, \frac{\sqrt{-1}}{2}(1+g^2)f\omega, gf\omega \right). \tag{3.3}$$

Let us denote by F, N, and r three positive real numbers such that

$$F = \int_{-1}^0 \sqrt{x(x^2-1)}\, dx, N = \int_{-1}^0 \frac{dx}{\sqrt{x(x^2-1)}}, r = \sqrt{\frac{2F}{N}}.$$

We will denote by G the meromorphic function r/w on $\bar{M}$. Then, we can show the following:

Lemma 3.2. *The pair $(G, 1/G^2)$ produces a complete minimal surfaces of genus one with one end and total curvature -12π by applying Lemma 3.1.*

Proof. It is easy to see that the pair $(G, 1/G^2)$ satisfies the condition (3.2) by (2.1).

The triplet

$$(\Phi_1, \Phi_2, \Phi_3) = \left(\frac{1}{2}\left(\frac{w\,dz}{Nr^2} - \omega\right), \frac{\sqrt{-1}}{2}\left(\frac{w\,dz}{Nr^2} + \omega\right), \frac{dz}{Nr}\right)$$

of meromorphic differentials on $\bar{M}$ is corresponding to $(G, 1/G^2)$ by (3.3). Since Φ_3 is exact, we see

$$\int_a \Phi_3 = \int_b \Phi_3 = 0.$$

We can see the real periods of Φ_1 and Φ_2 along a and b vanish. For example, the period of Φ_1 along a is calculated as follows:

$$\begin{aligned}\int_a \Phi_1 =& \frac{1}{2Nr^2}\int_{-1}^{0} \sqrt{x(x^2-1)}\,dx \\ &+ \frac{1}{2Nr^2}\int_{0}^{-1} -\sqrt{x(x^2-1)}\,dx - \frac{1}{2}\int_a \omega \\ =& \frac{F}{Nr^2} - \frac{1}{2} = 0.\end{aligned}$$

Hence, the pair $(G, 1/G^2)$ satisfies (3.1).

Thus the pair $(G, 1/G^2)$ produces a complete minimal surface of genus one with one end by applying Lemma 3.1. Since $\deg g$ is equal to 3, the total curvature of the corresponding minimal surface is equal to -12π. □

4. Existence of a family of minimal surfaces

In this section, we will show Theorem 1.1.

Proof of Theorem 1.1. We will denote by $\mathcal{L}$ the complex vector subspace of L spanned by the following basis:

$$\frac{1}{z^2}, \frac{w}{z}, \frac{1}{z^3}, \frac{w}{z^2}, \frac{1}{z^4}.$$

Let us define a set $\mathcal{M}$ of meromorphic functions on $\bar{M}$ by

$$\mathcal{M} := \left\{ g \,\middle|\, \frac{1}{g} - \frac{w}{r} \in \mathcal{L} \right\}.$$

We will consider $\mathcal{M}$ as $\mathbb{C}^5$ by identifying an element g of $\mathcal{M}$ such that

$$\frac{1}{g} - \frac{w}{r} = c_1\frac{1}{z^2} + c_2\frac{w}{z} + c_3\frac{1}{z^3} + c_4\frac{w}{z^2} + c_5\frac{1}{z^4}$$

with an element $(c_1, c_2, c_3, c_4, c_5) \in \mathbb{C}^5$. It is easy to see that each pair $(g, 1/g^2)$ $(g \in \mathcal{M})$ satisfies the condition (3.2).

We can see that the tuple $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_6)$ such that

$$\mathcal{F}_i(g) = \int_a \Phi_i, \quad \mathcal{F}_{i+3}(g) = \int_b \Phi_i \ (i = 1, 2, 3)$$

defines a holomorphic map from $\mathcal{M}$ to $\mathbb{C}^6$ where (Φ_1, Φ_2, Φ_3) is the triplet of meromorphic one-forms corresponding to $(g, 1/g^2)$ by (3.3). Then, each element $(g, 1/g^2)$ $(g \in (\operatorname{Re}\mathcal{F})^{-1}(0, \dots, 0))$ produces a complete minimal surface with finite total curvature by applying Lemma 3.1. In Section 3, we have already seen that G which corresponds to $(c_1, \dots, c_5)$ is an element of $(\operatorname{Re}\mathcal{F})^{-1}(0, \dots, 0)$.

We can see that the following holds:

$$\frac{\partial \mathcal{F}_i}{\partial c_j}(G) = \int_a \frac{\partial \phi_i}{\partial c_j}(G)\ dz,\ \frac{\partial \mathcal{F}_{i+3}}{\partial c_j}(G) = \int_b \frac{\partial \phi_i}{\partial c_j}(G)\ dz,$$

where $\phi_i = \Phi_i/dz$ $(i = 1, 2, 3, j = 1, \dots, 5)$. Since we obtain

$$\begin{aligned}
\left(\frac{\partial \phi_1}{\partial c_1}(G), \dots, \frac{\partial \phi_1}{\partial c_5}(G)\right) &= \left(\frac{1}{rz^2}, \frac{(z^2-1)}{rw}, \frac{1}{rz^3}, \frac{a(z^2-1)}{rzw}, \frac{1}{rz^4}\right), \\
\left(\frac{\partial \phi_2}{\partial c_1}(G), \dots, \frac{\partial \phi_2}{\partial c_5}(G)\right) &= \sqrt{-1}\left(\frac{\partial \phi_1}{\partial c_1}(G), \dots, \frac{\partial \phi_1}{\partial c_5}(G)\right), \\
\left(\frac{\partial \phi_3}{\partial c_1}(G), \dots, \frac{\partial \phi_3}{\partial c_5}(G)\right) &= \left(\frac{1}{z^2w}, \frac{1}{z}, \frac{1}{z^3w}, \frac{1}{z^2}, \frac{1}{z^4w}\right),
\end{aligned}$$

the Jacobian matrix of $\mathcal{F}$ at G becomes as follows:

$$\begin{pmatrix}
0 & C_1 & 0 & C_2 & 0 \\
0 & \sqrt{-1}C_1 & 0 & \sqrt{-1}C_2 & 0 \\
C_3 & 0 & C_4 & 0 & C_5 \\
0 & -\sqrt{-1}C_1 & 0 & \sqrt{-1}C_2 & 0 \\
0 & C_1 & 0 & -C_2 & 0 \\
-\sqrt{-1}C_3 & 0 & \sqrt{-1}C_4 & 0 & -\sqrt{-1}C_5
\end{pmatrix},$$

where

$$\begin{aligned}
C_1 &= 2\int_{-1}^{0} \frac{(x^2-1)}{r\sqrt{x(x^2-1)}}\,dx, C_2 = 2\int_{-1}^{0} \frac{(x^2-1)}{rx\sqrt{x(x^2-1)}}\,dx, \\
C_3 &= 2\int_{-1}^{0} \frac{dx}{x^2\sqrt{x(x^2-1)}}, C_4 = 2\int_{-1}^{0} \frac{dx}{x^3\sqrt{x(x^2-1)}}, \\
C_5 &= 2\int_{-1}^{0} \frac{dx}{x^4\sqrt{x(x^2-1)}}.
\end{aligned}$$

The rank of the Jacobian matrix of $\mathcal{F}$ is equal to 4 since the number C_i does not vanish $(i = 1, \dots, 5)$. Thus, $\mathcal{F}^{-1}(\mathcal{F}(G))$ is a complex submanifold of $\mathbb{C}^5$ of dimension one. The degree of any element of $\mathcal{M}$ except G is greater than 3 and less than 9. Thus, each pair $(g, 1/g^2)$ $(g \in \mathcal{F}^{-1}(\mathcal{F}(G)))$ except $(G, 1/G^2)$ produces a complete minimal surfaces of genus one with one end and total curvature less than -12π and greater than -36π □

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