Non-monotone Bifurcations Along an Algebraic Curve for Quadratic Rational Families

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1 Quadratic rational maps

1.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C}) \quad \text{for} \quad g \in \mathrm{PSL}_2(\mathbf{C}),\ f \in \mathrm{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \mathrm{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \mathrm{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f.

Milnor introduced in [Mil92] coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \le i \le 3)$. Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ with coordinates σ_1 and σ_2 (Lemma 3.1 in [Mil92]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. It is clear that $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{\langle f \rangle\,;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is called the **symmetry locus**.

For each $\mu \in \mathbf{C}$ let $\mathrm{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f which having a periodic point of period n and multiplier μ.

Each of $\mathrm{Per}_1(\mu)$ and $\mathrm{Per}_2(\mu)$ forms a straight lines as follows:

$$\begin{aligned} \mathrm{Per}_1(\mu) &= \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\} \\ \mathrm{Per}_2(\mu) &= \{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu\}, \end{aligned}$$

(Lemmas 3.4 and 3.6 in [Mil92]).

Proposition 1 *The symmetry locus $\mathcal{S}$ is defined by an irreducible algebraic curve in $\mathcal{M}_2(\mathbf{C})$ as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (1)$$

We give an proof in [FN], [FN2].

Corollary 1 *T he symmetry locus $\mathcal{S}$ is the envelope of the family of the lines $\mathrm{Per}_1(\mu)$.*

Milnor describes the curve (1) implicitly (compare Figure 15 in [Mil92]). Here we can give a defining equation (1) of this cubic curve.

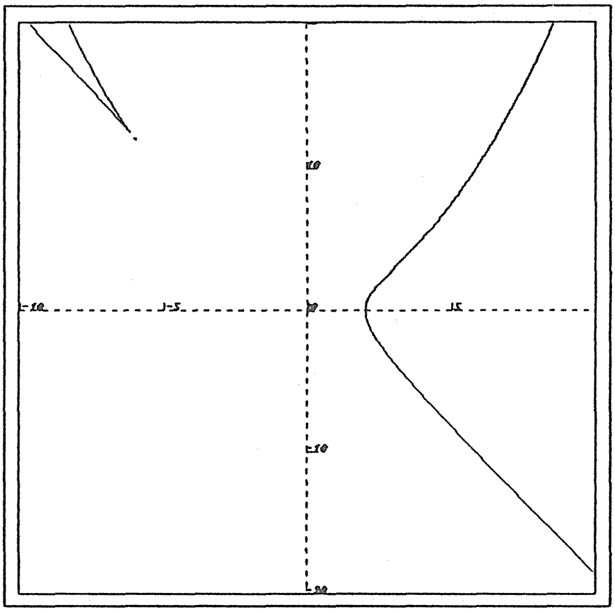

Figure 1: The real cut of the Symmetry locus.

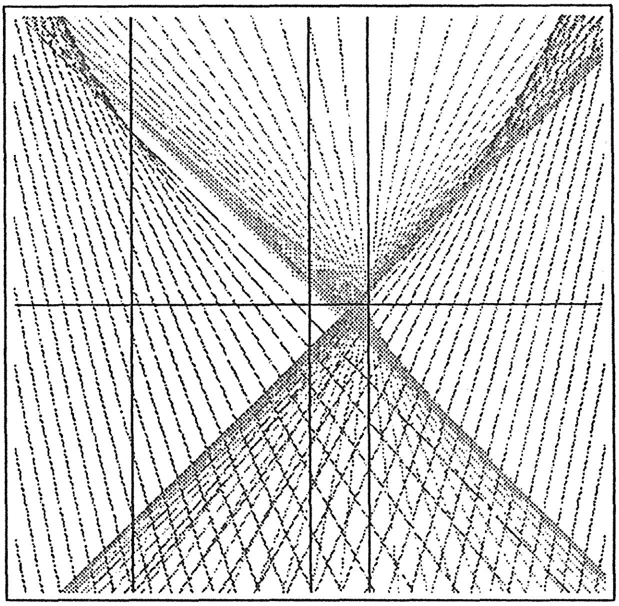

Figure 2: The lines $\mathrm{Per}_1(\mu)$.

1.2 Real moduli space

Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters σ_i $(1 \leq i \leq 3)$ are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ to be simply the real (σ_1, σ_2)-plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle\ \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\mathrm{Rat}_2(\mathbf{R})/\mathrm{PSL}_2(\mathbf{R})) \setminus \{\langle a(x+\frac{1}{x})\rangle_{\mathbf{R}}, \langle a(x-\frac{1}{x})\rangle_{\mathbf{R}}\}_{a\in\mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x})\rangle\}_{a\in\mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

2 A quadratic rational family with non-monotone bifurcations

let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increased, a parameter value λ_0 is called **orbit creating** if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called **orbit annihilating** if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called **neutral** if no periodic orbits are annihilated and no periodic orbits are created.

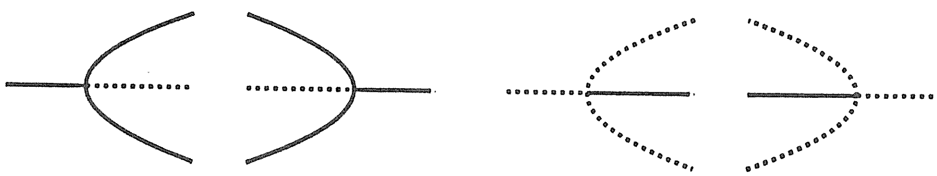

Figure 3: Regular period-doubling (-halving) bifurcations and irregular period-doubling (-halving) bifurcations.

A family $\{f_\lambda\}_\Lambda$ is said to be **monotone increasing** (resp. **decreasing**) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$

is called **non-monotone** if Λ contains both orbit creating and orbit annihilating parameter values.

Note that the sign of Schwarzian derivative $Sf = f'''(x)/f'(x) - \frac{3}{2}(f''(x)/f'(x))^2$ determines the type of local bifurcation: For a family of maps with negative Schwarzian derivative, a period-doubling bifurcation necessarily involves only an attracting (regular) orbit of period two, and not the reverse one which involves a repelling (irregular) orbit of period two ([?]). See Figure 3.

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [BB84] and rewritten by H. E. Nusse and J. A. Yorke ([NY88]):

$$\left\{f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

We note that quadratic rational maps have negative Schwarzian derivatives. Hence, only regular period-doubling (or -halving) bifurcations may occur in this family.

Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Since $\mathcal{M}_2(\mathbf{C})$ is isomorphic to $\mathbf{C}^2$ with coordinate σ_1 and σ_2, there is a natural compactification $\hat{\mathcal{M}}_2(\mathbf{C}) \cong \mathbf{CP^2}$, consisting of $\mathcal{M}_2(\mathbf{C})$ together with a 2-sphere of **ideal points** at infinity. Elements of this 2-sphere can be thought as limits of quadratic rational maps which degenerate towards a fractional linear or constant map ([Mil92]). Therefore for the case $m = 0$ of this family $f_{m,r}$, it makes sense that we should consider it as a degenerated limit.

Theorem 1 *In $\mathcal{M}_2(\mathbf{R})_{\mathbf{R}}$, the one parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve:*

For $r \neq \frac{1}{2}, 0$, this curve is of degree 4 defied by the equation

$$\begin{aligned} H_r(\sigma_1,\sigma_2) &= -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 \\ &\quad +((-32r^2+8)\sigma_2 + 512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 \\ &\quad +(512r^4 - 96r^2 - 12)\sigma_2 - 4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \quad (2) \end{aligned}$$

For $r = \frac{1}{2}$, the corresponding curve is of degree 3, i.e.,

$$H_{\frac{1}{2}}(\sigma_1,\sigma_2) = -\sigma_1^3 - 2\sigma_1^2 + (4\sigma_2 - 24)\sigma_1 + 8\sigma_2 - 64 = 0. \quad (3)$$

For $r = 0$, the corresponding curve is also of degree 3, i.e.,

$$H_0(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \qquad (4)$$

Proof. The three fixed points z_1, z_2, z_3 of $f_{m,r}$ are the roots of the equation

$$z^3 - mrz^2 + (1-m)z - mr = 0,$$

i.e.,

$$\begin{cases} z_1 + z_2 + z_3 = mr, \\ z_1z_2 + z_2z_3 + z_3z_1 = 1 - m, \\ z_1z_2z_3 = mr. \end{cases}$$

The multiplier μ_i of each fixed point z_i is given by

$$f'(z_i) = \mu_i = m\frac{z_i^2 - 1}{(z_i^2+1)^2} \qquad (i = 1, 2, 3).$$

By using "**Gröbner basis of Risa/Asir**, Symbolic and algebraic computation system by FUJITSU, we can obtain the coordinates $\sigma_1(= \mu_1 + \mu_2 + \mu_3)$ and $\sigma_2(= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$ as functions of m and r:

$$\begin{cases} 4m^2r^2 - m^2 + (\sigma_1 + 2)m - 4 = 0 \\ -4m^4r^4 + (m^4 - 12m^3 - 8m^2)r^2 + 2m^3 + (\sigma_2 - 5)m^2 + 4m - 4 = 0. \end{cases} \qquad (5)$$

Using "**Gröbner basis** again, we can remove m from (5) for each fixed r,and get the defining equation (2). We can check easily that (2)is irreducible if and only if $r \neq \frac{1}{2}$, from which follows the first and the last cases. In the case of $r = \frac{1}{2}$, substituting $r = \frac{1}{2}$ in (5) directory, then we obtain (3), which is clearly irreducible.

Conversely, to see any point on the curve $H_r(\sigma_1, \sigma_2) = 0$ comes from an $f_{m,r}$ for some m, observe carefully the process that m is removed from (5). Thus we can see that, except for finite number of points which annihilates the leading coefficients of some polynomial in m appearing in the course of the procedure, every point on the curve corresponds to an $f_{m,r}$ for some m. Then so does any point on the whole curve due to the continuity of the solution of (5), when regarded as equation of m. ∎

Remark 1 The equation of σ_1 in (5) is obtained by the following Program 2, which is suggested us by Takeshi Shimoyama, advaced researcher of ISIS, FUJITSU LABORATORIES LTD.

Program 2

```
if (vtype(gr)!=3) load("gr")$$
extern Ord$

def moduliS1()
{
        S1=nm(m*((z1^2-1)/(z1^2+1)^2
                +(z2^2-1)/(z2^2+1)^2+(z3^2-1)/(z3^2+1)^2)-s1);
        X=z1+z2+z3-m*r;
        Y=z1*z2+z2*z3+z3*z1-1+m;
        Z=z1*z2*z3-m*r;
        Ord=2;
        G=gr([S1,X,Y,Z],[z1,z2,z3,m,r,s1]);
        for (I=length(G)-1;I>=0;I--){
                E=G[I];
                if (vars(E)==[r,m,s1])
                      break;
        }
        return E;
}
end$
```

To say superfluously, the required equation (2) is obtained from following command of Risa/Asir.

Command of Risa/Asir

```
gr([4*m^2*r^2-m^2+(s1+2)*m-4-4*m^4*r^4,
+(m^4-12*m^3-8*m^2)*r^2+2*m^3+(s2-5)*m^2+4*m-4],[m,r]);
```

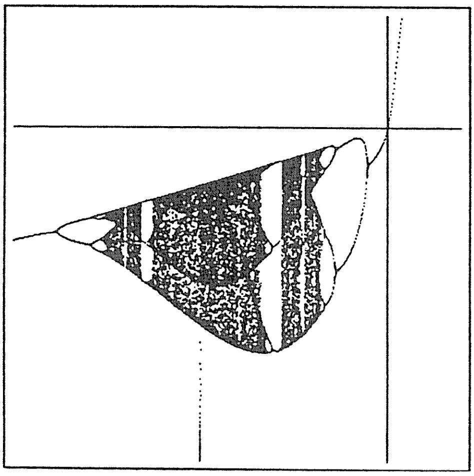

Figure 4: Non-monotone bifurcation; $-25.0 \leq m \leq 5.0$, $-3.0 \leq x \leq 1.0$, $r = 0.54$.

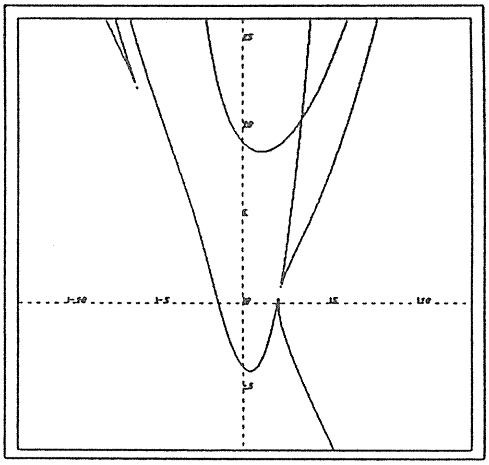

Figure 5: Algebraic curve of degree 4 and cubic curve in the moduli space. In the case of $r = 0.54$.

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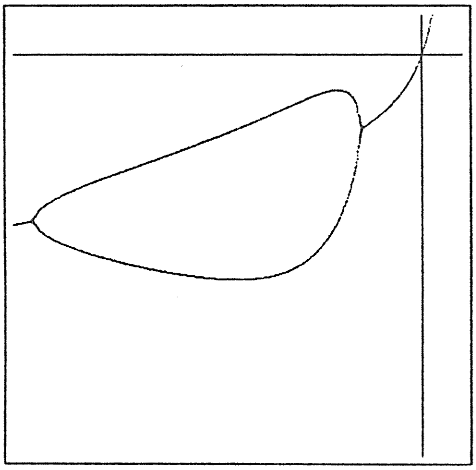

Figure 6: Period-bubbling bifurcation: $-10 \leq m \leq 1$, $-2 \leq x \leq 0.2$, Parameter $r = 0.58$.

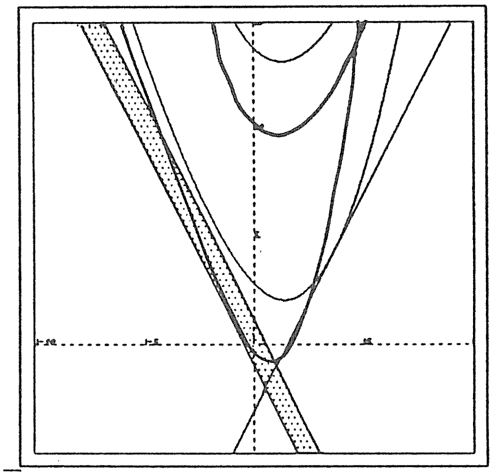

Figure 7: Algebraic curves of degree 4 in the "classified" moduli space. Thick curve corresponds with $r = 0.58$, thin curve corresponds with $r = 0.7$.

Example 1 Non-monotone bifurcation can occur at $r = 0.54$, See Figure 4. And its charasteristic curve is Figure 5.

We can analyze the non-monotone bifurcation by overwriting the algebraic curve of degree 4 on the $\mathcal{M}_2(\mathbf{R})$.

Example 2 One parameter family $\{f_{m,0.58}\}$ has non-monotone (period-bubbling) bifurcation. See Figure 6.

In Figure 7, the thick line indicates this family, and the gray belt is the region on which each map has attracting period 2 cycle. When algebraic curve of degree 4 through this gray belt, period-doubling bifurcation occurs. In this case, the curve intersects the gray belt (period-doubling occurs) and intersects again the period 1 region (period-halving occurs). Hence period-bubbling bifurcation occurs, as in Figure 6.

Theorem 2 *For a fixed parameter r, there are following three possibilities;*

1. *various bifurcations occur if* $0 < r \le \frac{1}{2}$,
2. *non-monotone bifurcations occur if* $\frac{1}{2} < r < \frac{3\sqrt{3}}{8}$, *or*
3. *any bifurcation can't occur if* $\frac{3\sqrt{3}}{8} \le r$.

A`oof is given in ([FN] and [FN2]).

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