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Moduli Space of Polynomial Maps with Degree Four

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Abstract. In this paper we give an answer and a correction by a more elementary method to the inverse problem stated in [2]. This method naturally leads us to concrete information about the moduli space of the polynomial maps of degree 4: for example, a defining equation of the singular part, called symmetry locus is obtained directly.

1. Introduction

Doing the same as the case of cubic polynomials (see [4]), we discussed about the geometry and topology of the polynomial maps of degree n ([2]). In this paper, we restrict our study to the case of degree 4, and supplement some results by a more elementary method than by one stated in Section 4.4 of [2].

We depend our calculations mainly on "Gröbner basis" of Risa/Asir, an experimental computer algebra system developed at FUJITSU LABORATORIES LIMITED.

First we must prepare some notations.

Let $\mathrm{Poly}_4(\mathbf{C})$ be the space of all polynomial maps of degree 4 from $\mathbf{C}$ to itself. The group $\mathfrak{A}(\mathbf{C})$ of all affine transformations acts on $\mathrm{Poly}_4(\mathbf{C})$ by conjugation:

$$g \circ p \circ g^{-1} \in \mathrm{Poly}_4(\mathbf{C}) \quad \text{for} \quad g \in \mathfrak{A}(\mathbf{C}),\ p \in \mathrm{Poly}_4(\mathbf{C}).$$

Two maps $p_1, p_2 \in \mathrm{Poly}_4(\mathbf{C})$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \mathfrak{A}(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_4(\mathbf{C})$

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under this action will be denoted by $\mathrm{M}_4(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle p\rangle$.

Let $\mathcal{P}_1(4)$ be an affine space of all monic centered polynomials of degree 4 with coordinate (c_0, c_1, c_2):

$$p(z) = z^4 + c_2 z^2 + c_1 z^1 + c_0.$$

Under conjugacy of the action of $\mathfrak{A}(\mathbf{C})$, we can take an element p of $\mathcal{P}_1(4)$ as a representative of a class $\langle p\rangle$. We note that p is determined up to the action of the group $G(3)$ of cubic roots of unity, where each $\eta \in G(3)$ acts on $p \in \mathrm{Poly}_4(\mathbf{C})$ by the transformation $p(z) \mapsto p(\eta z)/\eta$. Therefore the following three monic and centered polynomials belong to the same conjugacy class:

$$\begin{aligned} &z^4 + az^2 + bz + c \\ &z^4 + a\omega z^2 + bz + c\omega^2 \\ &z^4 + a\omega^2 z^2 + bz + c\omega \end{aligned}$$

where ω is a cubic root of unity.

We have a three-to-one canonical projection

$$\Phi : \mathcal{P}_1(4) \to \mathrm{M}_4(\mathbf{C}).$$

Thus we can use $\mathcal{P}_1(4)$ or $\{(c_0, c_1, c_2)\}$ as coordinates, called coefficients' coordinates for $\mathrm{M}_4(\mathbf{C})$ though there remains the ambiguity up to the group $G(3)$. On the other hand, we intended to introduce other coordinate, called multipliers' coordinates, in $\mathrm{M}_4(\mathbf{C})$ which is "smaller" than $\mathcal{P}_1(4)$ ([5]) ([2]): for each $p(z) \in \mathrm{Poly}_4(\mathbf{C})$, let $z_1, \cdots, z_4, z_5 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \le i \le 4)$, and $\mu_5 = 0$. Consider the elementary symmetric functions in the four multipliers,

$$\begin{aligned} \sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4, \\ \sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\ \sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \\ \sigma_4 &= \mu_1\mu_2\mu_3\mu_4 \\ \sigma_5 &= 0. \end{aligned}$$

Note that these are well-defined on the moduli space $\mathrm{M}_n(\mathbf{C})$, since μ_i's are invariant by affine conjugacy. Applying the Fatou index theorem, we have a linear relation ([5]):

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0. \tag{1}$$

Let $\Sigma(4)$ be an affine space with coordinates $(\sigma_1, \sigma_2, \sigma_4)$, so-called multipliers' coordinates.

We have a natural projection:

$$\Psi : \mathrm{M}_4(\mathbf{C}) \to \Sigma(4).$$

2. Inverse Problem

To serve $\Sigma(4)$ as good coordinates of $\mathrm{M}_4(\mathbf{C})$, we must investigate that the composition map $\Psi \circ \Phi$ is surjective or not. In this section, we shall give an answer to this problem via the following Proposition. This method is simpler and more clear than one shown in [2]. And we coorect a statement in Section 4.4 of [2].

Proposition 1

(transformation formula) *Between the spaces* $\mathcal{P}_1(4) = \{z^4 + c_2z^2 + c_1z + c_0\}$, *and* $\Sigma(4) = \{(\sigma_1, \sigma_2, \sigma_4)\}$, *there is a following transformation formula:*

$$\sigma_1 = -8c_1 + 12 \tag{2}$$

$$\sigma_2 = 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48 \tag{3}$$

$$\begin{aligned}\sigma_4 = {} & 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 + 128c_0)c_2 \\ & -27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3.\end{aligned} \tag{4}$$

Proof. Let p be $p(z) = z^4 + c_2z^2 + c_1z + c_0$ and four finite fixed points z_i $(0 \le i \le 3)$. By computing the symmetric function in four multipliers $\mu_i = p'(z_i)$ $(0 \le i \le 3)$, we get the relation (2),(3),(4). ∎

The practical procedures on using symbolic and algebraic computation systems are based on Gröbner basis with respect to lexical order,

`m0>m1>m2>m3>z0>z1>z2>z3>s4>s2>s1>c2>c1>c0.`

We can get the σ_i $(0 \le i \le 3)$ as the function in c_i $(0 \le i \le 2)$ variables.

We remark that by this procedure, eleven Gröbner basis are obtained but only three of them correspond to the transformation formula.

Now, we shall give a complete answer of the "inverse problem". Namely, for any $(\sigma_1, \sigma_2, \sigma_4)$ given, there exists (c_0, c_1, c_2) satisfying the transformation formula or not. A proof given below is more elementary than one stated in [2]. The result of the following proposition is due to the referee.

Proposition 2

The composition

$$\Psi \circ \Phi : \mathcal{P}_1(4) \longrightarrow \Sigma(4)$$

is not surjective: this map has no inverse image for any point on the curves $\mathcal{E}$, *except the only one point* $(4, 6, 1)$*:*

$$\begin{aligned}\mathcal{E}: \quad \sigma_1 &= 4, \\ \sigma_4 &= \tfrac{1}{4}(\sigma_2^2 - 8\sigma_2 + 16)\end{aligned}$$

Proof. Fix a point $(\sigma_1, \sigma_2, \sigma_4) \in \Sigma(4)$. The following equation is obtained by substituting the equation (2) to (3) of transformation formula:

$$4c_2^3 - 16c_0c_2 = -\sigma_2 - \frac{9}{32}\sigma_1^2 - \frac{3}{4}\sigma_1 + \frac{3}{2} \tag{5}$$

Let V be the value of the right hand of the relation (5):

$$V = \frac{1}{32}(-32\sigma_2 + 9\sigma_1^2 + 24\sigma_1 - 48) \tag{6}$$

First we start the case of $V = 0$. We put $c_1 = \frac{12-\sigma_1}{8}$ and $c_2 = 0$. Then c_0 is a one of the solutions of the equation given by (4):

$$1048576c_0^3 - 4096\sigma_4 - 27\sigma_1^4 + 432\sigma_1^3 - 1440\sigma_1^2 + 1792\sigma_1 - 768 = 0.$$

Second, we assume that $V \neq 0$. From the relation (5), we have $c_2 \neq 0$. Therefore dividing (3) by c_2, and substituting it into (4) we obtain the following equation:

$$Ac_2^6 + Bc_2^3 + C = 0 \tag{7}$$

where

$$\begin{aligned}&A = 262144(\sigma_1 - 4)^2, \\ &B = 1024(128\sigma_2^2 + (-144\sigma_1^2 + 384\sigma_1 - 256)\sigma_2 - 512\sigma_4 + 27\sigma_1^4 - 576\sigma_1^2 + 1280\sigma_1 - 768), \\ &C = -(32\sigma_2 - 9\sigma_1^2 - 24\sigma_1 + 48)^3.\end{aligned}$$

Now we note that $C = (32V)^3 \neq 0$.

Here, we make sure easily that the above equation (7) have solution(s) c_2 in cases where $A \neq 0$ or $B \neq 0$. Suppose, $A = 0$ and $B = 0$, equivalently that $\sigma_1 = 4$ and $\sigma_4 = (\sigma_2^2 - 8\sigma_2 + 16)/4$. We denote this curve on the plane $\sigma_1 = 4$ by $\mathcal{E}$.

Substituting these conditions into the transformation formula, we have a new relation $4c_0 - c_2^2 = 0$ by using Risa/Asir. Practical procedures are given in the boxed item (inverse problem). As this relation is a factor of the left hand of the equation (5), it contradicts to the condition $C \neq 0$. Hence for this case: $A = 0$,$B = 0$ and $C = (32V)^3 \neq 0$, corresponding solutions c_2 and c_0 can not exist.

We remark that for the case: $A = 0$, $B = 0$ and $C = (32V)^3 = 0$; namely $(4,6,1) \in \Sigma(4)$, there are infinitely many inverse solutions.

Therefore the equation (7) has solution(s) c_2 for any point in $\Sigma(4)$ except the curve $\mathcal{E} \setminus (4,6,1)$.

Once there exists c_2, substituting these c_2 to (3), c_0 is also obtained. The parameter c_1 depends only on σ_1. Now, c_0, c_1, c_2 are the coefficients of a monic and centered polynomial, therefore we can calculate fixed points, multipliers and its elementary symmetric functions from these coefficients. We remark that these elementary symmetric functions return to the initial values $\sigma_1, \sigma_2, \sigma_4$. ∎

For example, polynomials with degree 4 do not correspond to points $(4,0.4)$, $(4,4,0) \in \mathcal{E}$.

2.1. real inverse problem

Now we consider "real inverse problem", namely for any $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{R}^3$ given, whether there exists $(c_0, c_1, c_2) \in \mathbb{R}^3$ satisfying the transformation formula or not.

Let $\mathrm{Poly}_4(\mathbb{R})$ be the set of real polynomials of degree 4. Then we note that the parameters σ_i $(1 \leq i \leq 4)$ are all real.

Fix any $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{R}^3$. For the case $V = 0$ it is clear from a proof above that there exists $(c_0, c_1, c_2) \in \mathbb{R}^3$.

In the case of $V \neq 0$, put $c_2^3 = t$. If the discriminant $D = B^2 - 4AC$ of the quadratic equation (7) of t is negative, then the roots are not real numbers .

Here,

$$\begin{aligned} D = {} & 54\sigma_1^5 - 27(3\sigma_2 + \sigma_4 + 5)\sigma_1^4 + 36(\sigma_2^2 - 4\sigma_2 - 28)\sigma_1^3 + 4(-\sigma_2^3 + 90\sigma_2^2 + (36\sigma_4 + 744)\sigma_2 \\ & + 144\sigma_4 + 1048)\sigma_1^2 + 32(-5\sigma_2^3 - 68\sigma_2^2 + (-12\sigma_4 - 200)\sigma_2 - 40\sigma_4 - 168)\sigma_1 \\ & + 16(\sigma_2^4 + 28\sigma_2^3 + (-8\sigma_4 + 136)\sigma_2^2 + (16\sigma_4 + 240)\sigma_2 + 16\sigma_4^2 + 48\sigma_4 + 144) \end{aligned}$$

Therefore, for $\sigma_1 \ll -1$ this discriminant is negative and $c_2 \in \mathbb{C} \setminus \mathbb{R}$. Hence we conclude that for suitable $(\sigma_1, \sigma_2, \sigma_4) \in \mathbb{R}^3$, we can not find a real polynomial corresponding to this coordinate.

See a boxed item named (real inverse problem).

Procedure for computing (by Risa/Asir):(inverse problem) practical procedure No.1

```
[0] S1=-8*c1-s1+12$
[1] S2=-4*c2^3+16*c0*c2-18*c1^2+60*c1+s2-48$
[2] S4=16*c0*c2^4+(-4*c1^2+8*c1)*c2^3-128*c0^2*c2^2+
      (144*c0*c1^2-288*c0*c1+128*c0)*c2-27*c1^4+108*c1^3
      -144*c1^2+64*c1+256*c0^3-s4$
%% if c2 != 0
[3] Ctmp1=subst(S4,c0,(-18*c1^2+60*c1-4*c2^3+s2-48)/(-16*c2))$
[4] Ctmp2=nm(red(Ctmp1))$
[8] Ctmp3=subst(Ctmp2,c1,(12-s1)/8)$
[9] fctr(Ctmp3);
   [[1/524288,1],[-729*s1^6-5832*s1^5
   +(-27648*c2^3+7776*s2-3888)*s1^4+(41472*s2+48384)*s1^3
   +(-262144*c2^6+(147456*s2+589824)*c2^3-27648*s2^2
   -27648*s2+20736)*s1^2+(2097152*c2^6+(-393216*s2-1310720)*c2^3
   -73728*s2^2-221184*s2-165888)*s1-4194304*c2^6
   +(-131072*s2^2+262144*s2+524288*s4+786432)*c2^3+32768*s2^3
   +147456*s2^2+221184*s2+110592,1]]
[10] CC=car(car(cdr(@@)))$
[12] fctr(coef(CC,6,c2));
   [[-262144,1],[s1-4,2]]
[15] fctr(coef(CC,3,c2));
   [[1024,1],[-27*s1^4+(144*s2+576)*s1^2
   +(-384*s2-1280)*s1-128*s2^2+256*s2+512*s4+768,1]]
[16] fctr(coef(CC,2,c2));
   [[0,1]]
[17] fctr(coef(CC,1,c2));
   [[0,1]]
[18] fctr(coef(CC,0,c2));
   [[1,1],[-9*s1^2-24*s1+32*s2+48,3]]
```

Procedure for computing (by Risa/Asir):(inverse problem) practical procedure No.2

```
[21] fctr(subst(coef(CC,3,c2),s1,4));
[[131072,1],
 [-s2^2+8*s2+4*s4-16,1]]
%% Hence, if coef(CC,6,c2)=0 and coef(CC,3,c2)=0
%% then s1=4 and s4=(s2^2-8*s2+16)/4.....(*)
%% Since s1=4 <=> c1=1.
[22] SF2=subst(S2,c1,1);
-4*c2^3+16*c0*c2+s2-6
[23] SF4=subst(S4,c1,1);
16*c0*c2^4+4*c2^3-128*c0^2*c2^2-16*c0*c2+256*c0^3-s4+1
%% substitute these two relation to (*)
[24] fctr((-(-4*c2^3+16*c0*c2-6))^2-8*(-(-4*c2^3+16*c0*c2-6))+16
    -4*(16*c0*c2^4+4*c2^3-128*c0^2*c2^2-16*c0*c2+256*c0^3+1));
[[-16,1],[-c2^2+4*c0,3]]
```

Procedure for computing (by Risa/Asir): (real inverse problem)

```
[0] A=-262144*s1^2+2097152*s1-4194304$
[1] B=-27648*s1^4+(147456*s2+589824)*s1^2+(-393216*s2-1310720)*s1
      -131072*s2^2+262144*s2+524288*s4+786432$
[2] C=-729*s1^6-5832*s1^5+(7776*s2-3888)*s1^4+(41472*s2+48384)*s1^3
      +(-27648*s2^2-27648*s2+20736)*s1^2
      +(-73728*s2^2-221184*s2-165888)*s1+32768*s2^3+147456*s2^2
      +221184*s2+110592$
[3] D=B^2-4*A*C$
[4] fctr(D);
[[1073741824,1],
 [54*s1^5+(-81*s2-27*s4-135)*s1^4+(36*s2^2-144*s2-1008)*s1^3
  +(-4*s2^3+360*s2^2+(144*s4+2976)*s2+576*s4+4192)*s1^2
  +(-160*s2^3-2176*s2^2+(-384*s4-6400)*s2-1280*s4-5376)*s1
  +16*s2^4+448*s2^3+(-128*s4+2176)*s2^2+(256*s4+3840)*s2+256*s4^2
  +768*s4+2304,1]]
```

3. Symmetry locus

Next, we study the singular locus in the moduli space $\mathrm{M}_n(\mathbf{C})$. By an automorphism of a polynomial map p we will mean an affine transformation g that commutes with p. The collection $\mathrm{Aut}(p)$ of all automorphisms of p forms a finite group. The following characterization is given in [2]: *polynomial map of degree n has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$z^n + \sum_{\substack{k|(n-1), k \neq n-1 \\ 1 \leq p \leq [n/k]}} A(kp) z^{kp+1} + Bz$$

where $A(kp)$ are parameters in $\mathbf{C}$.

Let $\mathcal{S}_n$ ($\subset \mathrm{M}_n$), called symmetry locus, be the set consisting of all conjugacy classes $\langle p \rangle$ of polynomial maps admitting non-trivial automorphisms.

For the case of $n = 4$, we gave a defining equation of $\mathcal{S}_n$ as Proposition 6 in [2]. We shall give here another elementary proof, directly applying the transformation formula obtained above. Practical procedure is shown in a boxed item named symmetry locus.

Proposition 3

The symmetry locus $\mathcal{S}_4$ is in $\mathrm{M}_4(\mathbb{C})$ forms the following algebraic curve;

$$\begin{cases} \sigma_1 = s \\ \sigma_2 = 3(3s-4)(s+4)/32 \\ \sigma_4 = -(3s-4)^3(s-12)/4096 \end{cases}$$

Proof. The normal form of $\mathcal{S}_4$ is

$$\left\{z^4 + az\right\}_a.$$

Hence substituting the relations $c_0 = 0, c_1 = a, c_2 = 0$ into the transformation formula and removing the parameter a from these equations, we obtain the following two equations:

$$9\sigma_1^2 + 24\sigma_1 - 32\sigma_2 - 48 = 0$$
$$-27\sigma_1^4 + 432\sigma_1^3 - 1440\sigma_1^2 + 1792\sigma_1 - 4096\sigma_4 - 768 = 0.$$

A defining equation of the symmetry locus is given by the intersection of these two equations. ∎

Procedure for computing (by Risa/Asir): symmetry locus

```
[0] S1=-8*c1-s1+12$
[1] S2=4*c2^3-16*c0*c2+18*c1^2-60*c1-s2+48$
[2] S4=16*c0*c2^4+(-4*c1^2+8*c1)*c2^3-128*c0^2*c2^2+
       (144*c0*c1^2-288*c0*c1+128*c0)*c2-27*c1^4+108*c1^3-
       144*c1^2+64*c1+256*c0^3-s4$

[3] Sym1=subst(S1,c1,a);
-8*a-s1+12
[4] Sym2=subst(S2,c0,0,c1,a,c2,0);
18*a^2-60*a-s2+48
[5] Sym4=subst(S4,c0,0,c1,a,c2,0);
-27*a^4+108*a^3-144*a^2+64*a-s4

[6] Surf1=subst(Sym2,a,(12-s1)/8);
9/32*s1^2+3/4*s1-s2-3/2
[7] fctr(Surf1);
[[1/32,1],
 [9*s1^2+24*s1-32*s2-48,1]]

[8] Surf2=subst(Sym4,a,(12-s1)/8);
-27/4096*s1^4+27/256*s1^3-45/128*s1^2+7/16*s1-s4-3/16
[9] fctr(Surf2);
[[1/4096,1],
 [-27*s1^4+432*s1^3-1440*s1^2+1792*s1-4096*s4-768,1]]
```

Acknowledgements

Last, we express again that for obtaining results, we depend mainly on “Gröbner basis” of Risa/Asir, an experimental computer algebra system developed at FUJITSU LABORATORIES LIMITED.

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