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On New Method to Calculate Euler's Constant

Teluhiko HILANO *
Kanagawa Institute of Technology

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1. Introduction

Euler's constant is defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right). \tag{1}$$

There are many methods to calculate Euler's constant with arbitrary precision. See, for example, [3].

In this paper, we propose a new method to calculate Euler's constant with arbitrary precision. Comparing its efficiency to other methods and its implimentation will be reported llater. Our starting point is the following formula.

$$\gamma = -\log x + \int_0^x \frac{1-e^{-t}}{t}dt - \int_x^\infty \frac{e^{-t}}{t}dt. \tag{2}$$

This is well known, (See, [2]p.2 (1.1.3)). Changing the upper bound of its integral to x, the formula immediately follows. Expanding e^{-t} at $t=0$ and integrating it term by term, the second term of (2) is

$$\int_0^x \frac{1-e^{-t}}{t}dt = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n!n}. \tag{3}$$

As explained in [2, Chapter 1], the third term can be calculated using the asymptotic expansion. Since calculation using asymptotic expansion has predicted accuracy in its nature, we must choose large x to get larger precision. This produces to make the convergence of

*hilano@gen.kanagawa-it.ac.jp

(3) slower. To avoid this, we adopt the idea of [1] using the orthogonal polynomials in the interval $[0, 1]$ (Legendre polynomials) to calculate asymptotic value of an integral. We put

$$F(x) = \int_x^\infty \frac{e^{-t}}{t} dt. \tag{4}$$

and

$$G(x) = \int_x^{2x} \frac{e^{-t}}{t} dt = \int_0^1 \frac{e^{-x(1+t)}}{t} dt \tag{5}$$

We have

$$F(x) = \sum_{k=0}^{L-1} G(2^k x) + F(2^L x). \tag{6}$$

Now, we introduce the orthogonal polynomials on the interval $[0, 1]$.

Definition 1

For non-negative integer, we put

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^n (1-x)^n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k \tag{7}$$

We put

$$I_n(x) = \int_0^1 \frac{e^{-x(1+t)} P_n(t)}{1+t} dt. \tag{8}$$

For $x > 0$, the integral can be expressed as $I_n(x) = A_n G(x) + B_n(x)$, where

$$A_n = P_n(-1) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \tag{9}$$

and $B(x)$ is a function represented by e^x and polynomials of x. If $I_n(x)$ is small and A_n is rapidly increasing, $\frac{B_n(x)}{A_n}$ would be a good approximation of $G(x)$. Combining to (6), we can get a suitable approximate value of Euler's constant.

2. Properties of Legendre Polynomials

In this section, we show some properties of the polynomials defined by (7) Usually, Legendre polynomials are orthogonal polynomials on the interval $[-1, 1]$, the properties of our polynomials must be modified as ones in [8].

Theorem 2

The polynomials defined by (7) are satisfies the following properties.

1. We have

$$P'_n(x) = (1-2x)P'_{n-1}(x) - 2nP_{n-1}(x) \tag{10}$$

2. $P_n(x)$ satisfies the differential equation(Hypergeometric differential equation).

$$x(1-x)P''_n(x) + (1-2x)P'_n(x) + n(n+1)P_n(x) = 0. \tag{11}$$

or equivalently,

$$(x(1-x)P'_n(x))' + n(n+1)P_n(x) = 0. \tag{12}$$

3. We have

$$\begin{aligned} 2x(1-x)P'_n(x) &= (n+1)(P_{n+1}(x) - (1-2x)P_n(x)) \\ &= -n(P_{n-1}(x) - (1-2x)P_n(x)). \end{aligned} \tag{13}$$

4. We have the following recurrence relation:

$$(n+1)P_{n+1}(x) - (2n+1)(1-2x)P_n(x) + nP_{n-1}(x) = 0. \tag{14}$$

5. On the interval $[0,1]$, *we have*

$$|P_n(x)| \le 1. \tag{15}$$

Remark: Since our Legendre polynomials $P_n(x)$ are kind of hypergeometric function, namely $F(-n, n+1; 1; x)$, (11) is immediate result of the property of hypergeometric functions.

Proof 1. Applying Leibnitz rule to

$$\left(\frac{1}{n!}\left\{(x^n(1-x)^n)\right\}\right)' = \frac{1}{(n-1)!}\left\{(1-2x)(x^{n-1}(1-x)^{n-1})\right\},$$

we have

$$\begin{aligned} P'_n(x) &= \frac{1}{(n-1)!}\frac{d^n}{dx^n}\left\{(1-2x)(x^{n-1}(1-x)^{n-1})\right\} \\ &= (1-2x)\frac{1}{(n-1)!}\frac{d^n}{dx^n}\left\{x^{n-1}(1-x)^{n-1}\right\} - 2n\frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left\{x^{n-1}(1-x)^{n-1}\right\}. \end{aligned}$$

This proves (10).

2. Applying Leibnitz rule directly to the $n+1$ st derivative of the function $(x(1-x))\left(x^{n-1}(1-x)^{n-1}\right)$, we have

$$\begin{aligned} P'_n(x) &= \frac{1}{n!}\Bigg(x(1-x)\frac{d^{n+1}}{dx^{n+1}}\left(x^{n-1}(1-x)^{n-1}\right) + (n+1)(1-2x)\frac{d^n}{dx^n}\left(x^{n-1}(1-x)^{n-1}\right) \\ &\quad - n(n+1)\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1}(1-x)^{n-1}\right)\Bigg) \\ &= \frac{1}{n}x(1-x)P''_{n-1}(x) + \frac{n+1}{n}(1-2x)P'_{n-1}(x) - (n+1)P_{n-1}(x). \end{aligned}$$

Using (10) and rewriting n to $n+1$, we we (11).

3. Differentiating the formula multiplying $x(1-x)$ to (10) and using (12), we have

$$-n(n+1)P_n(x) = -2x(1-x)P'_{n-1}(x) - (1-2x)n(n-1)P_{n-1}(x) \\ - 2nx(1-x)P'_{n-1}(x) - 2n(1-2x)P_{n-1}(x).$$

Hence, we have

$$(2n+2)x(1-x)P'_{n-1}(x) = n(n+1)P_n(x) - n(n+1)P_{n-1}(x). \tag{16}$$

Dividing the both sides by $n+1$ and rewriting n to $n+1$, we get the first relation of (13). On the other hand, eliminating $P'_{n-1}(x)$ from (10) and (16), we have the second relation of (13).

4. (14) is the immediate consequence of (13).

5. The maximum of $|P_n(x)|$ on the interval $[0,1]$ attains at the points $x = 0, 1$ or ones where $P'_n(x) = 0$. We have $P_n(0) = 1$ and $P_n(1) = (-1)^{n-1}$ by (13). If $P'_n(x) = 0$, we also get

$$(1-2x)P_n(x) = P_{n-1}(x).$$

by (13). Hence we have $|P_n(x)| \leq 1$ by induction.

3. Properties of A_n and $I_n(x)$

In this section, we prove some properties of A_n and $I_n(x)$ defined by (9) and (8), respectively. Most of properties of A_n are already appeared in [6],[7].

Theorem 3

The sequence $\{A_n\}$ defined by (9) satisfies the following properties:

1. *$\{A_n\}$ satisfies the following recurrence relation:*

$$nA_n - (6n-3)A_{n-1} + (n-1)A_{n-2} = 0. \tag{17}$$

2. *The generating function $f(t)$ of $\{A_n\}$ defined by $\sum_{n=0}^{\infty} A_n t^n$ is*

$$f(t) = (1 - 6t + t^2)^{-1/2} \tag{18}$$

3. *For any positive ϵ, there exists some constant C_ϵ such that A_n satisfies the following inequality:*

$$A_n \geq C_\epsilon (3 + 2\sqrt{2})^{(1-\epsilon)n} \tag{19}$$

Proof. 1. Since $A_n = P_n(-1)$, this is the immediate consequence of (14).

2. Because

$$f'(t) = \sum_{n=1}^{\infty} nA_n t^{n-1},$$

we have

$$\sum_{n=0}^{\infty}(n+2)A_{n+2}t^n = \frac{f'(t)-A_1}{t},$$

$$-\sum_{n=0}^{\infty}(6n+9)A_{n+1}t^n = -6f'(t) - \frac{3(f(t)-A_0)}{t},$$

$$\sum_{n=0}^{\infty}(n+1)A_n t^n = tf'(t)+f(t).$$

Thus, we have

$$f'(t) - A_1 - 6tf'(t) - 3(f(t)-A_0) + t^2 f'(t) + tf(t) = 0,$$

from (17). Since $A_0 = 1$ and $A_1 = 3$, we get the differential equation of $f(t)$:

$$(t^2 - 6t + 1)f'(t) = -(t-3)f(t).$$

This implies $f(t) = \dfrac{1}{\sqrt{1-6t+t^2}}$, because $f(0) = 1$.

3. The nearest singularity to the origin of the generating function $f(t)$ is $3 - 2\sqrt{2}$. Hence

$$\limsup_{n\to\infty} \frac{1}{A_n^{1/n}} = 3 - 2\sqrt{2} = \frac{1}{3+2\sqrt{2}}.$$

This proves (19), because $\{A_n\}$ is monotone increasing.

Theorem 4

For any integer n and $x > 0$ we have

$$0 < |I_n(x)| < \min\left((\sqrt{2}-1)^{2n}\log 2, \frac{e^{-x}}{x}\right). \tag{20}$$

Proof The inequality $|I_n(x)| \le \dfrac{e^{-x}}{x}$ is an immediate result of (15). To apply the integral by parts n times to $I_n(x)$, we get

$$I_n(x) = (-1)^n \int_0^1 \frac{d^n}{dt^n}\left(\frac{e^{-x(1+t)}}{1+t}\right)\frac{t^n(1-t)^n}{n!}dt.$$

Using Leibnitz rule to the above integrand, we have

$$\frac{d^n}{dt^n}\left(\frac{e^{-x(1+t)}}{1+t}\right) = \sum_{k=0}^{n}\binom{n}{k}(-x)^k e^{-x(1+t)}\frac{(-1)^{n-k}(n-k)!}{(1+t)^{n-k+1}}$$
$$= (-1)^n\frac{n!}{(1+t)^{n+1}}e^{-x(1+t)}\sum_{k=0}^{n}\frac{(x(1+t))^k}{k!}.$$

Hence, we have

$$I_n(x) = \int_0^1 \frac{t^n(1-t)^n}{(1+t)^{n+1}}e^{-x(1+t)}\sum_{k=0}^{n}\frac{(x(1+t))^k}{k!}dt.$$

The integrand of this is non-negative and this produces the estimate

$$0 < I_n(x) < \int_0^1 \frac{t^n(1-t)^n}{(1+t)^{n+1}}dt.$$

The function $f(t) = \dfrac{t(1-t)}{1+t}$ has the maximum $(\sqrt{2}-1)^2$ at $t = -1+\sqrt{2}$ in the interval $[0,1]$. This produces

$$\int_0^1 \frac{t^n(1-t)^n}{(1+t)^{n+1}}dt \le (\sqrt{2}-1)^{2n}\int_0^1 \frac{dt}{1+t} \le (\sqrt{2}-1)^{2n}\log 2.$$

4. The Algorithm

Now, we propose the algorithm to calculate Euler's constant up to N digits.

- Step1: Calculate (3) for $x = 1$ upto $N+2$ digits.
- Step2: Choose L such that L satisfies the inequality $e^{-2^L} \le 10^{-N-1}$. $L = \log_2 N + 2$ is sufficient to our purpose.
- Step3: Calculate $G(2^k)$ for $k = 0, 1, \ldots, L$ upto $N+2+L/10$ digits to calculate $\dfrac{B_n(2^k)}{A_n}$ up to N+2+L/10 digits. Since $\left|\dfrac{I_n(2^k)}{A_n}\right| \le \dfrac{1}{(\sqrt{2}+1)^{2n}A_n}$ and $A_n \approx (\sqrt{2}+1)^{2n}$, we must calculate $\dfrac{B_n(2^k)}{A_n}$ upto $N+2+L/10$ digits, choosing $N+2+L/10 \approx 4n\log(\sqrt{2}+1) \approx 1.53n$.

To calculate $B_n(x)$, we remark that it consists of the terms

$$\int_0^1 e^{-x(1+t)}t^k dt = -\sum_{j=0}^{k}\frac{1}{x^{k-j+1}}\frac{k!}{j!}e^{-2x} + \frac{k!e^{-x}}{x^{k+1}}$$
$$= \frac{k!}{x^{k+1}}e^{-x}\left(1 - e^{-x}\sum_{j=0}^{k}\frac{x^j}{j!}\right), \tag{21}$$

which is estimated trivially by $\frac{e^{-x}}{x}$. Their coefficients have absolute value not greater than 1. Hence, it is sufficient to caluculate (21) upto $N + n/10 + 2 \approx 1.06N$ digits. There are some problems to culculate (21) because of cancellation. To consider the efficiency of our method, we must analyze the order of (21) in details.

References

[1] Beukers, F. A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.,* 11(1979), 268-272

[2] Bleistein, N. and Handelsman, R. A. *Asymptotic Expansions of Integrals* Dover, 1986

[3] Brent, R. and McMillan, E. M. Some New Algorithms for High-Precision Computation of Euler's Constant Math. Comp. 34(1980), 305–312

[4] de Bruijn, N. G. *Asymptotic Methods in Analysis,* reprinted, 1981, Dover

[5] Erdélyi, A.(Ed.) *Higher Transcendental Functions,* reprinted, McGraw-Hill, 1953

[6] van der Poorten, A. J., Some Wonderfull Formulae ... Footnotes to Apery's Proof of the Irrationarity of $\zeta(3)$, Sémimaire Delange-Pisot-Poitou, 20e année, 1978/1979,n°29

[7] van der Poorten, A. J., A Proof that Euler Missed ... Apéry's Proof of the Irrationality of $\zeta(3)$, *Mathematical Intelligencer,* **1**(1978), p.195-203

[8] Whittaker, E.T. and Watson, G.N. *A Course of Modern Analysis* 4th Edition, Cambridge Univ. Press, 1927