Sraffa in the Light of Marx, Revisited.*)

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Introduction.

Various schools of economics have attempted to construct the theory of production and distribution from various angles. What is most popular and important may be the neoclassical theory of marginal productivity.

Marx's economic theory started from his criticism of Ricardo, whilst Sraffa, based on Ricardo's economic theory, challenged the marginal productivity theory. The core of Sraffa's theory is, above all, the criticism of the neoclassical theory of distribution and price, but not a few economists called Sraffians, or neo-Ricardians, such as Steedman, have made Sraffa-based criticisms of Marx.

A defence of Marx against Sraffa-based criticism was fundamentally undertaken in Chapter I. The objective of this appendix is not to repeat it, but, in turn, to examine, in the light of Marx, Sraffa's theory of the standard commodity, and to investigate its fundamental features.

The major problem to be discussed in the following will be described in § 1 in the case of the Leontief economy. Though Sraffa's theory does not depend on the assumption of the linearity of the techniques, the framework of an economy with linear techniques will be applied in what follows. By so doing, the comparison of Marx and Sraffa will be much facilitated.

In §§ 1–2, Sraffa's theory of the standard commodity will be discussed in the case of the Leontief economy. This is the counterpart of Chapter I. In § 3, Sraffa's theory will be extended to the von Neumann economy case. This corresponds to the discussion of Chapters III and IV.

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*) This is a revised version of the author's paper which was published in this journal (1980) under the same title. In revising the previous one, however, full reference to Fujimori [82] is assumed, as this is rewritten as an appendix to Fujimori, Y.(1982) *Modern Analysis of Value Theory*, Springer.

The notation employed in this appendix is the same as used in Chapters I, III and IV, unless otherwise stated. Some additional notations will be introduced where they are necessary. The assumptions, albeit similar, will be repeated. As for wages, however, it is assumed that

(#8′) Wages are paid ex post.

§ 1. Sraffa's theory of the standard commodity in a Leontief economy.

1. Let us first construct the system of production prices with ex post wages. Let

r : rate of profit with ex post wages,

ω : wage rate,

and the production price system is described by

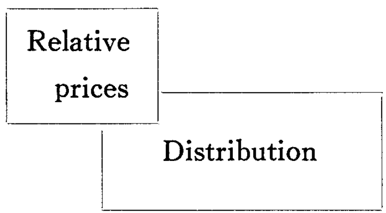
(1) $$p=(1+r)pA+\omega L,$$

(2) numeraire equation.[1]

Suppose that n types of good are produced in the economy, and the system (1) consists of n equations. Hence, the production price system, (1) and (2), has $n+1$ equations in $n+2$ unknowns, i. e., p, r, and ω. Consequently, if one of the unknowns is given, the remaining unknowns are all determined. Assume that, in addition to (2), the price of good 1, for instance, is given, and the prices of the remaining kinds of good, i. e., $p_2, p_3, \ldots, p_n$ and distribution, r and ω, are determined simultaneously. Or, if either r or ω is given, the remaining distributive factor and the prices of all kinds of goods are determined.

Reviewing the causal relationship between the relative prices and distribution, one can establish the following diagram:

DIAGRAM App.-1.



That is, the determination of the relative prices and distribution should be made simultaneously, but they are basically two different things.

Sraffa's theorem. Relative prices and distribution are simultaneously determined, but they are different.

A remark on the treatment of the wage rate in connection with (2) is necessary. Since the production price system here includes the numeraire equation, the real wage rate can be measured. That is, Sraffa normalised national income as

$$p(I-A)x=1.$$

Thus the real wage rate is represented by the ratio of wages to national income.

In what follows, the foundation of Sraffa's theorem will be discussed.

1) Production price with ex post wages is also denoted by p, but there will be no confusion.

2. At the outset, the set of basic assumptions will be described here, and the fundamental results of the production price system (1) will be stated. The following assumptions are made:

(A. 1) $A \geqq 0, \quad F \geq 0^n.$

(A. 2) $L > 0_n.$

(A. 3) A is indecomposable.

(A. 4) The economy is productive: $\rho(A) < 1$

Now, let

$\bar{R}$: the maximum profit rate,

and this is defined by

(3) $$p^M = (1+\bar{R})p^M A,$$

namely,

(4) $$\bar{R} = \frac{1}{\rho(A)} - 1.$$

Evidently, $\bar{R} > 0$ in view of (A. 4). Moreover,

Proposition 1.

(i) Productiveness is equivalent to $\bar{R} > 0$.

(ii) For $\forall \omega \geqq 0$ and $\forall r \in [0, \bar{R}]$ there exists a $p > 0_n$.[2)]

Proof.

(i) Trivial in the light of Frobenius' theorem

(ii) (1) can be rewritten as

(5) $$p = \omega L[I - (1+r)A]^{-1},$$

so that the conclusion soon follows. Q. E. D.

Now, based on Sraffa's theorem one can ask: if the relative prices and distribution are different, it should be possible to describe distribution even in terms of relative price. How is it done? The subsequent proposition gives the first clue for this problem.

The relative prices determined by (1) are independent of distribution, if and only if $d(p_j/p_i)/dr = 0$. Without loss of generality, one may put $i=1$. Then, this condition is equivalent to

(6) $$\frac{dp_j(r)/dr}{dp_1(r)/dr} = \frac{p_j(r)}{p_1(r)}$$

The converse is also true: (6) implies

$$\frac{d(p_j/p_1)}{dr} = \frac{p_i(dp_j/dr) - p_j(dp_1/dr)}{p_1^2} = 0.$$

Bearing this in mind, one can show:

Proposition 2. The relative prices p satisfying (1) are independent of distribution, if and only if p satisfies (3):

$$p(\lambda I - A) = 0_n, \qquad \lambda = 1/(1+\bar{R}).$$

2) If the wage rate ω is employed in lieu of F, it is easy to see that the productiveness of the economy entails immediately the existence of economically meaningful equilibrium: $p > 0_n$, $\omega > 0$ and $r > 0$. The same is true in the case of the production price system with ex ante wages.

Proof.

Sufficiency: from (1) and (3), one has $p=\omega L/\{1-\lambda(1+r)\}$. Differentiate this with respect to r, and one obtains

$$\frac{dp}{dr}=\frac{\lambda}{1-\lambda(1+r)}p,$$

i. e., (6),

$$\frac{dp_j/dr}{dp_1/dr}=\frac{p_j\lambda/\{1-\lambda(1+r)\}}{p_1\lambda/\{1-\lambda(1+r)\}}=\frac{p_j}{p_1}.$$

Necessity: differentiate (1) with respect to r, and one has

$$(7)\qquad \frac{d\tilde{p}}{dr}[I-(1+r)A]=\tilde{p}A+\frac{d\tilde{\omega}}{dr}L=0_n.$$

where $\tilde{p}=p/p_1$ and $\tilde{\omega}=\omega/p_1$.

Differentiate the middle part of (7) again with respect to r, and one has

$$\frac{d\tilde{p}}{dr}A+\frac{d^2\tilde{\omega}}{dr^2}L=0_n,$$

so that,

$$\frac{d^2\tilde{\omega}}{dr^2}=0.$$

Namely, $\frac{d\tilde{\omega}}{dr}$ is a constant, and one can write

$$\tilde{\omega}=\frac{d\tilde{\omega}}{dr}r+k,$$

where k is a temporary constant. In view of this,

$$\tilde{p}=(1+r)\tilde{p}A+\left(\frac{d\tilde{\omega}}{dr}r+k\right)L=\tilde{p}A+kL$$

owing to (7). Hence, from

$$(1+r)\tilde{p}A=\tilde{p}-\tilde{\omega}L=\left(1-\frac{\tilde{\omega}}{k}\right)\tilde{p}+\frac{\tilde{\omega}}{k}\tilde{p}A,$$

it follows that $p\propto pA$. That $\lambda=(1+R)^{-1}$ is derived from (A. 4). Hence, (3) hold.

Q. E. D

The value system is necessarily referred to in the dialogue between Marx and Sraffa. As discussed in Chapter I, value is defined by

$$(8)\qquad w=wA+L.$$

Then, the following proposition holds true with respect to (6):

Proposition 3. The following six conditions are all equivalent:

(i) prices determined by (1) satisfy (6),

(ii) prices are proportional to values: $p\propto w$,

(iii) L is the left-side eigen vector of A: $L\propto LA$,

(iv) the organic compositions of capital are uniform,

(v) value is proportional to direct labour input: $w\propto L$,

(vi) price is proportional to direct labour input: $p\propto L$.[3)]

3) Also see Nuti. As for the organic composition of capital, see Chapter I, p. 9, of Fujimori [82].

Proof.

First, the equivalence of (iii), (v) and (vi) will be shown. If $p=\omega L[I-(1+r)A]^{-1}\propto L$, then $\omega L\propto L-(1+r)LA$, and hence $L\propto LA$. Conversely, $L\propto LA$ implies

$$p=\omega L[I-(1+r)A]^{-1}=\omega L[I+(1+r)A+(1+r)^2A^2+\cdots]\propto L.$$

Thus, (iii)$\Longleftrightarrow$(vi). Likewise, one has (iii)$\Longleftrightarrow$(v). From these two, (iii)$\Longrightarrow$(ii) follows.

Nextly, (iii) implies $wA\hat{L}^{-1}=L(I-A)^{-1}\hat{L}^{-1}\propto L\hat{L}^{-1}$, i. e., (iv). If (iv) is true, then $wA\hat{L}^{-1}\hat{L}\propto L$, i. e.,

$$L(I-A)^{-1}A=L(A+A^2+A^3+\cdots)\propto L,$$

namely, (v). Thus, (iii)$\Longrightarrow$(iv)$\Longrightarrow$(v).

Thirdly, in view of Pronosition 2, it is obvious that (ii)$\Longrightarrow\frac{d\tilde{p}}{d\omega}=0\Longrightarrow$(i) ; on the other hand, if (i) holds, one has

$$p=\omega\left[\frac{1+\bar{R}}{\bar{R}-r}\right]L\propto L$$

from (1) and (3), so that (vi) holds. Hence, (ii)$\Longrightarrow$(i)$\Longrightarrow$(vi). Q. E. D.

Proposition 4. If any one of the above six conditions is true, the wage-profit curve becomes linear for an arbitrary $x\geqq 0^n$.

In fact, as shown in the proof of Proposition 2, one has

$$\frac{d\bar{\omega}}{dr}=\text{const.}$$

Thus, these two propositions show that the production structure as specified by condition (iv), i. e., the uniformity of the organic compositions of capital or capital-labour ratios, permits the description of distribution, for any output, without being influenced by relative prices. This condition, however, is no longer satisfied in general.

Consider a dual condition of (6).

Definition 1. (*Standard facter, stadard commodity*) A combination of commodities is called the standard commodity, if it is identical with or proportional to the combination of the aggregate means of production required for its own production. The ratio of the standard commodity to its input subtracted by unity is called the standard factor.

Let

R : standard factor,

x^s : standard commoidty,

and these two are defined by the *standard system* :

(9) $$x^s=(1+R)Ax^s\ ;\ Lx^s=1.$$

Proposition 5.

(i) The productiveness of the economy is equivalent to that there exists a nonnegative standard commodity : Pd. C.$\Longleftrightarrow\exists x^s\geqq 0^n$.

(ii) standard factor=maximum profit rate : $R=\bar{R}$.

(The proof is trivial.)

Adopt the normalization of the *standard national income* as :

$$px^s-pAx^s=1.$$

Substitute (9) into this, and it follows that

$$RpAx^s = p(I-A)x^s = 1.$$

The real wage rate is now measured as

(10) $$\hat{\omega} = \frac{\omega}{RpAx^s} = \omega,$$

which is called the *standard wage rate*. Then, one has:

Proposition 6. (On the basis of the above two normalisations)

(11) $$r = R(1-\hat{\omega}).$$

Proof.

Postmultiply (1) by x^s, and it follows that

$$r = \frac{px^s - pAx^s - \omega Lx^s}{pAx^s} = R - \frac{\omega Lx^s}{RpAx^s} R = R(1-\hat{\omega}).$$

In view of (10), the conclusion is derived. Q. E. D.

Note that (11) holds true with prices not necessarily satisfying (6), and that the domain of ω is

$$0 \leqq \omega \leqq 1.$$

Now, it is seen that if the standard commodity is employed as weights of aggregation, a remarkable formula of distribution is established.

3. The discussion of the standard commodity defined by (9) is dual to that of (6). With reference to this fact, investment and consumption in equilibrium growth will be considered.

Take a standard bundle of wage goods $f \geq 0^n$, and fix it in such a way that for a given ω there exists a unique $c > 0$ fulfilling

(12) $$cpf = \omega,$$

where c represents the number of wage goods units and hence the real wage, and that f is normalized as

(13) $$c(\max \omega) = 1.$$

Disregard capitalists' consumption, and equilibrium growth can be described by $(\delta^c ; q^c)$ such that

(14) $$q^c = (1+\delta^c)Aq^c + cfLq^c,$$

where

q^c $n \times 1$: capacity output with ex post wages,

δ^c : equilibrium growth rate with ex post wages.

For p^M determined by (3), write

$$k = cp^M fLq^c,$$

(15) $$\hat{k} = \frac{k}{\bar{R}p^M Aq^c},$$

and one can prove:

Proposition 7. Let $Lq^c = 1$ and $p^M f = 1$, and

(16) $$\delta^c = \bar{R}(1-\hat{k}).$$

In fact, premultiply (13) by p^M, and (16) soon follows in view of (3) and (15).

Thus, a linear investment-consumption frontier is obtained. It is easy to see that (11) and (16) are represented by one and the same line-segment in $\boldsymbol{R}_+^2$. It is also easy to see that as r varies from 0 to $\bar{R}$, δ^c also varies from 0 to R. Observe, however, that for a given ω, $\omega \neq \hat{k}$ in general.

4. It has been shown that either if price satisfies (6) or if the standard commodity is employed as weights of aggregation, the wage-profit curve becomes linear. Starting from the condition that price and distribution are independent of each other, one has obtained a sufficient condition concerning the aggregator, i. e., the employment of the standard commodity as weights of aggregation.

As shown by Proposition 6, the standard commodity makes it possible to describe distribution independent of prices even in an economy that does not fulfill the uniformity of the organic compositions of capital. What is the essence of the standard commodity with such a remarkable property? Since Sraffa exemplified his theorem by the use of the standard commodity, the question can be posed as: why can distribution differ from relative prices, and in what sense?

The standard commodity is contemplated in relation to the numeraire equation (2), namely the normalization of the standard national income, so as to measure the real wage rate.

Therefore, the implication of this normalization needs to be investigated. In the next section, an examination of the standard commodity from the angle of the transformation of values into prices will be made.

§ 2. The theory of transformation and the standard commodity.

1. Let us dissect the standard commodity according to Marx's idea of the transformation of values into prices.

Along the line of Marx's transformation theorems of the equallity, such as total price= total value, one has the following:

Theorem I.

(i) $p^{\dagger}x^s = wx^s$ is equivalent to $Rp^{\dagger}Ax^s = Lx^s$, where $p^{\dagger}$ satisfies (1).

(ii) If $p^{\dagger}x^s = wx^s$, then $r = R(1-\omega)$.

Proof.

(i) From (8) and (9), one has the following three:

$$wx^s = (1+R)wAx^s,$$

$$wx^s = wAx^s + Lx^s,$$

$$px^s = (1+R)pAx^s,$$

so that for $p^{\dagger}$ satisfying $p^{\dagger}x^s = wx^s$, the subsequent

$$Rp^{\dagger}Ax^s = Lx^s,$$

holds. The converse is also true.

(ii) Evaluating the profit rate, one gets

$$r=\frac{p^{\dagger}x^{s}-p^{\dagger}Ax^{s}-\omega Lx^{s}}{p^{\dagger}Ax^{s}}$$
$$=R-\frac{Lx^{s}}{p^{\dagger}Ax^{s}}\omega$$
$$=R(1-\omega).$$

Q. E. D.

Remark that ω in the above (ii) represents the real wage rate measured on the basis of $p^{\dagger}x^{s}=wx^{s}$.

Now, compare Proposition 6 with this theorem, and it is seen that to put $px^{s}-pAx^{s}=Lx^{s}=1$ and transform ω into $\hat{\omega}$ is tantamount to considering the existence of value behind price and the regulation of the total price by the total value.

2. The transformational iteration á la Marx-Okishio can be formulated with respect to the price system with ex post wages in the following manner: for $\forall z\epsilon \boldsymbol{R}_{+}^{n}$, consider the sequences $\{w^{t}\}$ and $\{r^{t}\}$

$$w^{t+1}=(1+r^{t})w^{t}\cdot A+\omega L,$$

(17) $$1+r^{t+1}=\frac{w^{t}\cdot z-\omega Lz}{w^{t}\cdot Az},$$

where $w^{0}=w$.

The convergence of the sequences generated by (17) is shown by the following proposition:

Proposition 8. For $z=x^{s}$, the sequences generated by (7) converge, yielding

$$p^{\dagger}=\lim_{t\to\infty} w^{t}=\omega L[I-(1+r)A]^{-1};$$

$$1+r^{t}=\frac{w^{0}\cdot x^{s}-\omega Lx^{s}}{w^{0}x^{s}}(1+R), \text{ for} \forall t.$$

Proof.

(17) can be rewritten as

(18) $$w^{t+1}=\frac{w^{t}\cdot z-\omega Lz}{w^{t}\cdot Az}w^{t}\cdot A+\omega L,$$

so that one has for $\forall z\epsilon \boldsymbol{R}^{n}$

(19) $$w^{t}\cdot z=w^{t-1}\cdot z=\cdots=w^{0}z.$$

and especially for x^{s}. Postmultiply (17a) by x^{s}, and one has

$$w^{t+1}x^{s}=(1+r^{t})w^{t}Ax^{s}+\omega Lx^{s},$$

and hence,

(20) $$1+r^{t+1}=\frac{wx^{s}-\omega Lx^{s}}{wx^{s}}(1+R)=\text{const.}$$

in view of (19). Therefore, (17) is written as

$$w^{t+1}=(1+r)w^{t}A+\omega L.$$

From this

$$w^{t+1}=(1+r)^{t+1}wA^{t+1}+\omega L\sum_{k=0}^{t-1}(1+r)^{k}A^{k}.$$

Since

$$\lim_{t\to\infty} A^t = 0$$

and

$$\sum_{t=0}^{\infty}(1+r)^t A^t = [I-(1+r)A]^{-1}$$

in the light of (A. 4) for $r<R$, it follows that

$$\lim_{t\to\infty} w^{t+1} = \omega L[I-(1+r)A]^{-1}.$$

Since r^t is constant, $1+r$ is derived from (20). Q. E. D.

Note that in the above iteration, ω is treated as given and kept constant throughout the transformation procedure. Hence, the first step of the transformation yields the exact level of the profit rate. In order to reach the price vector, however, the iteration should be repeated ad infinitum.

In order to see that the profit rate is evaluated in the first step of the iteration by values and a given real wage rate, let us compare distribution in the worlds of price and value.

Consider again the standard bundle of wage goods (12), and let

r' : value rate of profit,

$\omega' = cwf$: real wage rate in value terms.

Now, one has

(21) $$r' = \frac{wx^s - wAx^s - \omega' Lx^s}{wAx^s} = R(1-\omega')$$

It is easy to see that (11) and (21) represent one and the same line-segment in $\boldsymbol{R}^2_+$:

Proposition 9. Let $r' = r'(r)$. If the standard commodity is used as weights of aggregation, $r'(r)$ is a continuous increasing function of r satisfying $r'(R) = R$ and $r'(0) = 0$.

Proof.

It is trivial that the chain of mappings $r \to \omega \to c \to \omega' \to r'$ is continuous. It is not difficult to see that one can derive: $\frac{dr'}{dr} > 0$ and $r'(R) = R$.

Next, if $r=0$, then $p(A+fL)=1$, so that $r'=0$. i. e., $r'(0)=0$. Q. E. D.

Note that, however, in general $r' \neq r$ for a given c. In other words, distribution in value terms does not suffice to determine distribution in price terms in this case.

Nevertheless, the fundamental Marxian theorem of the production prices with ex post wages can be proved on the basis of the above proposition.

The rate of surplus value can be defined as

(22) $$\mu = \frac{1}{cwf} - 1.$$

Proposition 10. The rate of profit with ex post wages is positive, if and only if the rate of surplus value is positive: $r>0 \Longleftrightarrow \mu>0$.

Proof.

If $r=0$, then obviously $c=1$, so that $p=w$ in view of (5) and (13). Thus, one has $\mu=0$, because $cwf = pf = 1$.

Now, μ is a continuous, increasing functien of r, so that if $r > 0$, then $\mu > 0$.

The converse is also true, because the function $\mu = \mu(r)$ has its inverse because it is increasing. Q. E. D.

3. Is the parellelism of distribution in value and price terms characteristic of the production price with ex post wages? In order to see this, let us consider Marx's production price system with ex ante wages. It is represented by

(23) $$p = (1+\pi)(pA + \omega L).$$

Let

π' : value rate of profit with ex ante wages, and one has

$$w = (1+\pi')(wA + \omega' L).$$

Then, the following holds:

Proposition 11.

(i) $p^* x^s = w x^s$ is equivalent ts $R p^* A x^s = L x^s$, where p^* satisfies (23).

(ii) If x^s is employed as weights of aggregation, one has

$$\pi = \frac{R(1-\omega)}{1+\omega R}, \quad \pi' = \frac{R(1-\omega')}{1+\omega' R)}.$$

(As for the proof, the same algebraic manipulation as in the proof of Theorem I yields the conclusion.)

Although $\pi(\omega)$ and $\pi'(\omega')$ are not linear, they are of the same form, and are represented by the same curve $\boldsymbol{R}^2_+$. Hence, the parellelism is seen to be characteristic of the standard commodity.

4. It should be emphasized that the parellelism has been confirmed, but that the deterministic relationship between value and price is not obtained. That is, such an equality as Morishima-Seton's equality does not hold if the standard commodity is used as weights of aggregation: $\omega \neq \omega'$ in general.

A strong equality $\omega = \omega'$ holds, if any three of the following four equalities hold:

total price = total value,

total price of constant capital = total value of constant capital,

total price of variable capital = total value of variable capital,

total profit = total surplus value.

Namely, on the basis of (1) and (8), one can write

(24) $$\begin{aligned} px &= wx, \\ pAx &= wAx, \\ pFLx &= wFLx, \\ rpAx &= Lx - wFLx. \end{aligned}$$

It is easy to see that any three of these four constitute an independent system.

Except in the case where prices are proportional to values, one condition sufficient for

(24) to hold is given by

$$x = x^s,$$

and

$$f \propto x^s.$$

If these conditions are satisfied, one has that $r = r'$, and hence, r is expressed in teams of μ: from (22), $\omega = \omega'$ and Theorem I (ii), one obtains

(25) $$r = R\frac{\mu}{1+\mu}.$$

In view of the above, some neo-Ricardians often assert that Sraffa solved the transformation problem. (Cf. e. g. Eatwell.) However, the condition $f \propto x^s$ is extremely restrictive and fictitious, just as its dual condition $L \propto LA$ imposed on the production structure is hardly satisfied or realistic. If Sraffa's economic theory aims at criticising the neoclassical aggregate production function for not explaining distribution except in the case where $R \propto RA$, neo-Ricardian's interpretation concerning (25) is completely against their preceptor. In fact, μ in (25) is a function of x^s, and hence it does not reflect the actual rate of surplus value. Thus, the equality (25), as opposed to Morishima-Seton's equality, cannot be said to carry an important economic meaning. It is difficult to say that Sraffa solved the transformation problem.

The standard commodity should be grasped merely as weights of aggregation. As an aggregator, it links price with value as shown by Theorem I.

Neo-Ricardians may refuse Theorem I, for the role played by the standard commodity seems to be independent of value. Does the standard commodity, however, not need such an idea of transformation of values into prices as its basis outside the Leontief ecnomy? In the next section, an extension of the standard commodity will be contemplated in the von Neumann economy case.

§ 3. Joint-production and the Sraffa proportion.

1. Sraffa introduced joint-production into his economic theory with respect to fixed capital as von Neumann did. Sraffa and other neo-Ricardians, however, are inclined to discuss the joint-production system with square input and output matrices.

Considering the von Neumann economy described in Chapters III and IV, let us make the following assumptions:

(B. 1) $$A \geqq 0, \quad B \geqq 0, \quad L \geq 0_n, \quad F \geq 0^m.$$

(B. 2) $$x \geqq 0^n \text{ and } Dx \geqq 0^m \Longrightarrow Lx > 0.$$

(B. 3) $$\exists x \geqq 0^n : Dx \geq 0^m.$$

Now, the production price system (1) can be extended to

(26) $$pB = (1+r)pA + \omega L.$$

The value system is expressed as

(27) $$wB = wA + L.$$

R and x^s, satisfying the extended standard system,

(28) $$Bx^s=(1+R)Ax^s, \quad x^s\neq 0^n,$$

are called respectively the standard factor and the *Sraffa proportion*. The Sraffa proportion is now an intensity vector, which does not represent a composite commodity.

The maximum rate of profit is determined by the dual equation of (28) as

(29) $$p^M B=(1+\bar{R})p^M A, \quad p^M\geq 0_m.$$

Introduce here the subsequent three conditions:

(V. S) $$\text{rank } D=\text{rank}\begin{bmatrix} D \\ -L \end{bmatrix}$$

(S. S) $$\text{rank } D(R)<n,$$

where $D(R)=B-(1+R)A$.

(P. Pf. C′) $$\exists p^M\geq 0_m, \quad \bar{R}>0 : p^M B=(1+\bar{R})p^M A.$$

The condition S. S. guarantees the existence of a non-zero Sraffa proportion. The remaining two were introduced in Chapter III. (p. 48, p. 59.) In a Leontief economy, the productiveness of the economy ensures S. S. as V. S. In a von Neumann economy, however, the two are not mutually related.

In an economy with competitive techniques, namely the case where the number of the processes is greater than that of the goods, one may well think that S. S. is satisfied. However, $x^s\geq 0^n$ is not yet ensured. Therefore, it does not follow that

$$RpAx^s>0$$

or that

$$Lx^s>0.$$

Then, the normalization as $Lx^s=1$ will not make sense.

Nevertheless, the transformation theorem still holds:

Theorem II. Assume V. S., S. S. and P. Pf. C′.

(i) $p^\dagger Bx^s=wBx^s$ is equivalent to $Rp^\dagger Ax^s=Lx^s$, where $p^\dagger$ satisfies (26).

(ii) Suppose $p^\dagger Bx^s=wBx^s$. If x^s is used as weights of aggregation, then

$$r=R(1-\omega).$$

(As for the proof, the same mathematical manipulation as in the proof of Theorem I soon yields the results.)

This theorem says that in so far as $x^s\neq 0^n$ the wage-profit curve is linearly described. Since $R>0$ in view of (B. 3), the linear wage-profit curve thus obtained makes sense.

Consequently, in the von Neumann economy case it is made clear that Sraffa's theory of the standard commodity should rest on Marx's theory of transformation.

2. It is evident that the extension of the standard commodity in equality terms comes up against difficulties. Therefore, its generalisation á la von Neumann will be made in this subsection.

Consider the following two programming problems:

(30) $$\text{Max } \{R \mid Bx^s\geqq(1+R)Ax^s, \ x^s\geqq 0^n.\}$$

(31) $$\text{Min } \{\bar{R} \mid p^M B\leqq(1+\bar{R})p^M A, \ p^M\geqq 0_m.\}$$

Definition 2. (*Standard factor, Sraffa proportion*) The optimum solution of (30) is called the Sraffa proportion, and the maximum magnitude of R defines the standard factor.

This definition gives the most generalised description of the standard factor and the standard commodity. Write

R^M : standard factor,

x^s $n\times 1$: Sraffa proportion.

(31) defines the *maximum warranted profit rate*, min $\bar{R}$, with the *Sraffa price* as its optimum solution. Let

$\bar{R}^m$: maximum warranted profit rate,

p^M : Sraffa price vector.

It is obvious that there exist $R^M>0$ and $x^s\geq 0^n$ in a von Neumann economy satisfying (B.1) through (B.3). Hence, the expenditure of labour can be normalized as

(32) $$Lx^s=1.$$

Make furthermore the following:

(B.4) $$B1^n>0^m, \qquad 1_m A>0_n.$$

(B.5) $$\min\{\bar{R}\mid p^M B\leqq(1+\bar{R})p^M A,\ p^M\geqq 0_m.\}>0.$$

By dint of (B.4), it follows that

$$\bar{R}^m\leqq R^M,$$

and in view of (B.5), one has

$$\bar{R}^m>0.$$

Now introduce (12) again, and consider the equilibrium depicted by the following:

(33) $$\mathrm{Max}\{\delta^c\mid Bq^c\geqq(1+\delta^c)Aq^c+cfLq^c,\ q^c\geqq 0^n\}.$$

(34) $$\mathrm{Min}\{r^w\mid p^w B\leqq(1+r^w)p^w A+cp^w fL,\ p^w\geqq 0_m\}.$$

The optimum solutions of the two problems and their optimised magnitudes define the *von Neumann equilibrium with ex post wages*, represented by the quadruplex $(r^w, p^w, \delta^c, q^c)$, which is an extension of the state of the economy determined by (1) and (13). One may well call r^w in the above the *warranted profit rate with ex post wages*.

Propositions 6 and 7 are extended to:

Proposition 12.

(i) Let $Lx^s=1$. Then,

(35) $$r^w\geqq R^M(1-\omega).$$

(ii) Let $Lq^c=1$ and $p^m f=1$. Then,

$$\delta^c\leqq\bar{R}^m(1-\hat{k}).$$

Here,

$$\omega=\frac{\omega}{Rp^w Ax^s},\quad \hat{k}=\frac{k}{\bar{R}p^M Aq^c},\quad k=cp^M fLq^c.$$

Proof.

(i) Postmultiply (34) by x^s, and one has

$$r^w\geqq\frac{p^w Bx^s-\omega Lx^s}{p^w Ax^s}-1.$$

Since (30) entails

$$\frac{p^w B x^s}{p^w A x^s} \geqq 1 + R^M,$$

it follows that

$$r^w \geqq R^M(1-\omega).$$

(ii) Likewise, premultiply (35) by p^M. The rest of the proof is similar to that of (i).

Q. E. D.

Rewrite (35) and (36) respectively as:

(37) $$r^w{}_{\min} = R^M(1-\omega),$$

and

(38) $$\delta^c{}_{\max} = \bar{R}^m(1-\hat{k}).$$

It is easy to see that (37) and (38) are no longer represented by the same line-segment in $\boldsymbol{R}^2_+$, because in general $R^M \neq \bar{R}^m$.

3. Furthermore, consider the wage-profit curve in terms of optimum value. As discussed in Chapter IV, the optimum value is an optimum solution of the following:

(39) $$\mathrm{Max}\{c\Lambda f L x \mid \Lambda B \leqq \Lambda A + L,\ \Lambda \geqq 0_m\}.$$

The warranted rate of profit in terms of optimum value with ex post wages is defined, for an optimum value Λ^0, by

(40) $$\min\{r'^w \mid \Lambda^0 B \leqq (1+r'^w)\Lambda^0 A + \omega' L,\ \omega' = c\Lambda^0 f\}.$$

In the same manner as (35) is derived, one has

(41) $$r'^w \geqq R^M(1-\omega'),$$

where

$$\hat{\omega}' = \frac{\omega'}{R\Lambda^0 A x^s}.$$

Thus, one may write

$$r'^w{}_{\min} = R^M(1-\hat{\omega}')$$

instead of (41).

It appears that formally (37) and (42) are represented by one and the same line-segment in $\boldsymbol{R}^2_+$, but there is no definite correspondence between the two, such as that observed in Proposition 10.

In order to make clear the relationship between the two, let us introduce the rate of unpaid labour:

$$\mu' = \frac{1}{c\Lambda^0 f} - 1.$$

Obviously, μ' is a continuous decreasing function of c. Since r^w and r'^w defined by (34) and (40) respectively are monotone decreasing functions of c, one can say:

Proposition 13. r^w and r'^w are monotone increasing function of μ'.

(As for the proof, refer to Proposition IV-6.)

Finally, the fundamental Marxian theorem in this case will be stated as follows.

Introduce the rate of surplus labour η expressed by

$$\eta=\frac{Lx}{Lz^0}-1,$$

where, Lz^0 is determined by the dual of (39):

(43) $$\min\{Lz \mid Bz\geqq Az+cfLx,\ z\geqq 0^n\}.$$

Proposition 14. $r^w>0$ and $\delta^c>0$ are equivalent to $\eta>0$.

Proof.

Premultiply (43) by p^w and postmultiply (34) by z^0. In view of $Lx=(1+\eta)Lz^0$, it follows that

$$c\eta p^w fLz^0\leqq r^w p^w Az^0.$$

If $p^wf=0$, then $r^w>0$ follows from (B.5); if $p^wf>0$, $\eta>0$ implies $r^w>0$.

Nextly, premultiply (33) by Λ^0, and it follows that

$$\delta^c\Lambda^0Aq^c\leqq Lq^c\Lambda^0cf\mu.$$

Since $\Lambda^0Aq^c\geqq 0$, $\delta^c>0$ imlies $\mu'\geqq 0$.

If $\mu'=0$, then $\mu'=\eta=0$ in view of Proposition IV-2, i. e., $Lx=Lz^0$ for $\forall x$, so that invariably one has $z^0=x$, and

$$Bz^0\geqq Az^0+cfLx=Az^0+cfLz^0.$$

This implies $\delta^c=0$. Hence, $\delta^c>0$ implies $\eta>0$.

Finaly, $\delta^c\geqq r^w$ in general. This completes the proof. Q. E. D.

§ 4. Conclusions.

1. One of Sraffa's contribution to economics is that he gave a rigorous description of the classical production price such as (1), which is, although wages are paid ex ante, applied to Marxian economics in Chapter I.

On this basis Sraffa's theory starts from the distinction between distribution and relative prices, and the standard commodity is seen to be a weight of aggregation to exemplify this distinction: in an economy in which the organic composition of capital is not uniform, to describe distribution independently of changes in relative prices necessitates the standard commodity.

Since the productiveness of the Leontief economy ensures the existence of the standard commodity, the effectiveness of the standard commodity appears to be independent of the concept of value.

Recall that Marx's transformation has, as discussed in Chapter I, two aspects—the transformation of values into prices and that of the rate of surplus value into the profit rate. This concerns the newly created value portion, whilst that concerns the valuation of goods. Theorems I and II show that Sraffa's procedure, i. e., to normalize the amount of labour in the standard system and measure the wage rate in terms of the standard national income is equivalent to Marx' idea of transformation. This fact becomes clearer in the von Neumann economy case.

On the basis of Marxian transformation theorems, it is now seen that distribution, which concerns the newly created value portion, is different from relative prices, which are valuation of goods, because the newly created value portion itself does not depend on how to price goods.

The standpoint from which Sraffa's theory of the standard commodity is grasped as a theory of the Marxian transformation is also found in Meek [1] [2], Dobb [1]—[4], Roncaglia [1] [2], etc. Based on the Sraffian discussion in the Leontief economy case, they rather maintained that Sraffa solved Marx's transformation problem by way of Ricardo's problem of the invariable measure of value, or that Sraffa had different objectives of analysis.

As discussed above, however, Marx is rather fundamental to Sraffa.

2. Sraffa's standard commodity concerns a dual aspect of value in the sense that on a wage-profit curve the notion of value corresponds to the extreme point $(0, \omega_{\max})$, whereas the standard commodity corresponds to another extreme point $(\bar{R}, 0)$.

It is also easy to see that the standard commodity is the von Neumann proportion with $F=0^m$. Moreover, it must be observed that the case in which distribution can be expressed independently of relative prices is restricted to the case where wages are paid ex post, unless the organic composition of capital is uniform. In general cases, including the ex ante wage case, only the parellelism between the value and price worlds, instead of the linearity of the wage-profit curve, can be observed, as shown by Propositions 11 and 12. These facts mean not only that the standard commodity is a specialised von Neumann proportion, but also that the theory of the standard commodity should be extended on the basis of von Neumann theory.

3. Sraffa did not pose the problem of why the profit rate can be positive. Since most of his theoretical thrusts are based on the wage-profit curve, it is worth mentioning that the discussion made on the basis of the wage-profit curve makes sense provided that the wage-profit curve exists in $\boldsymbol{R}^2_+$. This appendix provided the fundamental Marxian theorems pertaining to Sraffa's price system.

4. Sraffa's theorem was given a firm basis and an attempt to extend the standard commodity to the Sraffa proportion was made. Thus, Sraffa's theory of price is inserted into a wider context of Marx and Marx-von Neumann. This does not negate, but reinforces, the theoretical importance of Sraffa's contribution to economics.