A formal and complete overhaul of the generalised fundamental Marxian theorem[*)]

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§ 1. Introduction.

The capitalist society exists on the exploitation of workers by capitalists——this is a well-known proposition by Marx, and in the past quarter of century, formal analysis of this proposition has been made considerably As well-known, the first rigorous formulation of the statement, " profits imply exploitation ", was made by Okishio [63], which was later named, together with its converse, the fundamental Marxian theorem by Morishima [73], and further it was developed by Morishima [74] in the von Neumann framework. Morishima established the theorem from the standpoint of the duality——the duality between growth and profit: it is called the generalised fundamental Marxian theorem.

Recently, however, Washida [86b] pointed out that the growth aspect should be separated from exploitation, and made an attempt to reconstruct the fundamental Marxian theorem.

The objective of this short memorandum is to review some formal aspect of the discussion concerning the generalised fundamental Marxian theorem. Our subsequent discussion concentrates rather on the " overhaul " of the generalised fundamental Marxian theorem, and confirms an exact formulation of the fundamental Marxian theorem. We shall first prove four propositions which state facts related to exploitation, profit and growth. In order to avoid possible ambiguity, we shall repeat proofs of all propositions.[1)] Secondly, we shall discuss the implications of those four propositions separately: which one represents Marx' theory of exploitation and how remaining ones support the core of the fundamental Marxian theorem. The Marx-Okishio type fundamental Marxian theorem will be regarded as legiti-

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*) This is originally written as comments on Washida [86b], and the author benefitted much by the interchange with Mr. Washida. Needless to say, however, the author alone is solely responsible for remaining errors.

1) We note here some of the mathematical notations employed in our discussion.

We sometimes employ logical notations such as $\forall$, $\exists$, $\wedge$ and $\sim$. They indicate " for all ", " there exist(s) (a(n))", " and ", and " not " (negation) respectively.

As for the vector comparison, we promise:

$$x>y \Leftarrow\Rightarrow (\forall i) x_i > y_i \,;$$
$$x \geq y \Leftarrow\Rightarrow (\forall i) x_i \geqq y_i \wedge x \neq y \,;$$
$$x \geqq y \Leftarrow\Rightarrow (\forall i) x_i \geqq y_i \,;$$

where x_i indicates the i-th component of a vector x. Also refer to Fujimori [82].

mate one. This means that the duality as declared by Morishima should be discarded in so far as the explanation of exploitation is concerned. Our review indicates the imporatnce of the "overhaul" of the theorem rather than its "reconstruction."

§ 2. Overhauling the generalised fundamental Marxian theorem.

A joint-production economy with n processes, which, being represented by linear techniques, produces m types of goods, and with homogeneous labour is called the *Marx-von Neumann-Morishima economy*. The economy is fully described with the following, if m and n are finite:

A $m\times n$: input matrix.
B $m\times n$: output matrix.
L $1\times n$: labour vector.
F $m\times 1$: wage good bundle.

Definition 1. A quadruplet $(x, p, \delta, \gamma)\in R^n\times R^m\times R\times R$, defined by

(1) $$\mathrm{Max}\{\delta|Bx\geqq\delta(A+FL)x,\ x\geqq 0^n\},$$

and

(2) $$\mathrm{Min}\{\gamma|pB\leqq\gamma p(A+FL),\ p\geqq 0_m\},$$

is called an *equilibrium* of the given Marx-von Neumann-Morishima economy. We often denote an epuilibrium by $(x^c, p^w, \delta^c, \gamma^w)$, where δ^c is called the *potential growth factor*, γ^w the *warranted profit factor*.

Definition 2. The amount of labour necessary to reproduce the wage good bundle for a unit of labour expenditure, defined by

(3) $$\mathrm{Min}\{Lz|Bz\geqq Az+F,\ z\geqq 0^n\},$$

is called *necessary labour*.

It is convenient to consider the dual problem of this:

(4) $$\mathrm{Max}\{\Lambda F|\Lambda B\leqq\Lambda A+L,\ \Lambda\geqq 0_m\}.$$

In the subsequent discussion, we denote the optimum solutions of (3) and (4) respectively by z^o and Λ^o, when they exist.

Definition 3. The subsequent ratio is called the *rate of surplus labour*:

(5) $$\eta=\frac{1}{Lz^o}-1.$$ [2)]

We place the following assumption on our discussion:

A1 $$A\geq O,\ B\geqq O,\ L\geq 0_n,\ F\geq 0^m.$$

Remark that we do not mention A1 in stating propositions and theorems.

In our subsequent discussion, we shall first consider the following two conditions:

(Pd) $$\exists\, x\geqq 0^n : Bx>Ax$$

(wIL) $$x\geqq 0^n\wedge Bx\geq Ax+F \Rightarrow Lx>0.$$

Since the meanings of these two conditions are well-known, we do not mention them

2) From the angle of formalism, our discussion is obtained by considering $Lx^a=1$, where $x^a\in R^n$ is the actual operation vector, in Morishima's discussion.

here.[3)]

The following two lemmas are trivial:

Lemma 1. $$\eta>0 \Rightarrow 1>Lz^{o}\geqq 0.$$

Lemma 2. $$Lz^{o}=\Lambda^{o}F.$$

We also require:

Lemma 3. Let K be an $m\times n$ matrix. The following two are exclusive:

$$\exists x\geqq 0^{n} : Kx>0^{m}.$$

$$\exists p\geq 0_{m} : pK\leqq 0_{n}.$$

(Gale[60], Theorem 2.10, p. 49.)

Now, let us first consider profitability.

Proposition 1. $$\gamma^{w}>1 \Rightarrow \eta>0.$$

Proof)

Suppose $\eta\leqq 0$. Then, $Lz^{o}>1$ by Definition 3 (and A1). It follows that by Lemma 2 one has, for some $\Lambda^{o}\geqq 0_{m}$,

$$\Lambda^{o}B\leqq\Lambda^{o}A+(\Lambda^{o}F)L.$$

This implies that $\gamma^{w}\leqq 1$ in (2). q. e. d.

Remark 1. The first key point here is that we can prove this proposition without any additional conditions.

Proposition 2. Suppose (Pd) and (wIL). Then, $\eta>0 \Rightarrow \gamma^{w}>1$.

Proof)

Premultiply (3) by p^{w}, and one has

(6) $$p^{w}Bz^{o}\geqq p^{w}Az^{o}+p^{w}F(1+\eta)Lz^{o}$$

in view of (5). Postmultiply (2) by z^{o}, and one has

(7) $$p^{w}Bz^{o}\leqq\gamma^{w}p^{w}(A+FL)z^{o}.$$

From (6) and (7), one has

(8) $$(\gamma^{w}-1)(p^{w}Az^{o}+p^{w}FLz^{o})\geqq\eta p^{w}FLz^{o}.$$

Now, (wIL) implies $Lz^{o}>0$. As for $p^{w}F$, we can consider two cases:

(i) if $p^{w}F=0$, then $\exists p\geq 0_{n} : pB\leqq\gamma pA$. This implies, by Lemma 3, $\sim\exists x\geqq 0^{m} : Bx>\gamma Ax$. In the light of (Pd), it follows that $\gamma^{w}>1$.

(ii) if $p^{w}F\neq 0$, then trivially $p^{w}F>0$ (by A1), so that $\gamma^{w}>1$. q. e. d.

Thus, the above two can be integrated into one:

Corollary 1. (Marx-Okishio.) Suppose (Pd) and (wIL). Then,

$$\gamma^{w}>1 \Leftrightarrow \eta>0.$$

Remark 2. It is important that this corollary does not depend on the potential growth factor.

Now, let us turn to the potential growth:

Proposition 3. Suppose (Pd). Then, $\eta=0 \Rightarrow \delta^{c}>1$.

Proof)

3) The first one is a productiveness condition of the economy and the second indicates that labour is necessary to reproduce wage goods.

In view of Lemma 1, $\eta>0$ implies that for some $z\geqq 0^n$, holds $1>Lz\geqq 0$, and hence

(9) $$Bz\geqq Az+F\geq Az+FLz.$$

By rearranging the order of goods, one can write $F=\begin{bmatrix}F_1\\0\end{bmatrix}$. Let A and B be split correspondingly as $A=\begin{bmatrix}A_1\\A_2\end{bmatrix}$ and $B=\begin{bmatrix}B_1\\B_2\end{bmatrix}$.

Then. one has

(10) $$B_1z>A_1z+F_1(Lz),$$
$$B_2z\geqq A_2z.$$

Meanwhile, because of (Pd), one has

$$X\equiv\{x\geqq 0^n|Bx>Ax+F\}\neq\phi.$$

so that there exists a $y\in V(z)$ (near z) such that $By>Ay+FLy$, Hence, $\delta^c>1$. q. e. d.

Remark 3. It should be observed that the above proposition does not needs (wIL). This is the second key point in our discussion.

In order to prove the converse of Proposition 3, let us make the following condition :

(sIL) $$Lx^c>0.$$

Proposition 4. Suppose (sIL). Then, $\delta^c>1 \Rightarrow \eta>0$.

Proof)

Postmultiply (4) by x^c, and one has

(11) $$\Lambda^oBx^c\leqq\Lambda^oAx^c+Lx^c,$$

while premultiply (1) by Λ^o, one has

(12) $$\Lambda^oBx^c\geqq\delta^c\Lambda^oAx^c+\delta^c\Lambda^oFLx^c$$
$$\geqq\Lambda^oAx^c+\delta^c\Lambda^oFLx^c.$$

Hence, by combining (11) and (12), one has

(13) $$(1-\delta^c(\Lambda^oF))Lx^c\geqq\Lambda^oAx^c.$$

Since $\Lambda^oAx^c\geqq 0$ (by A1), (sIL) implies that $1-\delta^c(\Lambda^oF)\geqq 0$. In view of Lemma 2, one can infer

$$\eta=\frac{1}{\Lambda^oF}-1\geqq\delta^c-1>0.$$ q. e. d.

Remark 4. Note that the supposition implies (Pd), and hence ensures the solvability of (3), so that η can be defined.

Remark 5. The proof of the above proposition gives the information concerning the possibility of the case in which $\delta^c>1$ but $\eta\leqq 0$. This case is possible, when

(14) $$Lx^c=0$$

and

(15) $$\Lambda^oAx^c=0.$$

as will be seen in (13).

§ 3. Concluding remarks.

1. By way of conclusion, let us make some remarks out of our formal review on the

generalised fundamental Marxian theorem. We mentioned the two key points in our discussion. We shall develop in the following what they mean.

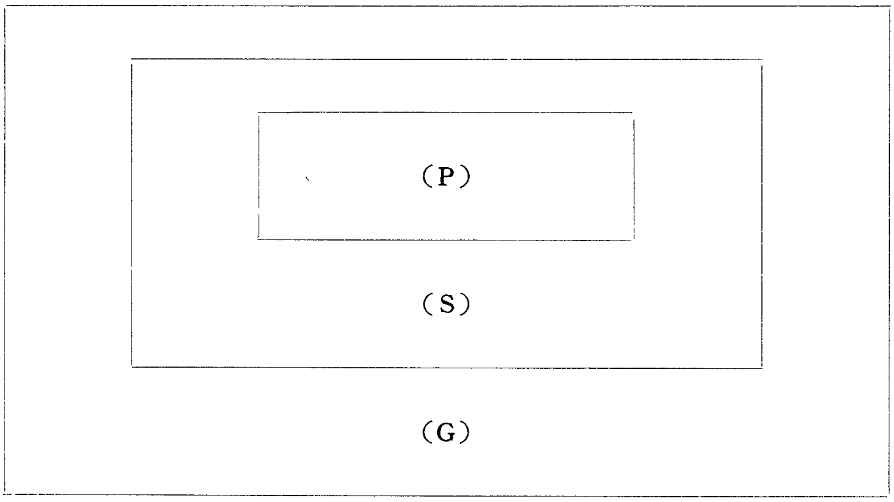
At the outset, if we consider four propositions separately, we see that each proposition is proved by different sets of assumptions. That is to say, they have different economic contents. For the sake of brevity, let us denote three statements as: (P) $\gamma^w>1$; (S) $\eta>0$; and (G) $\delta^c>1$.

By combining Propositions 1 and 3, we can show, like the Venn diagram in the theory of sets, the relationship among the three statements as follows, where (Pd) is assumed:

Note that all those regions (P), (S) and (G) are all in the region where (Pd) is true, as seen by Remark 4.

It must be observed that there is a gap between regions in which (P), (S) and (G) are true. Propositions 2 and 4 show that those gaps can be covered exactly by conditions (wIL) and (sIL) respectively: Proposition 2 shows that condition (wIL) is sufficient to cover the gap between (S) and (P), while condition (sIL) is sufficient to cover the gap between (G) and (S), as shown by Proposition 4.

FIGURE 1.



Although integrating propositions makes their meanings rather ambiguous, we have Corollary 1 by combining the first two propositions. As opposed to the Leontief economy case, however, (P) and (S) are not equivalent without additional conditions in a Marx-von Neumann-Morishima economy. The core of Corollary 1 clearly lies in the part stated by Proposition 1, which is originally shown by Okishio [63] in the Leontief economy case. Corollary 1 shows that exploitation is made clear by looking at the price equilibrium alone, as asserted by Washida [86b]. In this sense, Corollary 1, or more precisely Proposition 1, should be regarded as the fundamental Marxian theorem. It is worth noting that the overhaul of the generalised fundamental Marxian theorem will bring about the Marx-Okishio type fundamental Marxian theorem.

If we combine Propositions 3 and 4 to make Corollary 2, and compare it with Corollary 1, we see that there is a discrepancy between them: Corollary 2 will be stated as:

Corollary 2. Suppose (Pd) and (sIL). $\eta>0 \Leftrightarrow \delta^c>1$.

They require different conditions, i. e., (sIL) or (wIL), on the expenditure of labour. In this sense, the duality between growth and distribution does not go hand in hand: we can confirm with ease that the duality between growth and distribution is redundant, in so far as we are concerned with exploitation. Therefore, the generalised fundamental Marxian theorem itself needs not be reconstructed, as opposed to Washida [86b]'s intention, in relation to the explanation of exploitation.

As we have pointed out above, these three statements are logically different in its validitity. This is why we say "overhaul", and not "reconstruction."

2. Furthermore, if we take Proposition 3 alone into consideration, then we see that there is a possibility of potential growth without surplus labour and hence without exploitation. Since we do not exclude the system such as the automation system of production, this does not negate the validity of the core of the fundamental Marxian theorem.

On the other hand, Proposition 4, as remarked at the end of its proof with (14) and (15), gives rather a clue to understand the situation of such an automation system than a clue to exploitation.[4)]

With regards to the gaps between (G) and (S), and between (P) and (S), it is worth mentioning that our overhaul of the generalised fundamental Marxian theorem indicates an important possibility of designing a new society in which growth is possible without assigning profits.[5)]

3. Nevertheless, it is worth asking whether we should abandon the duality approach. The point is the place of (sIL) and (wIL). It is interesting to note that the importance of conditions such as (sIL) is already known in the discussion of the duality between the wage-profit and investment-consumption frontiers. If they are accepted, one can combine Corollaries 1 and 2, and thus the warranted profit factor and the potential growth factor.[6)]

Evidently, it will make sense to discuss the duality between growth and distribution by excluding the case of such an automation system as specified by (14) and (15). With (sIL) and (wIL), a combination of Corollaries 1 and 2 will virtually result in Morishima's theorem as:

4) We do not discuss a rigorous definition of the automation system of production here.

5) Washida [86a] made an attempt to show that the fundamental Marxian theorem is not valid in a noncapitalist economy. His new formulation should be regarded, however as such that involves a different definition of necessary labour. Therefore, his idea that the noncapitalist economy can grow without surplus labour whenever there is a saving is still based on the existence of these gaps, provided that the fundamental Marxian theorem is regerded as a logical and formal condition to explain growth and profitability of the economy.

6) In the activity analysis, it is often assumed that the land of Cockaigne is impossible. In a Marx-von Neumann-Morishima economy, this can be interpreted as

$$x\geqq 0 \wedge Bx \geq Ax \Leftrightarrow Lx>0.$$

This is a stronger condition, implying both (sIL) and (wIL). See Fujimori [82], p. 49.

Corollary 3. (Morishima.) Suppose (Pd), (wIL) and (sIL). Then, $\eta>0$, $\gamma^w>1$ and $\delta^c>1$ are all equivalent.

As already pointed out in the above, Corollary 3 involves some redundant element, and hence should be regarded as an appendix to the theory of exploitation, but this corollary is still important, as it states that both growth and distribution of economies with (sIL) and (wIL) depends on exploitation. Morishima's theorem can be applied to the greatest region in which (P) matters.

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