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An Extended Portfolio Theory in a Capital Asset Pricing Model

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Abstract

In this paper, standard market portfolio formalism has been extended using the Lagrange multiplier method. The standard market portfolio has been found to be a special case where the multiplier parameter value $v = 0$ in the present formalism. A maximum efficiency investment curve (MEIC) was derived, and the curve has been proven to be a hyperbolic function. The properties of the fluctuation of the stock issues were studied, and it has been found that all the values of stock issues considered are restricted to the domain created by the MEIC.

Key Words: Extended Portfolio Theory, CAPM

Introduction

Previously in this research, the fluctuation of stock price indices[1-5] has been studied in light of oscillation theory. In those papers, the time development for the Nikkei 255, a typical index, was analyzed. Following from that, in this paper the properties of a portfolio formed by a combination of stock issues[6, 7] are studied. In the capital asset pricing model (CAPM), it is assumed that the stock issues under investigation form an equilibrium market. Therefore, the market portfolio is determined by maximizing the risk price. In the present paper, the research is begun with standard market portfolio theory and then is extended to non-standard portfolio theory.

This paper is organized as follows. In Section 1 an extended portfolio theory in terms of the Lagrange multiplier method is presented, and in Section 2 the present theory is applied to Nikkei 225 data from the Tokyo Stock Market. The properties of the portfolio are then studied.

1. Extended Portfolio Theory in a Capital Asset Pricing Model

1.1 *The Market Portfolio*

First, a review of the properties of the market portfolio in the capital asset pricing model (CAPM) is in order. The issues under consideration are assumed to be in an equilibrium market. The variable m is defined as a stock issue, and r_m^n is its profit rate at a month n, where $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$. The variables μ_m and σ_m are then defined as the average value of r_m^n over the N months period and its standard deviation, respectively. Suppose a portfolio is made in terms of M stock issues and w_m is defined as the investment rate for the stock issue m. Then the average value of the profit rate of this portfolio is written as

$$\mu_P = \sum_{m=1}^{M} w_m \mu_m \qquad (1)$$

and the standard deviation of the profit rate is

$$\sigma_P = \sqrt{\sum_{mm'} \mathrm{Cov}\,(r_m, r_{m'}) w_m w_{m'}}, \qquad (2)$$

where $\mathrm{Cov}\,(r_m, r_{m'})$ is the covariance of the fluctuation of the stock prices between issues m and m'. Moreover, μ_P is found in the following expression:

$$\mu_P - r_f = \sum_{m=1}^{M} w_m (\mu_m - r_f), \qquad (3)$$

where r_f is the monthly rate of the risk free bond.

Suppose these M issues form an equilibrium market. The market portfolio is then defined as the one which maximizes the risk price $(\mu_P - r_f)/\sigma_P$. Namely,

$$\begin{aligned} \frac{\partial}{\partial w_m}\left(\frac{\mu_P - r_f}{\sigma_P}\right) &= \frac{\mu_m - r_f}{\sigma_P} - \frac{\mu_P - r_f}{\sigma_P^3} \sum_{m'} \mathrm{Cov}\,(r_m, r_{m'}) w_{m'} \\ &= \frac{\mu_m - r_f}{\sigma_P} - \frac{(\mu_P - r_f)\sigma_m}{\sigma_P^3} \sum_{m'} \rho_{mm'} \sigma_{m'} w_{m'} = 0, \end{aligned} \qquad (4)$$

where $\rho_{mm'}$ is the correlation function of the fluctuation of the stock prices between issues m and m'. The correlation function is related to the covariance as follows:

$$\mathrm{Cov}\,(r_m, r_{m'}) = \rho_{mm'} \sigma_m \sigma_{m'}. \qquad (5)$$

Therefore, the investment rate w_m satisfies the following equation:

$$\frac{\mu_P - r_f}{\sigma_P^2} \sum_{m'} \rho_{mm'} \sigma_{m'} w_{m'} = \frac{\mu_m - r_f}{\sigma_m}. \qquad (6)$$

Note that Equation (4) determines the extreme value for the risk price, where the value sometimes becomes minimum but not maximum. This point is discussed in more detail in Section 2.

1.2 *The Extended Portfolio Theory*

Suppose a portfolio of M stock issues is considered. Then the average value of the profit rate of this portfolio μ_P and the standard deviation of the profit rate σ_P vary depending on the investment rate for each issue. However, the value (σ_P, μ_P) of this portfolio is found to be located in a limited domain in this two dimensional space. The boundary curve of this domain can be obtained by maximizing the value of μ_P under the condition that the value of σ_P is fixed. This curve is called the maximum efficiency investment curve (MEIC). A method to obtain the MEIC is proposed in this next section.

The standard approach of determining the investment ratio w_m for each issue is extended by introducing the Lagrange multiplier method. The Lagrange multipliers are denoted as λ and v, and the following two constraints are set:

$$\sigma_P = \sqrt{\sum_{mm'} \mathrm{Cov}\,(r_m, r_{m'})\, w_m w_{m'}} = \sigma_{P_0} \tag{7}$$

and

$$W = \sum_m w_m = 1. \tag{8}$$

Under these constraints, the investment ratios w_m which maximize the value of μ_P are identified, namely

$$\mu_P - r_f = \sum_{m=1}^{M} w_m(\mu_m - r_f). \tag{9}$$

Assume that the investment ratios w_m are maximized:

$$S = \mu_P - r_f - \lambda\sigma_P - vW. \tag{10}$$

Namely,

$$\begin{aligned} \frac{\partial S}{\partial w_m} &= \mu_m - r_f - \frac{\lambda}{\sigma_P}\sum_{m'} \mathrm{Cov}\,(r_m, r_{m'})\, w_m - v \\ &= \mu_m - r_f - \frac{\lambda\sigma_m}{\sigma_P}\sum_{m'} \rho_{mm'}\sigma_{m'} w_{m'} = 0. \end{aligned} \tag{11}$$

Therefore, w_m satisfies the following equation:

$$\frac{\lambda}{\sigma_P}\sum_{m'} \rho_{mm'}\sigma_{m'} w_{m'} = \frac{\mu_m - r_f - v}{\sigma_m}. \tag{12}$$

A new variable x_m is introduced by

$$x_m = \frac{\lambda\sigma_m w_m}{\sigma_P}. \tag{13}$$

Then Equation (12) is written as

$$\sum_{m'} \rho_{mm'} x_{m'} = u_m. \tag{14}$$

By solving the simultaneous equations of Equation (14), the investment ratios are obtained:

$$w_m = \frac{\sigma_P}{\lambda\sigma_m} x_m, \tag{15}$$

under the constraint $\sigma_P =$ constant. In Equation (14), u_m is defined by

$$u_m = \frac{\mu_m - r_f - v}{\sigma_m}. \tag{16}$$

Since the sum of the investment ratios is

$$\sum_m w_m = \frac{\sigma_P}{\lambda}\sum_m \frac{x_m}{\sigma_m} = 1, \tag{17}$$

one of the Lagrange multiplier is determined as

$$\frac{\sigma_P}{\lambda} = \frac{1}{\sum_m \frac{x_m}{\sigma_m}}. \tag{18}$$

From Equation (12),

$$\sum_m (\mu_m - r_f - v) w_m = \frac{\lambda}{\sigma_P} \sum_{mm'} \rho_{mm'} \sigma_m w_m \sigma_{m'} w_{m'}. \tag{19}$$

This next equation is finally obtained

$$\mu_P - r_f - v = \lambda \sigma_P, \tag{20}$$

and therefore

$$\frac{\mu_P - r_f}{\sigma_P} = \lambda + \frac{v}{\sigma_P}. \tag{21}$$

In the present approach, the investment ratio w_m is obtained by solving the simultaneous equations of Equation (14), where μ_m is shifted by the common parameter v (called the chemical potential in physics). Note that the case for $v = 0$ in Equation (21) corresponds to the standard market portfolio.

1.3 *The Maximum Efficiency Investment Curve*

In this section a proof is given showing that the maximum efficiency investment curve (MEIC) is hyperbolic function for (σ_P, μ_P). Starting with the simultaneous equations of Equation (14) in the extended portfolio,

$$\sum_{m'} \rho_{mm'} x_{m'} = \frac{\mu_m - v}{\sigma_m}, \tag{22}$$

where the risk free bond $r_f = 0$ for simplicity. Then the solution of Equation (22) is written as follows:

$$x_m = \sum_{m'} (\rho^{-1})_{mm'} \frac{\mu_{m'}}{\sigma_{m'}} - v \sum_{m'} (\rho^{-1})_{mm'} \frac{1}{\sigma_{m'}}, \tag{23}$$

where $(\rho^{-1})_{mm'}$ is the inverse matrix element which is symmetrical for indices m and m'. Then the investment ratio is

$$w_m = \frac{x_m}{\sigma_m \sum_{m'} \frac{x_{m'}}{\sigma_{m'}}}. \tag{24}$$

Next, the following constants are defined:

$$A \equiv \sum_{mm'} \frac{\mu_m}{\sigma_m} (\rho^{-1})_{mm'} \frac{\mu_{m'}}{\sigma_{m'}}, \tag{25}$$

$$R \equiv \sum_{mm'} \frac{1}{\sigma_m} (\rho^{-1})_{mm'} \frac{\mu_{m'}}{\sigma_{m'}} = \sum_{mm'} \frac{\mu_m}{\sigma_m} (\rho^{-1})_{mm'} \frac{1}{\sigma_{m'}}, \tag{26}$$

$$E \equiv \sum_{mm'} \frac{1}{\sigma_m} (\rho^{-1})_{mm'} \frac{1}{\sigma_{m'}}. \tag{27}$$

Then,

$$\sum_m \frac{x_m}{\sigma_m} = R - vE \tag{28}$$

and

$$w_m = \frac{x_m}{\sigma_m (R - vE)}. \tag{29}$$

Therefore, μ_P is expressed in terms of these constants as follows:

$$\mu_P = \sum_m \mu_m w_m = \frac{A - vR}{R - vE} \tag{30}$$

On the other hand, σ_P is also expressed by these constants as follows:

$$\begin{aligned} \sigma_P^2 &= \sum_{mm'} \sigma_m w_m \rho_{mm'} \sigma_{m'} w_{m'} \\ &= \frac{1}{(R - vE)^2} \sum_{mm'} \left(\frac{\mu_m}{\sigma_m} - v \frac{1}{\sigma_m} \right) (\rho^{-1})_{mm'} \left(\frac{\mu_{m'}}{\sigma_{m'}} - v \frac{1}{\sigma_{m'}} \right) \\ &= \frac{A - 2vR + v^2 E}{(R - vE)^2}. \end{aligned} \tag{31}$$

In deriving Equation (31), Equations (23) through (29) were used. In addition, one can eliminate the variable v by combining Equations (30) and (31). After a straightforward calculation, the following hyperbolic function for (σ_P, μ_P) is finally obtained:

$$\frac{\sigma_P^2}{\sigma_P^{\infty 2}} - \frac{(\mu_P - \mu_P^\infty)^2}{\dfrac{\sigma_P^{\infty 2}}{\sigma_P^{0^2} - \sigma_P^{\infty 2}} (\mu_P^0 - \mu_P^\infty)^2} = 1, \tag{32}$$

where the parameters are expressed by the constants as follows:

$$\mu_P^0 \equiv \mu_P (v = 0) = \frac{A}{R}, \tag{33}$$

$$\mu_P^\infty \equiv \mu_P (v = \infty) = \frac{R}{E}, \tag{34}$$

$$\sigma_P^{0^2} \equiv \sigma_P^2 (v = 0) = \frac{A}{R^2}, \tag{35}$$

$$\sigma_P^{\infty 2} \equiv \sigma_P^2 (v = \infty) = \frac{1}{E}. \tag{36}$$

Note that the value of the standard market portfolio (σ_P^0, μ_P^0) is a solution for this hyperbolic function. This extended portfolio formalism is applied to Tokyo Nikkei 255 issues in the next section.

2. Application to Nikkei 225 Data from the Tokyo Stock Market

In this section the extended portfolio formalism is applied to the data of the Tokyo Stock Market, and the properties of the MEIC and the standard market portfolio are studied. For example, eight stock issues which had relatively high risk prices in 2003 and 2004 have been chosen. They are Sanyo Electric, Hino Motors, Taisei Corporation, Asahi Kasei Corporation, Kaneka Corporation, Furukawa Electric, Tsugami Corporation, and SoftBank Corporation. Table 1 shows the investment ratios w_m, the average profit rate μ_P, the standard deviation σ_P, and the risk price μ_P/σ_P for the market portfolio.

Table 1 The values of w_m, μ_P, σ_P and μ_P/σ_P in the market portfolio

w_m	2003	2004
Sanyo Electric (6764)	0.355	4.556
Hino Motors (7205)	2.233	−0.417
Taisei Corp (1801)	−1.732	1.120
Asahi Kasei Corp (3407)	0.867	0.464
Kaneka Corp (4118)	−0.569	0.277
Furukawa Electric (5801)	0.446	−3.532
Tsugami Corp (6101)	−1.393	−1.374
SoftBank Corp (9984)	0.793	−0.094
μ_P	0.096	−0.368
σ_P	0.030	0.058
μ_P/σ_P	3.250	−6.387

Note that the risk price of the market portfolio in 2004 becomes a negative value, whereas it was a positive value in 2003.

Table 2 The values of w_m, μ_P, σ_P and μ_P/σ_P in the extended portfolio theory

v	−0.02	0	0.02	0.06	∞	−0.2	−0.15	−0.1
Sanyo Electric	6.118	4.556	3.743	2.908	1.165	−0.410	−1.320	−4.717
Hino Motors	−0.588	−0.417	−0.328	−0.236	−0.045	0.128	0.228	0.600
Taisei Corp	1.524	1.120	0.910	0.695	0.244	−0.162	−0.397	−1.275
Asahi Kasei Corp	0.686	0.464	0.349	0.231	−0.016	−0.239	−0.368	−0.849
Kaneka Corp	0.064	0.277	0.387	0.501	0.738	0.952	1.075	1.537
Furukawa Electric	−4.915	−3.532	−2.813	−2.073	−0.531	0.863	1.668	4.675
Tsugami Corp	−1.795	−1.374	−1.155	−0.930	−0.460	−0.035	0.210	1.125
SoftBank Corp	−0.094	−0.094	−0.094	−0.095	−0.095	−0.095	−0.096	−0.097
μ_P	−0.508	−0.368	−0.295	−0.220	−0.063	0.078	0.159	0.464
σ_P	0.080	0.058	0.046	0.036	0.024	0.034	0.045	0.094
μ_P/σ_P	−6.337	−6.387	−6.347	−6.106	−2.653	2.283	3.528	4.943

Table 2 shows the investment ratios w_m, the average profit rate μ_P, the standard deviation σ_P, and the risk price μ_P/σ_P for the extended portfolio in 2004. Each column corresponds to $v = -0.02, 0, 0.02, \ldots$ Note that the case for $v = 0$ corresponds to the standard market portfolio of Table 1. In Figure 1, the values μ_P and σ_P for eight stock issues and the market portfolio value for 2004 have been plotted, along with a curve connecting the points of the maximum efficiency investment curve (MEIC). As demonstrated in the previous section, the connected curve is a hyperbolic curve defined by Equation (32). It is clear that all the points for given eight stock issues are inside the domain formed by the MEIC and that the market portfolio value is a point on this curve.

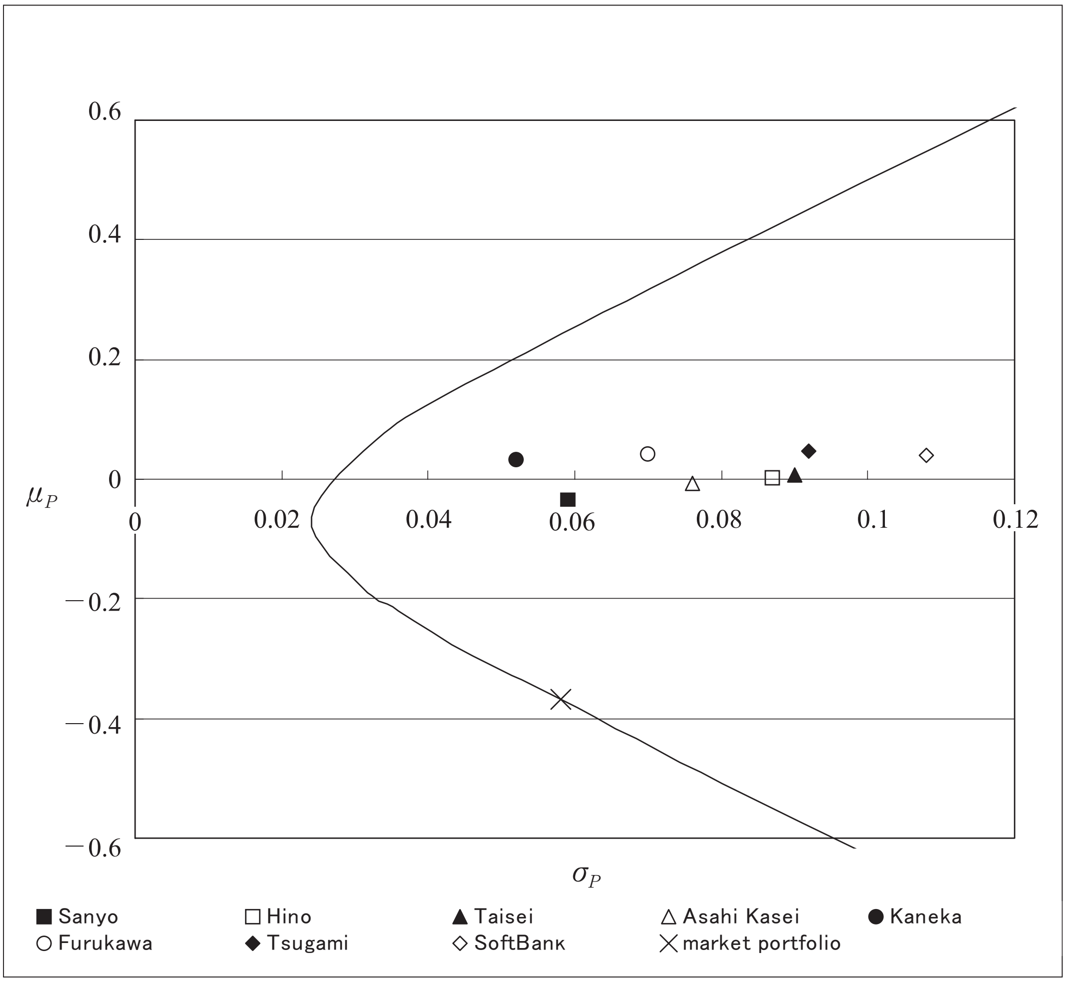

Figure 1. The values of μ_m and σ_m for eight stock issues and the maximum efficient investment curve

In summary, standard market portfolio formalism has been extended using Lagrange multiplier method. The standard market portfolio is a special case where the parameter value $v = 0$ in the present formalism. The maximum efficiency investment curve (MEIC) has been obtained and proven to be a hyperbolic function. All the values of the stock issues considered are restricted to the domain defined by the MEIC.

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