Conditional stability of Kleinian groups

Katsuhiko Matsuzaki *
Department of Mathematics, Ochanomizu University
Tokyo, Japan

Abstract

We prove that any algebraically small quasiconformal deformation of a finitely generated Kleinian group is induced by a quasiconformal automorphism of the Riemann sphere with a small dilatation.

This is a research announcement on a problem of stability of Kleinian groups. More detailed arguments are contained in a forthcoming monograph [5].

Let Γ be a finitely generated non-elementary Kleinian group, which is identified with a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, with a fixed system of generators $\Gamma = \langle \gamma_1, \ldots, \gamma_N \rangle$. We consider the set of $\mathrm{PSL}_2(\mathbf{C})$-representations

$$\mathrm{Hom}(\Gamma) = \{\rho \mid \rho : \Gamma \to \mathrm{PSL}_2(\mathbf{C}) \text{ is a homomorphism}\}.$$

This is regarded as an analytic subset of $\mathrm{PSL}_2(\mathbf{C})^N$ by the correspondence

$$\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_N)) \in \mathrm{PSL}_2(\mathbf{C})^N.$$

We also consider an analytic set of all representations sending any parabolic element to a parabolic one or the identity, that is

$$\mathrm{PHom}(\Gamma) = \{\rho \in \mathrm{Hom}(\Gamma) \mid \mathrm{tr}^2\rho(\gamma) = 4 \text{ for any parabolic } \gamma \in \Gamma\}.$$

Let $\Omega(\Gamma)$ be the region of discontinuity of Γ in $\hat{\mathbf{C}}$. We consider the Teichmüller space $T(\Omega(\Gamma)/\Gamma)$ of the union of orbifolds $\Omega(\Gamma)/\Gamma$. For every

*E-mail: matsuzak@math.ocha.ac.jp

$[\mu] \in T(\Omega(\Gamma)/\Gamma)$, we denote by f_μ a quasiconformal automorphism of $\hat{\mathbf{C}}$ that gives the deformation $[\mu]$ of the complex structure of $\Omega(\Gamma)/\Gamma$ and that satisfies a suitable normalization condition. Then a holomorphic map

$$\tilde{\Psi} : T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C}) \to \mathrm{PHom}(\Gamma)$$

is defined by

$$\rho(\gamma) = (A \circ f_\mu)\gamma(A \circ f_\mu)^{-1}$$

for any pair $([\mu], A) \in T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C})$. This map $\tilde{\Psi}$ is well-defined independently of the choice of the representative μ. Moreover, the Sullivan rigidity theorem implies that the image of $\tilde{\Psi}$ coincides with the set of the whole representations induced by quasiconformal automorphisms of $\hat{\mathbf{C}}$ (cf. [5, Chapter 5]). This set is called the quasiconformal deformation space and denoted by $\mathrm{QHom}(\Gamma)$ ($\subset \mathrm{Hom}(\Gamma)$).

For a torsion-free geometrically finite Kleinian group Γ, Marden [4] proved that $\mathrm{QHom}(\Gamma)$ is a complex *regular* submanifold of $\mathrm{PSL}_2(\mathbf{C})^N$. In this note, we prove that this is satisfied for any finitely generated Kleinian group Γ. It is known that the derivative $d\tilde{\Psi}$ of $\tilde{\Psi}$ at any point is injective. Hence $\tilde{\Psi}$ is a holomorphic immersion onto $\mathrm{QHom}(\Gamma)$. Thus our problem is just compatibility of the Teichmüller topology of $T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C})$ and the topology of $\mathrm{QHom}(\Gamma)$, which is the algebraic topology for $\mathrm{PSL}_2(\mathbf{C})$-representations:

Problem If ρ_n converge to id in $\mathrm{QHom}(\Gamma)$, do there always exist $(t_n, A_n) \in \tilde{\Psi}^{-1}(\rho_n)$ such that t_n converge to the base point $0 \in T(\Omega(\Gamma)/\Gamma)$ and A_n converge to id as $n \to \infty$?

This problem is originated in Bers [1, p.578]. See also [2]. Later Krushkal published a series of papers (cf. [3]) concerning this problem. A finitely generated Kleinian group is called *conditionally stable* or *quasi-stable* if it satisfies the property in the problem above.

We will show that a result on geometric convergence of Kleinian groups yields the affirmative answer to this problem.

Theorem 1 *Any finitely generated Kleinian group* Γ *is conditionally stable. Hence* $\mathrm{QHom}(\Gamma)$ *is a complex regular submanifold of* $\mathrm{PSL}_2(\mathbf{C})^N$.

We first remark the following two facts.

Lemma 2 Γ *is conditionally stable if any component subgroup of* Γ *is conditionally stable.*

Proof. This follows easily from the definition of conditional stability. ■

Lemma 3 *Let Γ be a finitely generated Kleinian group and Γ' a subgroup of Γ of finite index. If Γ' is conditionally stable, then so is Γ.*

Proof. Suppose that Γ is not conditionally stable. Then there is a sequence $\rho_n \in \mathrm{QHom}(\Gamma)$ converging to *id* such that the maximal dilatation of the extremal quasiconformal automorphism f_n inducing ρ_n does not tend to 1 as $n \to \infty$. Here the extremal quasiconformal map is the one with the smallest maximal dilatation among quasiconformal maps with the required property.

We restrict ρ_n to the subgroup Γ' and have $\rho'_n \in \mathrm{QHom}(\Gamma')$. Then ρ'_n converges to *id* and f_n induces ρ'_n. Since Γ' is of finite index in Γ, f_n is also the extremal quasiconformal automorphism that induces ρ'_n (cf. [6]). But this contradicts the assumption that Γ' is conditionally stable. Thus we see that Γ is also conditionally stable. ■

By these facts, it suffice to consider torsion-free function groups Γ for proving Theorem 1. It is known that such Γ is constructed from elementary groups, quasifuchsian groups and totally degenerate groups without APT by a finite number of applications of the Maskit combination theorem. Moreover we can see that conditional stability is preserved under the Maskit combination theorem:

Lemma 4 *Assume that a torsion-free function group Γ is constructed from Γ_1 and Γ_2 (as the amalgamated free product or the HNN–extension) by the Maskit combination theorem. If both Γ_1 and Γ_2 are conditionally stable, then so is Γ.*

Proof. See [5, Section 7.3]. ■

Therefore Theorem 1 will complete if it is solved for totally degenerate groups without torsion nor APT. The crucial fact for this step is the following result due to Thurston (cf. [5, Section 7.2]).

Proposition 5 *Let Γ_0 be a finitely generated torsion-free Fuchsian group and $\theta_n : \Gamma_0 \to \Gamma_n$ a sequence of type-preserving isomorphisms onto Kleinian groups, which converges algebraically to a type-preserving isomorphism $\theta : \Gamma_0 \to \Gamma$. If Γ is a totally degenerate group, then Γ_n also converge geometrically to Γ.*

Applying this proposition, we can assert:

Lemma 6 *Under the same circumstances as in Proposition 5, if* Γ_n *and* Γ *are totally degenerate groups, then the marked complex structures* t_n *of* $\Omega(\Gamma_n)/\Gamma_n$ *converge to* t *of* $\Omega(\Gamma)/\Gamma$. *In particular, any torsion-free, totally degenerate group without APT is conditionally stable.*

Proof. Let C_n be the convex core of the hyperbolic manifold $\mathbf{H}^3/\Gamma_n$ and ∂C_n the relative boundary of C_n, which is regarded as a pleated surface with a marked hyperbolic structure s_n. By Proposition 5, we can see that s_n converge to the marked hyperbolic structure s of the boundary surface of the convex core of $\mathbf{H}^3/\Gamma$. By Sullivan's theorem (cf. [5, Section 7.1]), s_n and t_n are in a bounded Teichmüller distance independent of n. Hence $\{t_n\}$ is a bounded sequence in the Teichmüller space and there is a subsequence $\{t_{n'}\}$ which converges to some t'. Then $\Gamma_{n'}$ converge algebraically to a b-group Γ' such that the marked complex structure of D'/Γ' is t', where D' is the invariant component of $\Omega(\Gamma')$. However, Γ' should coincide with Γ, and hence $t' = t$. ■

Thus we obtain Theorem 1.

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